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TECHNICAL MEMORANDUM

No. 1092

ON LAMINAR AND TURBULENT FRICTION

By Th. von Kármán

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ON LAMINAR AND TURBULENT FRICTION[1]

By Th. von Kármán

The theoretical treatment of surface friction of liquids or gases at a solid wall encounters serious difficulties as soon as the processes are no longer defined by the viscosity of the fluid alone, but also involve the forces of inertia with the probable exception of the flow phenomena in capillaries and the problems of lubricant friction — as is the case in nearly all practical problems. All the same, two substantial advances have been achieved in this domain within the last decades; namely, by Prandtl's "boundary layer theory" and Blasius' confirmation of the previously suspected nature of the friction loss in smooth pipes.

Unfortunately, the results of Prandtl's theory have remained confined to a comparatively narrow range, first for the more obvious reason, that the paper work involved for specific cases is enormous, but then also because its physical range of validity is, like the theory of pure friction flow in pipes, restricted to narrow limits. Just as the pure friction flow, the so-called laminar flow in pipes, is replaced by a "turbulent flow" at higher velocities, so the laminar boundary layer is replaced by a "turbulent" boundary layer.

The present report deals, first with the theory of the laminar friction flow, where the basic concepts of Prandtl's boundary layer theory are represented from mathematical and physical points of view, and a method is indicated by means of which even more complicated cases can be treated with simple mathematical means, at least approximately. An attempt is also made to secure a basis for the computation of the turbulent friction by means of formulas through which

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[1] "Über laminare und turbulent Reibung." Z.f.a.M.M., Vol. 1, No. 4, Aug. 1921, pp. 233-252.

the empirical laws of the turbulent pipe resistance can be applied to other problems on friction drag.

MATHEMATICAL IMPORT OF THE BOUNDARY LAYER THEORY[1]

The problem is restricted to two-dimensional flows; the axis $y \neq 0$ is chosen as fixed boundary to which the fluid adheres.

The differential equations of two-dimensional flow with friction can, by introduction of the stream function ψ by means of the formula

$$u = \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}$$

(u and v velocity components in x- and y-direction) and elimination of the pressure, be expressed by the single equation

$$\frac{\partial\Delta\psi}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y} = \nu\Delta\Delta\psi \qquad (1)$$

where Δ denotes the operation $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\nu = \frac{\mu}{\rho}$ kinematic viscosity (μ = viscosity, ρ = density of fluid). The boundary layer theory refers to flow phenomena for which at same distance from the wall the friction shall exert no perceptible effect on the velocity field, so that for great values of y the stream function changes into an assumedly known potential function ψ_0 (x, y, t). At the wall itself both velocity components u and v are to disappear. In order to meet both conditions, first put

$$\psi = \psi_0 - y\cdot\left(\frac{\partial\psi_0}{\partial y}\right)_{y=0} + \sqrt{\nu\psi_1}\left(\frac{y}{\sqrt{\nu}}, x, t\right) \qquad (2)$$

It is clear that for small values of y the first two terms cancel out, leaving only the stream function $\sqrt{\nu\psi_1}$ (the stream function of the boundary layer flow). This is

[1]A list of references on boundary layer theory is given in reference 1.

then so defined that at the wall $\frac{\partial\psi_1}{\partial y} = \frac{\partial\psi_1}{\partial x} = 0$. On the other hand, when ν represents a small quantity, $\frac{y}{\sqrt{\nu}}$ becomes very great for all values of y differing appreciably from zero; hence, to comply with the first condition — transition to potential flow — it is sufficient to determine ψ_1, in such a way that $\sqrt{\nu}\,\frac{\partial\psi_1}{\partial y} = \left(\frac{\partial\psi_0}{\partial y}\right)_{y=0}$ for $\eta = \frac{y}{\sqrt{\nu}} = \infty$.

Thus it is apparent that within the boundary layer (η = finite) the first two terms, outside the boundary layer (η very great) the last two terms annul each other. (Strictly taken, the u component of the boundary layer becomes the u — velocity of the potential flow; for the v component the boundary layer flow gives a quantity of the order of magnitude $\sqrt{\nu}$, which is not contained in the potential flow.)

Introduce formula (2) in equation (1), arrange in powers of $\sqrt{\nu}$ and retain only the highest terms with $\frac{1}{\sqrt{\nu}}$. Then the introduction of $\eta = \frac{y}{\sqrt{\nu}}$ as variable instead of y, the expansion of $\frac{\partial\psi_0}{\partial y}$ and of $\frac{\partial\psi_0}{\partial x}$ according to the formulas

$$\frac{\partial\psi_0}{\partial y} = \left(\frac{\partial\psi_0}{\partial y}\right)_{y=0} + \left(\frac{\partial^2\psi_0}{\partial y^2}\right)_{y=0} \eta\sqrt{\nu},$$

$$\frac{\partial\psi_0}{\partial x} = \left(\frac{\partial^2\psi_0}{\partial x\partial y}\right)_{y=0} \eta\sqrt{\nu}$$

and lastly, considering that $\Delta\Delta\psi_0 = 0$, $\left(\frac{\partial\psi_1}{\partial x}\right)_{y=0} = 0$, affords

$$\frac{1}{\sqrt{\nu}}\left[\frac{\partial^3\psi_1}{\partial t\partial\eta^2} + \frac{\partial\psi_1}{\partial\eta}\frac{\partial^3\psi_1}{\partial x\partial\eta^2} - \frac{\partial\psi_1}{\partial x}\frac{\partial^3\psi_1}{\partial\eta^3} - \frac{\partial^3\psi_1}{\partial\eta^3}\right] = 0 \qquad (3)$$

and, after one integration:

$$\frac{\partial^2\psi_1}{\partial t\partial\eta} + \frac{\partial\psi_1}{\partial\eta}\frac{\partial^2\psi_1}{\partial x\partial\eta} - \frac{\partial\psi_1}{\partial x}\frac{\partial^2\psi_1}{\partial\eta^2} - \frac{\partial^3\psi_1}{\partial\eta^3} = f\,(x,\ t)$$

or with $\frac{\partial \psi_1}{\partial \eta} = u$, $\sqrt{\nu}\,\frac{\partial \psi_1}{\partial x} = -v$, the variable $y = \eta\sqrt{\nu}$ being introduced again,

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} - \nu\,\frac{\partial^2 u}{\partial y^2} = f\,(x,\ t) \tag{4}$$

in agreement with the Prandtl equations.

The function $f\,(x,\ t)$ is determined by the condition for $y = 0$. Since u must change to $u_o = \left(\frac{\partial \psi_o}{\partial y}\right)_{y\,=\,0}$, there is obtained

$$\frac{\partial u_o}{\partial t} + u_o\frac{\partial u_o}{\partial x} = f\,(x,\ t) \tag{4a}$$

for $\eta = \infty$.

The frictionless potential flow follows Bernoulli's equation differentiated along the boundary as streamline (p_o = pressure along the wall)

$$\frac{\partial u_o}{\partial t} + u_o\frac{\partial u_o}{\partial x} = -\,\frac{1}{\rho}\,\frac{\partial p_o}{\partial x} \tag{4b}$$

hence

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\,\frac{1}{\rho}\,\frac{\partial p}{\partial x} + \nu\,\frac{\partial^2 u}{\partial y^2} \tag{4c}$$

The significance of (4) and (4b) obviously is that the assumedly known pressure distribution p_o along the wall which arises from the potential flow is to a certain extent regarded as impressed field (of force) for the boundary layer flow; the pressure differences perpendicular to the wall within the boundary layer being disregarded. It is this very essential hypothesis in Prandtl's theory that leads to the reduction of the number of equations and the arrangement of the entire problem.

THE MOMENTUM THEOREM OF THE BOUNDARY-LAYER THEORY

To bring out the physical sense of the boundary layer theory the evidence contained in the foregoing equations is formulated as follows:

(a) A boundary layer thickness δ (as a function of x) is to exist such that for $y \geqq \delta$ there no perceptible deviation occurs in the flow pattern relative to the potential flow; especially the x-component u of the velocity can be put equal to the wall velocity of the potential flow u_o for $y = \delta(x)$.

(b) Within the boundary layer itself the pressure is only dependent on x and equal to the pressure that corresponds to the potential flow along the wall.

By virtue of the two assumptions (a) and (b) the momentum theorem in the x direction can be applied to a fluid volume bounded by the wall, a short piece of the line $y = \delta(x)$ and two cross sections perpendicular to the wall at x and $x + dx$ (fig. 1.) The increase of momentum is equated to the resultant of the outside forces, which involve the pressure difference, and the friction R at the wall as outside forces. Since for $y = \delta$ the flow changes into frictionless potential flow, the friction at the transitional area between the boundary layer and the outer field can be ignored.

Hence

$$\frac{\partial}{\partial t}\int_0^\delta \rho u dy + \frac{\partial}{\partial x}\int_0^\delta \rho u^2 dy - u_o \frac{\partial}{\partial x}\int_0^\delta \rho u dy = -\delta \frac{\partial p}{\partial x} - R \qquad (5)$$

$\frac{\partial}{\partial t}\int_0^\delta \rho u dy$ is the time rate of change of the momentum contained in the considered volume; $\frac{\partial}{\partial x}\int_0^\delta \rho u^2 dy$ is the excess of momentum leaving the front surface over the amount of momentum entering at the rear; $\frac{\partial}{\partial x}\int_0^\delta \rho u dy$ is the inflow

volume per unit length of the side area $y = \delta(x)$, so that $u_0 \frac{\partial}{\partial x} \int_0^\delta \rho u dy$ indicates the momentum entrained with this volume of fluid.

In laminar flow the frictional force R referred to unit surface is $R = \tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y = 0}$ (τ = shearing stress in the fluid.) Later on, it is shown that equation (5) is practical also for turbulent flow conditions, if u and p are regarded as the average values of the velocity with respect to time and τ_0 is expressed by a corresponding empirical formula.

Equation (5) can, of course, be derived also by integration with respect to y from equation (4) with due regard to (4a) and (4b). It obviously yields, on the basis of plausible assumptions for the velocity profile u (y) in the boundary layer $(0 < y < \delta)$, simply a differential equation for δ, that is, for the boundary layer thickness as a function of x and t. Limited to stationary processes it affords an ordinary differential equation of the first order for δ as a function of x, so that the development of the boundary layer can be followed by comparatively simple calculations. The subsequent report by K. Pohlhausen (reference 1) contains the calculations for a number of practically important cases, so that this method need not be gone into further. His calculations show that in all cases computed by Prandtl's partial differential equations the approximate method ensures results commensurate for all practical purposes. In this manner a further development of the theory is made possible even where the solution of the partial differential equations is extremely tedious, if not impossible.

LAMINAR AND TURBULENT BOUNDARY LAYER

The simplest and practically most important case that the boundary layer theory deals with is the frictional resistance of a plate towed in a fluid at rest parallel to its own plane. Taking the case of two-dimensional motion and referring the motion to the assumedly static plate, the problem is as follows: The parallel flow with the uniform

velocity U is given as a potential motion, the frictional boundary is to start in the origin of the coordinates $x = y = 0$ and for $x \geqq 0$ be given by the axis $y = 0$. The boundary layer thickness and the wall friction is to be computed as a function of x. This problem has already been solved by Blasius (reference 2); he found that the boundary layer thickness increases with $\sqrt{x}$. Computing the friction drag for a plate of length l and width 1 yields the frictional force (for friction on both sides)

$$W = 1.327 \sqrt{\mu \rho l U^3} \tag{6}$$

or, if put, as usual,

$$W = c_f \gamma F \frac{U^2}{2g} \tag{7}$$

where the resistance is referred to the velocity head $\frac{U^2}{2g}$, the surface F and the specific weight of the fluid $\gamma = \rho g$, i

$$W = 1.327 \sqrt{\frac{\nu}{U l}} F \gamma \frac{U^2}{2g} \tag{8}$$

The coefficient of the frictional drag c_f is a function of the Reynolds number, or "reduced velocity" R, in case the nondimensional quantity: velocity x plate length divided by coefficient of kinematic viscosity is introduced as such, so that

$$c_f = 1.327 \frac{1}{\sqrt{R}} \tag{8a}$$

Blasius indicated, in a later report (reference 3) based upon measurements, that formulas (8) and (8a) are no longer valid for large Reynolds numbers, that rather a sudden change occurs in the nature of the resistance and presumably in the state of flow, similar to that occurring in pipe flow at the critical limit. On the other side of the sudden change the resistance increases at more than the 3/2 power of the velocity; hence the resistance coefficient in equation (7) decreases slower than $\frac{1}{\sqrt{R}}$.

Next, it is assumed that the laminar boundary layer, for which the Prandtl-Blasius theory gives the afore-mentioned results, is replaced by a "turbulent boundary layer," in which - as for the turbulent flow in pipes - the velocity is subjected to continuous fluctuations in magnitude and direction. The first consequence of the fluctuations - when plotting the streamlines of the average flow - is that the shearing stress is not caused by the sliding of the adjacent fluid portions alone; the portion of the shearing stress corresponding to the friction becomes small relative to the momentum transport owing to the irregular convection of the supplementary velocities. Up to now, it has not succeeded to explore in some way the nature of this momentum convection - apparently obeying statistical laws - and to make the fluctuation phenomena accompanying the turbulent flow amenable to a theoretical study. In this respect the present article contributes nothing to the solution of the puzzle. The task undertaken here merely involves the introduction of plausible assumptions for the distribution of the average values of the velocity within the boundary layer, which are based on the empirical law of turbulent motion in pipes and the application of the previously derived momentum equation to the equilibrium of the boundary layer. It results in relations of the turbulent friction at a towed plate which are in very good agreement with experience.

THE TURBULENT FLOW IN SMOOTH PIPES

The laws of flow resistance in pipes have been the subject of an unusually large number of experiments. But the empirical material has not improved much up to within recent date because the different degrees of wall roughness had been frequently ignored and the tests were not referred to the physically correct parameter, that is, the Reynolds number. (The only resistance formula, so far, which allows for the relative roughness and the Reynolds number is that by R. von Mises (reference 4.) In many instances no consideration was given to the fact that the constant velocity profile in the pipe is formed only after a fairly long "convection path." Blasius merits the credit of having found an empirical formula by analyzing the material and comparing the best experiments for smooth pipes which very accurately reproduces the nature of the flow resistance over a wide range. According to it the pressure drop for a circular pipe is referred to the velocity head of the average velocity $\frac{v^2}{2g}$

$$h = \lambda \frac{l}{d} \frac{v^2}{2g}, \quad \lambda = 0.316 \sqrt{\frac{\nu}{vd}} \tag{9}$$

where l is pipe length, and d pipe diameter.

By this formula, which represents the experiments extremely well over a wide velocity range the pressure drop is proportional to the 7/4 power of the average velocity as against the previously held conception that the resistance law above the critical velocity would approach the square law fairly soon. R. von Mises incidentally conjectured that contemporarily with increasing velocity, the velocity distribution over the section becomes consistently more uniform so that the measured parabolic velocity profiles merely form a transitional phenomenon and that the profile varies continuously with increasing velocity (reference 4.) In the technical literature a parabolic distribution independent of the velocity is for the most part tacitly assumed. The writer concurs with Von Mises to the extent of assuming a distribution varying with the Reynolds number but with the difference of assuming a well defined distribution function as asymptotic form rather than the uniform distribution, which the velocity distribution approaches at large Reynolds numbers and on perfectly smooth walls. Hence, the assumption that in the turbulent as in the laminar zone, at least, for large Reynolds numbers for which the resistance law (9) holds true, a similar remaining velocity distribution over the cross section exists, so that for increasing throughflow volume all velocities increase in proportion. Prandtl raised the question whether conclusions could be drawn from the empirical law (9) regarding this velocity distribution. He found on the basis of a dimensional analysis that, under certain plausible assumptions, the resistance law definitely defines the distribution of the velocity in the direct vicinity of the wall. The suggestion for the following analysis goes back to a conversation with Prandtl in the fall of 1920. The publication is with his consent, though the process of derivation is somewhat different from his.

Consider a pipe of circular cross section. If the velocity in the pipe axis $(r = 0)$ is indicated by u_{max}, the assumption of a velocity profile independent of the throughflow volume and increasing similarly implies that the ratio $\frac{u}{u_{max}}$ is a definite function of $\frac{r}{a}$ only (r = distance from pipe axis, a = pipe radius).

Hence, the first assumption reads: the velocity at distance r from the pipe axis can be put

$$u = u_{max}\,\varphi\left(\frac{r}{a}\right) \tag{10}$$

$\varphi\left(\frac{r}{a}\right)$ being independent of u_{max}. On doubling the velocity in the center all velocities are doubled.

The second assumption states: the velocity distribution in vicinity of the wall, that is, near $r = a$, is to depend, aside from the physical constants μ and ρ, only on the distance from the wall $\eta = a - r$, and further, on the shearing stress (frictional force) τ_o transferred to the wall. Hence, for small values of η

$$u = f\,(\mu,\ \rho,\ \tau_o,\ \eta) \tag{11}$$

Specifically, the quantity u is to be independent of the pipe dimensions, that is, of a, for small values of η. This assumption is based upon the plausible concept that the velocity distribution next to the wall is independent of the other boundaries of the flow, so that a definite relation exists between the friction on a wall element and the immediately adjacent velocity distribution. Visualize equation (11) developed by increasing powers of η; the first term of the development to read

$$u = f_1(\mu,\ \rho,\ \tau_o)\eta^x \tag{11a}$$

x to be defined later.

The third assumption contains the empirical resistance law: on doubling the velocity the pressure drop and the shearing stress at the wall τ_o is to be increased as $1:2^{7/4}$.

The dimensional equality of the left and right side of (11a) can obviously be maintained only when f contains the quantities μ, ρ, τ_o also only in powers; for, on bearing in mind that $\sqrt{\frac{\tau_o}{\rho}}$ and $\frac{\nu}{\eta}$ have the dimensions of

velocities, it is readily apparent that the only possible dimensionally correct combination is

$$u = B\left(\frac{\tau_0}{\rho}\right)^{\frac{1+x}{2}}\left(\frac{\eta}{\nu}\right)^{x} \tag{12}$$

where B is a nondimensional constant.

On the other hand, since u increases according to (10) in proportion to the throughflow volume; whereas, τ_0 according to the resistance law increases with the 7/4 power of the throughflow volume, the relation

$$\frac{1+x}{2} = 4/7, \quad x = 1/7 \tag{13}$$

must apply.

The first term of a development of the velocity is thus obtained as a function of the wall distance

$$u = B\left(\frac{\tau_0}{\rho}\right)^{4/7}\left(\frac{\eta}{\nu}\right)^{1/7} \tag{12a}$$

or, with $u(\eta)$ denoting the velocity distribution in proximity of the wall, the shearing stress

$$\tau_0 = \frac{1}{B^{4/7}}\rho\nu^{1/4}\lim_{\eta = 0}\left(\frac{u^{7/4}}{\eta^{1/4}}\right) \tag{12b}$$

B is a universal constant valid for smooth walls the magnitude of which is obviously contingent upon the statistical law of the turbulent fluctuation equilibrium.

It is somewhat surprising at first to find the differential quotient at the wall to be infinitely great. Since no momentum convection can occur on a smooth wall because both velocity components disappear, the shearing force must be equal to the frictional force $\mu\frac{\partial u}{\partial \eta}$. This expression should be infinite according to equation (12a). The matter is explained, however, by the fact that the equations (12)

and (12a) must be regarded as an asymptotic expression for the velocity distribution at infinitely large Reynolds numbers just as the power law for the flow resistance represents an asymptotic law for absolutely smooth walls and for very large Reynolds numbers. The true velocity distribution is obtained by drawing a tangent with finite slope, say, to the velocity curve, so that $\tau = \mu \frac{\partial u}{\partial \eta}$. (Compare, for example, the interesting measurements in reference 5.) It is readily apparent that the point of contact of this tangent is shifted to the point $\eta = 0$ with increasing Reynolds number. But it appears that equation (12) itself represents the velocity distribution with sufficient accuracy for moderate Reynolds numbers.

The best experiments on the velocity distribution in a circular pipe were undoubtedly those made by T. E. Stanton (reference 6), first, because he originated the use of very fine pitot tubes in velocity measurements, and second, he employed a very long straight entrance section ahead of the test section, thus ensuring that the measurements fell in the zone where the velocity profile no longer varied perceptibly. Figure 2 shows Stanton's velocity values (ratio of local velocity to pipe axis velocity) against the wall distance, both on a logarithmic scale. It is seen that – apart from the first test point, 0.25 millimeter from the wall, so that the indication of the pitot tube of 0.33 millimeter in diameter no longer seems reliable – that the test points lie very accurately on a straight line of 1/7 slope.[1]

For the further applications quantity B in (12a) and (12b), which according to the assumptions for smooth surfaces signifies a universal constant of the turbulent flow regime, must be determined next. For this purpose it is really necessary to know the total velocity distribution from the wall vicinity to the center of the pipe, whereas the formulas (12a) and (12b) are valid, for the present, only in wall proximity. The chosen method of calculation included the use of several appropriate interpolation formulas which satisfactorily reproduce the velocity distribution, as measured by various experimenters, and change to equation (12a) at the wall.

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[1]It is to be noted that Christen proposed a velocity distribution formula according to which the velocity is proportional to the 1/8 rather than 1/7 power of the distance from the wall (reference 7). A detailed presentation of the various distribution formulas is found in Forchheimer's work (reference 8) as well as in Gümbel's report (reference 9).

(a) An extreme case occurs when the formulas which indicate the velocity is proportional to the 1/7 power of the distance from the wall are continued to the pipe center. Therefore

$$u = u_{max}\left(\frac{a - r}{a}\right)^{1/7} = u_{max}\left(1 - \frac{r}{a}\right)^{1/7} \tag{13}$$

or for wall proximity

$$u = B\left(\frac{\tau_o}{\rho}\right)^{4/7}\left(\frac{\eta}{\nu}\right)^{1/7} = u_{max}\frac{\eta^{1/7}}{a^{1/7}}$$

Considering the relation

$$\frac{dp}{dx}\pi a^2 = 2\pi a\tau_o \quad \text{and} \quad \tau_o = \gamma\frac{\eta}{2l}a$$

existing between the pressure drop and the wall stress, the calculation of the flow resistance by Blasius's formula gives

$$B\left(\frac{\lambda v^2}{8}\right)^{4/7}\frac{1}{\nu^{1/7}} = \frac{U_{max}}{a^{1/7}}$$

The ratio of average velocity v, occurring in Blasius's formula, to maximum velocity is, by (13),:

$$\frac{v}{u_{max}} = 0.816$$

Thus with $\lambda = 0.316\left(\frac{\nu}{vd}\right)^{1/4}$ the value for the constant B is

$$B = 2^{1/7}\left(\frac{8}{0.316}\right)^{4/7} 0.816 = 8.57$$

(b) A better approximation to the measurements is afforded in the case where the velocity profile at the pipe center is "rounded off" a little. This is best obtained by the formula

$$u = u_{max}\left(1 - \left(\frac{r}{a}\right)^n\right)^{1/7} \tag{13a}$$

where the exponent n can be chosen arbitrarily; $n = 1$ obviously leads back to (13). Figure 3 contains a number of measurements of different experimenters along with the three curves for $n = 1$, 1.25, and 2. The test points lie almost without exception between curves $n = 1$ and $n = 2$. Repeating the above calculating process with $n = 1.25$ and $n = 2$, the constant B amounts to

$$B = 8.62 \quad \text{with} \quad n = 1.25$$

$$B = 8.82 \quad \text{with} \quad n = 2.00$$

The values of the ratio $\frac{\text{average velocity}}{\text{maximum velocity}}$ are 0.838 for $n = 1.25$ and 0.875 for $n = 2$. The most reliable measurements give 0.84.[1] From this it is concluded that (13) with $n = 1.25$ to 2 represents the conditions fairly accurately, so that hereinafter $B = 8.7$ is generally used.

Thus equation (12a) must be written

$$u = 8.7\left(\frac{\tau_0}{\rho}\right)^{4/7}\left(\frac{\eta}{\nu}\right)^{1/7} \tag{14a}$$

If the shearing stress τ_0 is expressed as function of the velocity, equation (12b) reads

$$\tau_0 = \frac{1}{B^{7/4}}\,\rho \lim_{\eta = 0}\left\{u^2\left(\frac{\nu}{u\eta}\right)^{1/4}\right\}$$

With the values of B obtained on the basis of the three interpolation formulas, the equation would read

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[1]Also worthy of mention are the measurements by G. J. Williams (reference 10) and Gümbel (reference 9), where the proportionality factor decreases a little with increasing Reynolds numbers and then approaches the limiting value 0.811. This would favor the simple interpolation formula under (a). But substantially higher values (up to 0.87) also occur, where the effect of entrance length and roughness have not yet been fully explained.

$$\tau = 0.0225\ \rho \lim_{\eta = 0} \left\{ u^2 \left(\frac{\nu}{u\eta} \right)^{1/4} \right\} \tag{14b}$$

as the general expression for the wall friction in case the velocity distribution $u(\eta)$ is known in the vicinity of the wall. The constant in equation (14b) amounts to 0.0233 for the velocity distribution in the pipe by (14), as against 0.0231 and 0.0221 with $\eta = 1.25$ and $\eta = 2$, by (14a).

APPLICATION TO HEAT TRANSFER

For comparing these formulas with the representations expressing the turbulent friction by an apparent increase in friction coefficient (reference 11) the shearing stress transmitted in a layer distant η from the wall is

$$\tau = g(\eta, \mu, \rho, u) \frac{d}{dy} (u\rho) \tag{15}$$

If u as a function of η and the pressure gradient in the pipe are known, the function g can be explicitly calculated. Near the wall τ must become τ_0.

Considering (14a) and especially the relation: $7\eta \frac{du}{dy} = u$, yields for g:

$$g(\eta, \mu, \rho) = 0.805\ \rho \left(\frac{\tau_0}{\rho} \right)^{3/7} \nu^{1/7}\ \eta^{6/7}$$

and putting $\tau = \tau_0 \frac{r}{a}$ as follows from the condition of equilibrium for the circular pipe, gives

$$g(\eta, \mu, \rho) = 0.805\ \rho \left(\frac{\tau_0}{\rho} \right)^{3/7} \nu^{1/7}\ Y^{6/7} \tag{15a}$$

Y being solely a function of η which becomes η for small η. The relation (15a) is applicable to any cross section if it is assumed that the ratio of shearing stresses $\frac{\tau}{\tau_0}$ is independent of the velocity and is only a function of the location.

Quantity g is a kind of "turbulent friction coefficient" or better expressed a "turbulence factor."

The portion of true friction at larger Reynolds numbers, up to an extremely thin layer at the wall, is vanishingly small, hence the shearing stress is to be regarded almost exclusively as an average value of the momentum convection. This identification is of interest because it makes it possible to develop further the analogy between frictional resistance and heat transfer in turbulent flow, discovered by Reynolds (reference 12) and Prandtl (reference 13). Assuming that the momentum transport and the heat transfer is accomplished by the same mechanism of the irregular molar fluctuating motion, evidently results in two analogous formulas for the shearing force transmitted perpendicular to the flow by "turbulent momentum conduction" per unit of surface and for the heat volume transferred by "turbulent heat conduction:"

$$\tau = 0.805 \left(\frac{T_0}{\rho}\right)^{3/7} \nu^{1/7} Y^{6/7} \frac{d(\rho u)}{dy}$$

$$q = 0.805 \left(\frac{T_0}{\rho}\right)^{3/7} \nu^{1/7} Y^{6/7} \frac{d(c\rho\theta)}{dy} \tag{15a}$$

where c = specific heat, θ = temperature, hence $c\theta$ = heat content per unit mass. Formula (15a) may be continued up to the wall with good approximation if the same proportionality, assumed for the mechanism of the "turbulent momentum and heat transfer," exists for the transfer of molecular momentum and heat, that is, for the laminar internal friction and for the true heat conduction. As previously pointed out by Prandtl, this is evidenced by the fact that for the respective fluid, the relation $\frac{c\mu}{\lambda} = 1$ exists between heat conduction λ, friction coefficient μ, and specific heat c. This condition is approximately complied with, in gases. If $\frac{c\mu}{\lambda}$ differs very much from unity, as, for example, for water, the formula may be extended only to the boundary of the laminar layer next to the wall; while the effect of this layer — as will be explained elsewhere — can be expressed by a limiting condition.

The formula (15a) enables the heat transfer to be computed in all cases, where the "velocity field" of the turbulent flow is applicable to the average values in time, and hence g is known. With this formula, H. Latzko (reference 14) worked out a number of technically important cases of heat transfer to turbulent flows. It succeeded, in particular, in showing, that it is incorrect to speak of a "heat transfer factor," as is customary in engineering, that the heat transfer is rather conditional upon the total arrangement. It also succeeded in explaining the effect of the individual factors and so to organize the occasionally contradictory experimental material. In this respect the calculating possibilities of heat transfer processes appear substantially extended beyond the Prandtl analogy conclusions, since for the latter a complete agreement in velocity and temperature field had to be assumed, while the formula of the present report makes the differences between both also amenable to calculation.

TURBULENT BOUNDARY LAYER ON THE FLAT PLATE

The subsequent calculations are based on the previously derived equations (14a) and (14b) according to which the velocity distribution as a function of the wall distance is

$$u = 8.7\left(\frac{\tau_0}{\rho}\right)^{4/7}\left(\frac{\eta}{\nu}\right)^{1/7} \tag{14a}$$

if τ_0, the shearing stress transferred to the wall is given, while the shearing stress τ_0 is

$$\tau_0 = 0.0225\ \rho u^2\left(\frac{\nu}{u\eta}\right)^{1/4} \tag{14b}$$

$u(\eta)$ being the velocity distribution in the neighborhood of the wall. To apply these relations to the "turbulent boundary layer" requires a corresponding formula for the velocity distribution. With δ = boundary layer thickness, U = velocity in undisturbed flow, and y = distance from the wall, the elementary formula reads

$$u = U\left(\frac{y}{\delta}\right)^{1/7} \tag{16}$$

Equating (16) to (14a) obviously gives

$$8.7\left(\frac{\tau_o}{\rho}\right)^{4/7} \frac{1}{\nu^{1/7}} = \frac{U}{\delta^{1/7}}$$

that is, the shearing stress

$$\tau_o = 0.0225\ \rho U^2\left(\frac{\nu}{U\delta}\right)^{1/4} \tag{17}$$

Equation (17) yields the formula that must be used in the momentum equation of the boundary layer as the frictional force in order to obtain a theory of the turbulent boundary layer which is to replace the Prandtl-Blasius theory for the laminar boundary layer.

Placing, in fact, equation (17) in equation (5) gives

$$\frac{d}{dx}\int_0^\delta \rho u^2 dy - U\frac{d}{dx}\int_0^\delta \rho u dy = 0.0225\ \rho U^2\left(\frac{\nu}{U\delta}\right)^{1/4}$$

Determination of the integrals $\int_0^\delta u dy$ and $\int_0^\delta u^2 dy$ by means of (16) gives the differential equation of the boundary layer thickness

$$\frac{7}{72}\frac{d\delta}{dx} = 0.0225\left(\frac{\nu}{U\delta}\right)^{1/4}$$

The solution of this equation reads

$$\delta = \left(\frac{90}{7}\right)^{4/5}(0.0225)^{4/5}\left(\frac{\nu}{U}\right)^{1/5} x^{4/5} \tag{18}$$

or, for the length l

$$\delta_l = 0.37\ l\left(\frac{\nu}{Ul}\right)^{1/5} \tag{18a}$$

The laminar boundary layer grows proportional to $\sqrt{x}$, the turbulent layer proportional to $x^{4/5}$, according to equation (18).

Now the frictional resistance of a plate of length l can be calculated, either by integration of the frictional forces along the plate or by applying the momentum theorem to the end section for $x = l$. The resistance (both sides) follows as

$$W = \frac{7}{36}\,\rho U^2 \delta l = 0.036\,\rho U^2 l \left(\frac{\nu}{Ul}\right)^{1/5} \tag{19}$$

Referring the resistance through the formula

$$W = c_f F \gamma \frac{U^2}{2g}$$

to the velocity head, gives the resistance coefficient c_f

$$c_f = 0.072\,\frac{1}{R^{0.2}} \tag{19a}$$

the Reynolds number R being put at $R = \frac{Ul}{\nu}$.[1]

Figure 4 contains the test data by Gibbons and Wieselsberger (reference 15) on comparatively smooth plates, the line for c_f according to (19a), the Reynolds number and the resistance coefficient are given on a logarithmic scale. The agreement is exceptionally good. Gebers (reference 16) obtained a slightly higher exponent. It is suspected that towing of very long plates is accompanied by inevitable vibrations which permit the resistance to increase rapidly.

[1]According to a communication by letter, Prandtl possessed formula (19a) before the writer did. He indicates (cf. Ergebnisse, vol. I) a similar formula with a supplementary term which allows for the possible existence of laminar flow at the front edge of a suitably sharp plate. After determination of the numerical factors from older tests by Gebers, he gives the formula $c_f = 0.073\,\frac{1}{R^{0.2}} - \frac{1600}{R}$, where the numerical factor in the second term generally depends upon the degree of sharpening, and should be practically vanishingly small for a rounded-off leading edge.

An equally good confirmation is afforded by the velocity measurements in the vicinity of a towed board. Thus the points in figure 5 represent the measured velocity distribution perpendicular to a board towed in water as a function of the wall distance and specifically in a section 8.56 meters behind the front edge (reference 17). The solid line gives the velocity distribution according to equation (16), the boundary layer thickness was computed by equation (18). A comparison of the test data with the curve according to equation (16) discloses, above all, that it is in no way necessary to assume a velocity jump at the wall, as commonly reported in the technical literature. The present formulas rather represent the rapid decrease in velocity next to the wall by the variation of the power curve with the exponent 1/7 correctly and unrestrictedly.

LAMINAR FLOW ON A ROTATING DISK

As a further illustrative example of applying the methods obtained for the calculation of laminar and turbulent frictional resistances, the case of a uniformly rotating disk is to be analyzed. The laminar state of flow caused by a rotating flat disk is of special interest for the reason that it represents one of the rare cases in which the differential equations of the viscous fluids can be integrated without ommissions. It offers an immediate check on the accuracy with which Prandtl's boundary layer equations yield an approximation.

The problem is posed as follows:

The half space $x > 0$ shall be filled with liquid. The boundary plane $x = 0$ rotates about the x-axis with the uniform rotational speed ω. What is the state of motion in the half space $x > 0$ with consideration to the fluid friction?

Introducing cylindrical coordinates r, ϑ, x and denoting with c_r, c_t, c_x the velocity components in radial, tangential and axial direction, and with p the hydrostatic pressure, the differential equation of the flow in cylindrical coordinates – when, as follows from reasons of symmetry, all velocities are independent of ϑ – read:

$$\left.\begin{aligned}
c_r \frac{\partial c_r}{\partial r} + c_x \frac{\partial c_r}{\partial x} - \frac{c_t^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left\{\frac{\partial^2 c_r}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{c_r}{r}\right) + \frac{\partial^2 c_r}{\partial x^2}\right\} \\
c_r \frac{\partial c_t}{\partial r} + c_x \frac{\partial c_t}{\partial x} + \frac{1 c_t c_r}{r} &= \nu\left\{\frac{\partial^2 c_t}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{c_t}{r}\right) + \frac{\partial^2 c_t}{\partial x^2}\right\} \\
c_r \frac{\partial c_x}{\partial r} + c_x \frac{\partial c_x}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left\{\frac{\partial^2 c_x}{\partial r^2} + \frac{1}{r}\frac{\partial c_x}{\partial r} + \frac{\partial^2 c_x}{\partial x^2}\right\}
\end{aligned}\right\} \quad (20)$$

and the equation of continuity

$$\frac{\partial c_r}{\partial r} + \frac{c_r}{r} + \frac{\partial c_x}{\partial x} = 0 \qquad (20a)$$

The construction of the equations shows that the system (20) and (20a) can be satisfied by the formula

$$c_r = rf(x),\ c_t = rg(x),\ c_x = h(x),\ p = p(x) \qquad (21)$$

thus yielding for the three functions f, g, h the ordinary simultaneous differential equations

$$f^2 - g^2 + h\frac{df}{dx} = \nu\frac{d^2f}{dx^2},\quad 2fg + h\frac{dg}{dx} = \nu\frac{d^2g}{dx^2},\quad \frac{dh}{dx} + 2f = 0 \qquad (22)$$

while the equation

$$h\frac{dh}{dx} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{d^2h}{dx^2} \qquad (23)$$

arising from the third equation of the system (20) defines the pressure distribution $p(x)$.

Since the fluid is to possess no rotation at infinity, but is to adhere to the rotating wall for $x = 0$, the system of the boundary conditions reads

$$\begin{aligned} f(0) &= 0 \\ g(0) &= \omega \\ h(0) &= 0 \end{aligned} \qquad\qquad \begin{aligned} f(\infty) &= 0 \\ g(\infty) &= 0 \end{aligned}$$

The function $h(x)$ has a finite limiting value for $x = \infty$. This means that there is a steady inflow against the rotating wall as is to be expected for reasons of continuity. Owing to the adherence of the fluid, the rotating wall acts like a kind of centrifugal fan. Next to the wall the fluid is continuously carried to the outside, to be replaced by axial inflow.

For nondimensional representation

$$\xi = x\sqrt{\frac{\omega}{\nu}} \tag{24}$$

is introduced as independent variable and in place of f, g, h the functions $\underline{f}$, $\underline{g}$, $\underline{h}$

$$\underline{f} = \frac{f}{\omega}, \quad \underline{g} = \frac{g}{\omega}, \quad \underline{h} = \frac{h}{\sqrt{\nu\omega}} \tag{24a}$$

so that (22) becomes

$$\underline{f}^2 - \underline{g}^2 + \underline{h}\frac{df}{d\xi} = \frac{d^2f}{d\xi^2}, \qquad 2\underline{f}\underline{g} + \underline{h}\frac{d\underline{g}}{d\xi} = \frac{d^2\underline{g}}{d\xi^2}, \qquad \frac{d\underline{h}}{d\xi} + 2f = 0 \tag{22a}$$

with the boundary conditions

$$\underline{f} = 0, \ \underline{g} = 1, \ \underline{h} = 0 \text{ for } \xi = 0$$

$$\underline{f} = 0, \ \underline{g} = 0 \text{ for } \xi = \infty$$

so that the equations are independent of all special data of the problem. The similarity laws of the problem are readily apparent. Since $\underline{g}(\xi)$ indicates the proportionality factor of the rotative speed at distance $x = \xi\sqrt{\frac{\nu}{\omega}}$ from the wall to speed of rotation ω, it is clear that with increasing velocity only one layer at the wall manifests perceptible rotational speeds, which decrease with increasing velocity and decreasing viscosity as $\sqrt{\frac{\nu}{\omega}}$. On the other hand, it follows from the last of equation (24a) that the axial inflow velocity increases at infinity as $\sqrt{\nu\omega}$.

The equation system (22) can be solved by any numerical method or by series expansion. However, it is preferred to apply Pohlhausen's method described in reference 1, for a first approximation.

It is assumed that the function $\underline{f}$ and $\underline{g}$ at a distance δ from the wall are already very little different from zero. From (21) it follows that the "boundary layer thickness" in the present case is constant along the wall, hence δ is independent of ξ.

Integration of the first two equations of (22a) between $\xi = 0$ and $x = \delta$, that is, $\xi = \delta\sqrt{\frac{\omega}{\nu}} = \xi_o$, gives

$$\left.\begin{aligned} &\int_0^{\xi_o} (\underline{f}^2 - \underline{g}^2)\, d\xi + \int_0^{\xi_o} \underline{h}\, \frac{df}{d\xi}\, d\xi = \frac{df}{d\xi}\Big]_0^{\xi_o} \\ &\int_0^{\xi_o} 2\underline{fg}\, d\xi + \int_0^{\xi_o} \underline{h}\, \frac{dg}{d\xi} = \frac{dg}{d\xi}\Big]_0^{\xi_o} \end{aligned}\right\} \quad (25)$$

Partial integration of the second integral, while bearing in mind that according to the last equation of the system (22a) $\frac{d\underline{h}}{d\xi}$ can be replaced by $-2f$, finally affords

$$3\int_0^{\xi_o} \underline{f}^2 d\xi - \int_0^{\xi_o} \underline{g}\, d\xi = -\left[\frac{d\underline{f}}{d\xi}\right]_0 \qquad 4\int_0^{\xi_o} \underline{fg}\, d\xi = -\left[\frac{d\underline{g}}{d\xi}\right]_0 \qquad (26)$$

Put

$$\left.\begin{aligned} \underline{f} &= a\,\frac{\xi}{\xi_o}\left(1 - \frac{\xi}{\xi_o}\right)^2 \left(1 + 2\,\frac{\xi}{\xi_o}\right) - 1/2\left(\frac{\xi}{\xi_o}\right)^2 \left(1 - \frac{\xi}{\xi_o}\right)^2 \\ \underline{g} &= 1/2\left(2 + \frac{\xi}{\xi_o}\right)\left(1 - \frac{\xi}{\xi_o}\right) \end{aligned}\right\} \quad (27)$$

as approximate expressions for $\underline{f}$ and $\underline{g}$, where a signifies a constant that is to be determined.

It was borne in mind that

$$\underline{f} = 0, \quad \underline{g} = 1 \quad \text{for} \quad \xi = 0$$

$$\underline{f} = \frac{d\underline{f}}{d\xi}, \quad \underline{g} = \frac{d\underline{g}}{d\xi} \quad \text{for} \quad \xi = \delta$$

and also, as is readily seen from (22a), that

$$\frac{d^2\underline{f}}{d\xi^2} = -1, \quad \frac{d^2\underline{g}}{d\xi^2} = 0 \quad \text{for} \quad \xi = 0$$

Numerical calculation of the integrals contained in (26) gives

$$\left.\begin{aligned}
\int_0^{\xi_0} \underline{f}^2 d\xi &= \xi_0 \left[0.0301a^2 - 0.00326a + 0.00159\right] \\
\int_0^{\xi_0} \underline{g}^2 d\xi &= \xi_0\, 0.2357 \\
\int_0^{\xi_0} \underline{fg} d\xi &= \xi_0 \left[0.0607a - 0.00567\right]
\end{aligned}\right\} \quad (28)$$

which, entered in (26), gives two ordinary equations for a and ξ_0

$$\left.\begin{aligned}
0.0903a^2 - 0.00978a - 0.23093 &= -\frac{a}{\xi_0^2} \\
0.2428a - 0.02328 &= \frac{3}{2\xi_0^2}
\end{aligned}\right\} \quad (29)$$

The numerical solution gives

$$a = 1.026, \quad \xi_0 = 2.58 \qquad (30)$$

On the basis of these data the boundary layer thickness δ and the axial inflow velocity c_∞ at infinity can then be computed. Obviously

$$\delta = \xi_0 \sqrt{\frac{\nu}{\omega}} = 2.58 \sqrt{\frac{\nu}{\omega}}, \quad -c_\infty = \omega \int_0^{\xi_0} 2\underline{f}d\xi = 0.708\sqrt{\nu\omega} \tag{31}$$

Assume, for example, air as fluid with $\nu = 0.14$ square centimeter per second and an rpm = 600 per minute, that is, $\omega = \frac{2\pi n}{60} = 62.8$ per second; then the boundary layer thickness, according to (31), would be

$$\delta = 0.122 \text{ cubic centimeter}$$

and the axial inflow velocity

$$c_\infty = -7.6 \text{ centimeters per second}$$

The most important problem is the calculation of the frictional resistance. Assuming the wall bounded by $r = a$, the case is obviously that of a rotating disk with radius a. However, the fact that the outer parts of the plane $x = 0$ are missing, cannot be without some effect on the motion of the fluid, although it is to be presumed that this effect remains insignificant, when the thickness of the boundary layer relative to disk radius is very small, as is almost always the case in practice. On these assumptions, the moment of the shearing forces acting on the disk is simply integrated from $r = 0$ to $r = a$, or what amounts to the same thing, the angular momentum leaving in unit time with the fluid at the cylindrical surface $r = a$ is computed and equated to the moment of the frictional forces. The latter process is preferred. The angular momentum of the fluid leaving at the cylindrical surface in unit time is

$$D = 2\pi a^2 \rho \int_0^{\delta} c_x c_r dx = M \tag{32}$$

or, by (21), (24), and (24a)

$$M = 2\pi a^4 \rho \omega^{3/2} \nu^{1/2} \int_0^{\xi_0} \underline{fg}d\xi \tag{32a}$$

The integral $\int_0^{\xi_0} \underline{fg}d\xi$ has already been computed. Entering its value from (28) gives

$$M = 0.92a^4 \rho \nu^{1/8} \omega^{3/8} \tag{33}$$

or

$$M = 1.84a^3 \frac{U^2}{2g} \frac{1}{\sqrt{R}} \tag{33a}$$

where $U = a\omega$ is the circumferential velocity and $R = \frac{Ua}{\nu}$ is the Reynolds number. Equation (33) must, of course, be used twice, to obtain the resistance of the disk exposed to flow from both sides.

FRICTIONAL RESISTANCE OF ROTATING DISK IN TURBULENT FLUID MOTION

The relation governing (33): a frictional moment proportional to the 3/2 power of the rotational speed at higher rotational velocities is not borne out in practice. On the contrary, a substantially quicker increase in frictional moment is recorded with the rotational velocity. So the assumption is made again as for the towed flat plate that a turbulent boundary layer is involved, and an attempt is made to secure an approximate value for the boundary layer thickness and the frictional resistance by applying the momentum equation.

For the rotating disk two equilibrium conditions are required, one in the radial, the other in the tangential direction.

Employing the same notation as in the preceeding section and adding τ_r and τ_t (fig. 6) for the frictional forces per unit surface at the wall, the momentum quantities in the radial direction are:

(a) Excess of outgoing momentum quantity at the cylindrical surface $(r + dr)\delta$ (for an arc element of opening angle 1) over to the incoming momentum quantity at area $r\delta$

$$\frac{d}{dr}\left\{ r\rho \int_0^\delta c_r^2 dx \right\} dr$$

(b) The radial component of a respectively ingoing and outgoing momentum quantity at the front surface (equal to the centrifugal force of the rotating fluid volume)

$$-\left(\rho \int_0^\delta c_t^2 dx\right) dr$$

These momentum quantities must be in equilibrium with the shearing force $\tau_r r dr$; hence

$$\rho \frac{d}{dr}\left\{r \int_0^\delta c_r^2 dx\right\} - \rho \int_0^\delta c_t^2 dx = -\tau_r r \tag{34}$$

In tangential direction the difference of the turning moment of the momentum leaving at the cylindrical surface $2\pi(r + dr)\delta$ and entering at the surface $2\pi r d$ can be computed and equated to the turning moment of the frictional forces acting on the circular surface.

$$2\pi \frac{d}{dr}\left\{r^2 \int_0^\delta c_r c_t dx\right\} = -\tau_t 2\pi r^2 \tag{35}$$

The formulas for the velocity distribution are according to the results of the section Turbulent Flow in Smooth Pipes.

$$c_r = c_o \left(\frac{x}{\delta}\right)^{1/7} \left(1 - \frac{x}{\delta}\right), \quad c_t = r\omega \left[1 - \left(\frac{x}{\delta}\right)^{1/7}\right] \tag{36}$$

with due consideration that

for $x = 0$ $c_r = 0$, $c_t = r\omega$; for $x = \delta$ $c_r = c_t = 0$

Now the integrals in (34) and (35) can be evaluated:

$$\int_0^\delta c_r^2 dx = 0.207\, c_o^2 \delta \Big|, \quad \int_0^\delta c_r c_t dx = 0.0681\, r\omega c_o \delta, \quad \int_0^\delta c_t^2 dx = 0.0276\, r^2 \omega^2 \delta \tag{37}$$

Further, put, in conformity with the assumptions on the measure of turbulent friction equation (14b).

$$\tau_r = 0.0225\ \rho\ \frac{c_o^{7/4}\nu^{1/4}}{\delta^{1/4}}\left[1 + \left(\frac{r\omega}{c_o}\right)^2\right]^{3/8}$$

$$\tau_t = 0.0225\ \rho(r\omega)^{7/4}\ \frac{\nu^{1/4}}{\delta^{1/4}}\left[1 + \left(\frac{c_o}{r\omega}\right)^2\right]^{3/8}$$

by combining the velocity components at the wall and applying the friction formula to the resultant.

With this equations (34) and (35) give the two differential equations

$$\frac{d}{dr}\left\{0.207\ c_o^2 r\delta\right\} - 0.0278\ r^3\omega^2\delta = -\ 0.0225\ c_o^2 r\left(\frac{\nu}{c_o\delta}\right)^{1/4}\left[1 + \left(\frac{r\omega}{c_o}\right)^2\right]^{3/8}$$

$$\frac{d}{dr}\left[0.0681\ r^3\omega c_o\delta\right] = 0.0225\ r^4\omega^2\left(\frac{\nu}{r\omega\delta}\right)^{1/4}\left[1 + \left(\frac{c_o}{r\omega}\right)^2\right]^{3/8}$$

The equations are satisfied if the relationship between boundary layer thickness and axial distance r is put as

$$\left.\begin{aligned} c_o &= \alpha r\omega \\ \delta &= \beta r^{3/5} \end{aligned}\right\} \qquad (38)$$

and which gives two ordinary equations for α and β analogous to the equation system (29).

The equations

$$0.7456\ \alpha^2\beta - 0.0278\ \beta = -\ 0.0225\ \alpha^2\left(\frac{\nu}{\alpha\beta\omega}\right)^{1/4}\left(1 + \frac{1}{\alpha^2}\right)^{3/8}$$

$$0.3133\ \alpha\beta = 0.0225\left(\frac{\nu}{\beta\omega}\right)^{1/4}\ (1 + \alpha^2)^{3/8} \qquad (39)$$

given after division

$$1.0589\ \alpha^2 - 0.0278 = 0$$

The numerical solution gives

$$\alpha = 0.162 \quad \text{and hence} \quad \beta = 0.462\left(\frac{\nu}{\omega}\right)^{1/5} \tag{40}$$

With these figures the boundary layer thickness becomes

$$\delta = 0.462\ r\left(\frac{\nu}{r^2\omega}\right)^{1/5}$$

The section modulus can be computed by the method given in the preceding chapter or else based on equation (35):

$$M = 2\pi a^2\rho\int_0^\delta c_r c_t dx = 0.0364\ a^5\omega^2\rho\left(\frac{\nu}{a^2\omega}\right)^{1/5} \tag{41}$$

and the friction for both sides of the disk

$$M = 0.0728\ a^5\omega^2\rho\left(\frac{\nu}{a^2\omega}\right)^{1/5} \tag{41a}$$

In accord with the calculations on the towing resistance of plates all frictional resistances are then referred to the velocity square and velocity head, respectively. With U = circumferential speed of the disk, the moment is

$$M = 0.146\ \gamma\frac{U^2}{2g}a^3\left(\frac{\nu}{Ua}\right)^{1/5} \tag{42}$$

or the resistance coefficient c_f as a function of the Reynolds number of the disk $R = \frac{Ua}{\nu}$

$$c_f = 0.146\frac{1}{\sqrt[5]{R}} \tag{43}$$

Figure 7 shows c_f plotted against the Reynolds numbers R – both on a logarithmic scale. The plot also contains the resistance coefficient

$$c_f = \frac{3.68}{\sqrt{R}} \tag{44}$$

secured from the calculation of the laminar boundary layer by equation (33a).

The experimental data were taken from a recently published report on frictional resistance of smooth disks in water by W. Schmidt (reference 18). The agreement is good. Of particular interest is the fact that the measurements at smaller Reynolds numbers fall exactly in the transitional zone between laminar to turbulent flow.[1]

NOTES ON ROUGHNESS

While for perfectly smooth pipes the Blasius resistance law is apparently applicable over a wide range of velocities, so that it seems more than an interpolation formula, pipes with rough sides soon exhibit after exceeding the critical point an approximately square relationship between gradient and velocity. For this state the pressure gradient may be put at

$$h = \lambda \left(\frac{\epsilon}{d}\right) \frac{v^2}{2g} \frac{l}{d} \tag{45}$$

---

[1] The experiments by Odell (Engineering, Vol. 77, 1904, p. 33 and by A. Stodola (steam turbines, 4th ed., Berlin 1910, pp. 120–129) on the friction of rotating disks in air give 20 to 30 percent higher values and a more rapid increase in friction with the circumferential velocity. Odell shows $\sim \omega^{2+\epsilon}$ with ϵ a small positive digit, Stodola $\omega^{1.9}$ instead of $\omega^{1.8}$). Odell's tests are certainly doubtful, because the paper disks which he used, flutter and thus simulate greater frictional resistance. In Stodola's report the higher exponent appears to correspond to the roughness of the disk.

λ is a function of the relative roughness $\frac{\epsilon}{d}$; ϵ denotes a quantity with the dimension of a length, which, to a certain extent measures the average increase in the wall roughness. The ratio of this quantity to the pipe diameter is termed the "relative roughness" (reference 4).

The square law for rough walls is plausible for the reason that the frictional resistance is visualized as being built up from the individual resistances of the wall protuberances, which singly obey the square law. The mechanism of frictional resistance is in these cases obviously caused by uniform shedding of vortices of well-defined intensity and dimensions, as is the case for flows on resistance bodies.

The flow resistance in perfectly smooth pipes might be visualized such that in this instance vortices of dissimilar magnitude are separated and float at random in the turbulent flow, the frequency of the vortices of different intensity and size being controlled by some unknown, statistical law.

By this concept the frictional resistance in smooth pipes can be regarded as a fictitious combination of resistances that correspond to the individual kinds of vortices. Assuming that a relationship exists between size of vortices and roughness, it may be said that the frictional resistance in smooth pipes can be obtained by superposition of the individual resistances observed on rough pipes and increasing with the square of the velocity, if the individual squared resistances are entered with correct weights in the calculation.

It is not without interest that conclusions can be drawn about the form of the function $\lambda\left(\frac{\epsilon}{d}\right)$ on the basis of this conception, and specifically, without knowing the law of frequency and the weight function of the individual resistances.

In particular, it can be shown that, if the Blasius law for smooth pipes holds true, the function $\lambda\left(\frac{\epsilon}{d}\right)$ must have the formula $\lambda_0\left(\frac{\epsilon}{d}\right)^{2/7}$ at least for small values of $\frac{\epsilon}{d}$, λ_0 denoting a constant.

On superposing the squared resistances by (45) on assumption of a weight function $\varphi(\epsilon)$, the resistance law for smooth pipes reads[1]:

$$h = \frac{\int_0^\infty \lambda\left(\frac{\epsilon}{d}\right)\varphi(\epsilon)d\epsilon}{\int_0^\infty \varphi(\epsilon)d\epsilon}\,\frac{l}{d}\,\frac{v^2}{2g} \tag{46}$$

Next, it is assumed that the function $\varphi(\epsilon)$ for small values of ϵ compared to pipe diameter is dependent only on the physical constants and the velocity distribution in the immediate proximity of the wall elements. Specifically, $\varphi(\epsilon)$ is to be independent of the pipe diameter. On the other hand, the velocity distribution directly adjacent to the wall is according to earlier assumptions entirely contingent upon the shearing stress at the particular wall element. Therefore, put

$$\varphi(\epsilon) = \varphi(\epsilon, \mu, \rho, \tau_0)$$

From the four quantities only one dimensionless combination can be formed: namely,

$$z = \left(\frac{\tau_0}{\rho}\right)^{1/2}\frac{\epsilon}{\nu} \tag{47}$$

Hence, write:

$$\varphi(\epsilon) = \varphi\left[\frac{\epsilon}{\nu}\sqrt{\frac{\tau_0}{\rho}}\right] \qquad h = \frac{l}{d}\,\frac{v^2}{2g}\,\frac{\int_0^\infty \lambda\left(\frac{\epsilon}{d}\right)\varphi\left(\sqrt{\frac{\tau_0}{\rho}}\,\frac{\epsilon}{\nu}\right)d\epsilon}{\int_0^\infty \varphi\left(\sqrt{\frac{\tau_0}{\rho}}\,\frac{\epsilon}{\nu}\right)}$$

---

[1] The extent of the integration to ∞ is merely a matter of form, $\varphi(\epsilon)$ decreases very substantially with increasing ϵ, and from a certain value of ϵ on equals zero.

which, with z introduced as variable gives

$$h = \frac{l}{d}\,\frac{v^2}{2g}\,\frac{\int_0^\infty \lambda\left(\frac{\epsilon}{d}\right)\varphi(z)dz}{\int_0^\infty \varphi(z)dz}$$

and with $\lambda = \lambda_o\left(\frac{\epsilon}{d}\right)^m$ gives

$$h = \frac{l}{d}\,\frac{v^2}{2g}\,\lambda_o\,\frac{\int_0^\infty z^n\varphi(z)dz}{\int_0^\infty \varphi(z)dz}\,\frac{v^m\rho^{m/2}}{d^m\tau_o{}^{m/2}} = K\,\frac{l}{d}\,\frac{v^2}{2g}\,\frac{v^m\rho^{m/2}}{d^m\tau_o{}^{m/2}} \qquad (48)$$

$K = \lambda_o\,\dfrac{\int_0^\infty z^n\varphi(z)dz}{\int_0^\infty \varphi(z)dx}$ is a pure number.

Considering the relation

$$\gamma h\,\frac{d^2\pi}{4} = d\pi\tau_o\,l \quad \text{or} \quad \tau_o = \frac{\gamma h d}{4l}$$

existing between h and τ_o there is obtained

$$h\left(1 + \frac{m}{2}\right)\frac{\gamma^{m/2}d^{m/2}}{4^{m/2}l^{m/2}} = K\,\frac{l}{d^{l+m}}\,\frac{v^2}{2g}\,v^m\rho^{m/2}$$

which, solved for h leaves

$$h = \left(\frac{K}{2}\right)^{\frac{s}{s+m}} \frac{l}{d} \frac{v^2}{2g} \left(\frac{\nu}{dv}\right)^{\frac{sm}{s+m}} \tag{49}$$

This law corresponds exactly to the Blasius resistance law for smooth pipes provided

$$\frac{2m}{2+m} = 1/4$$

or

$$m = 2/7^{1}$$

By introducing a specific formula for the weight function φ - say, after the type of the law of error - the relationship between the constants of Blasius' law for smooth pipes and the constants of the law of roughness can be ascertained. The writer hopes to be able to return to the further development of these arguments.

If, on the other hand, the law $\lambda = \lambda_0 \left(\frac{\epsilon}{d}\right)^{2/7}$ is introduced in (45) the result with $\lambda_0 \epsilon^{2/7} = \zeta$ is

$$h = \zeta \frac{l}{d^{9/7}} \frac{v^2}{2g}$$

or

$$v \cong \sqrt{\frac{h}{l}}\, d^{9/7} \cong \left(\frac{h}{l}\right)^{0.5} d^{0.64}$$

According to the analogy between pipes and channels the velocity in a channel with gradient J and hydraulic radius P would be

$$v = \text{constant}\ J^{0.5}\ P^{0.64}$$

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[1]Mr. Prandtl states that he has arrived at the same result by an entirely different process.

Incidentally, according to R. Manning, the empirical formula $v = \text{constant}\ J^{0.5}\ P^{0.66}$, according to Forchheimer, the formula $v = \text{constant}\ J^{0.5}\ P^{0.7}$, and according to Hermanek, the formula $v = \text{constant}\ J^{0.5}\ P^{0.6}$ gives a good representation of the test data in rough channels (reference 8, p. 70).

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

REFERENCES

1. Pohlhausen, K.: Zur näherungsweisen Integration der Differentialgleichung der laminaren Grenzschicht. Z.f.a.M.M., Bd. 1, Heft 4, Aug. 1921, pp. 252-268.

2. Blasius, H.: Zeitschr. für Mathematik und Physik, Z.V.D.I., Bd. 56, 1908, p. 13.

3. Blasius, H.: Z.V.D.I., Heft 131, 1913, p. 1.

4. von Mises, R.: Elemente der technischen Hydromechanik (Leipzig), 1914, p. 50.

5. Stanton, T. E., Marshall, D., and Bryant, C. N.: Proc. Roy. Soc., London, vol. 97, 1920, p. 413.

6. Stanton, T. E.: Proc. Roy. Soc., London, vol. 85, 1911, p. 369.

7. Christen: Z.f. Gewasserkunde, Bd. 6, 1904, p. 175.

8. Forchheimer, Ph.: Hydraulik (Leipzig), 1914, pp. 93-119.

9. Gümbel: Schiffbautechnische Gesellschaft Jahrbuch, Bd. 14, 1913, p. 393.

10. Williams, Gardner J.: Trans. Am. Soc. Civ. Eng., 1902.

11. Boussinesq, B.: Mémoires des savants étrangers, Bd. 23, 1877, p. 1, and ff
Hahn, Herglotz, and Schwarzschild: Z. f. Math. u. Phys., 51, 1904, p. 411.

12. Reynolds: On the Passage of Heat between Metal Surfaces and Liquids in Contact with Them. Phil. Trans. Roy. Soc., London, vol. 190, 1874, p. 9.

13. Prandtl, L.: Phys. Zeitschr., vol. 11, 1910, p. 1072.

14. Latzko, H.: Der Wärmeübergang an einen turbulenten Flüssigkeits- oder Gasstrom. Z.f.a.M.M., Bd. 1, Heft 4, Aug. 1921, pp. 268-290.

15. Wieselsberger, C., Betz A., and Prandtl, L.: Ergebnisse der aerodynamischen Versuchsanstalt zu Göttingen, I. Lfg. R. Oldenbourg (München and Berlin) 1921, p. 123.

16. Gebers, Friedrich: Law of Similitude for the Surface Resistance of Lacquered Planes Moving in a Straight Line through Water. NACA TM No. 308, 1925.

17. Gümbel: Das Problem der Oberflächenwiderstandes, Schiffbautechnische Gesellschaft Jahrbuch, vol. 14, 1913, p. 478.

18. Schmidt, W.: Ein einfaches Messverfahren für Drehmomente, Z.V.D.I., Bd. 65, Nr. 17, April 23, 1921, pp. 441-444.

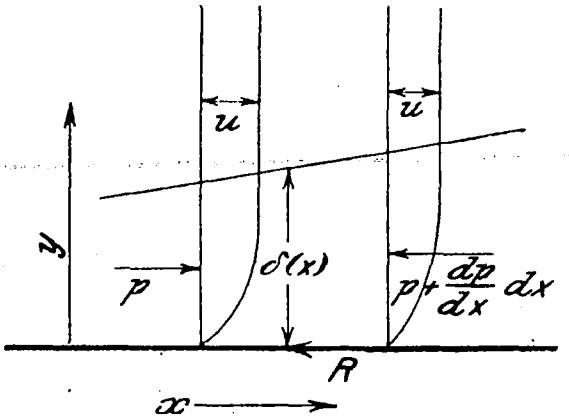


Figure 1.

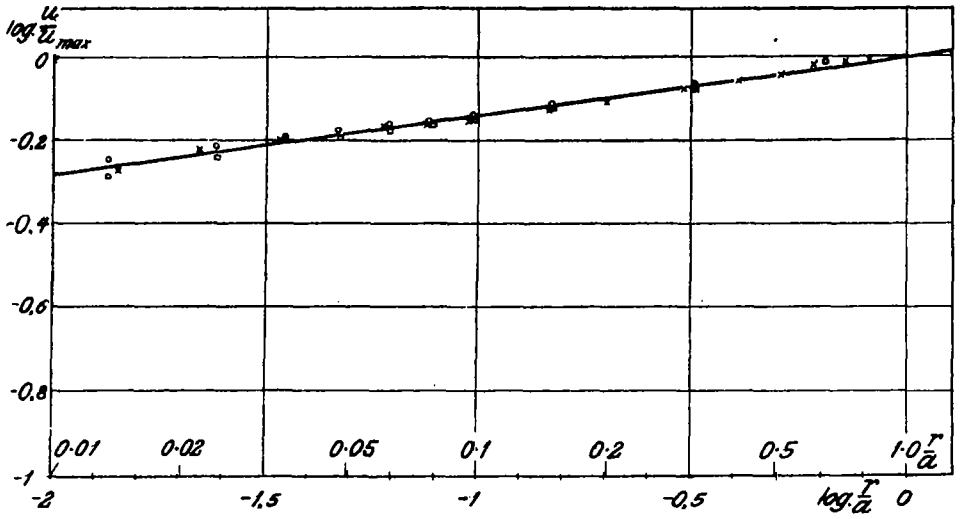
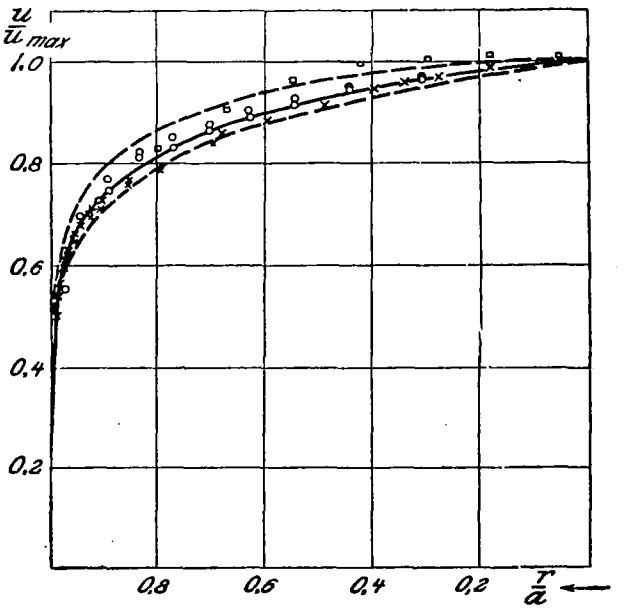


Figure 2.



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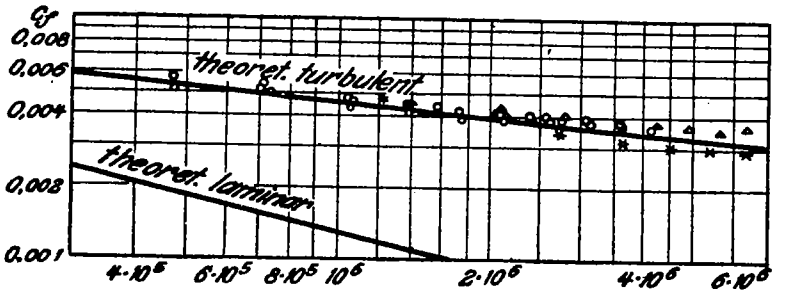

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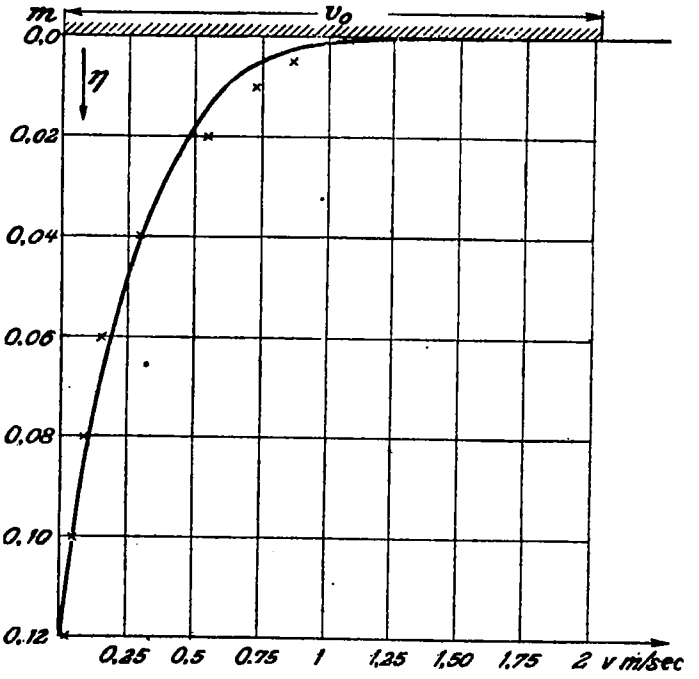
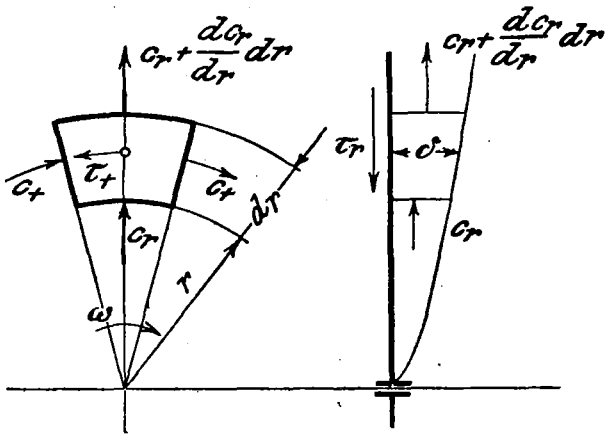


Figure 5.



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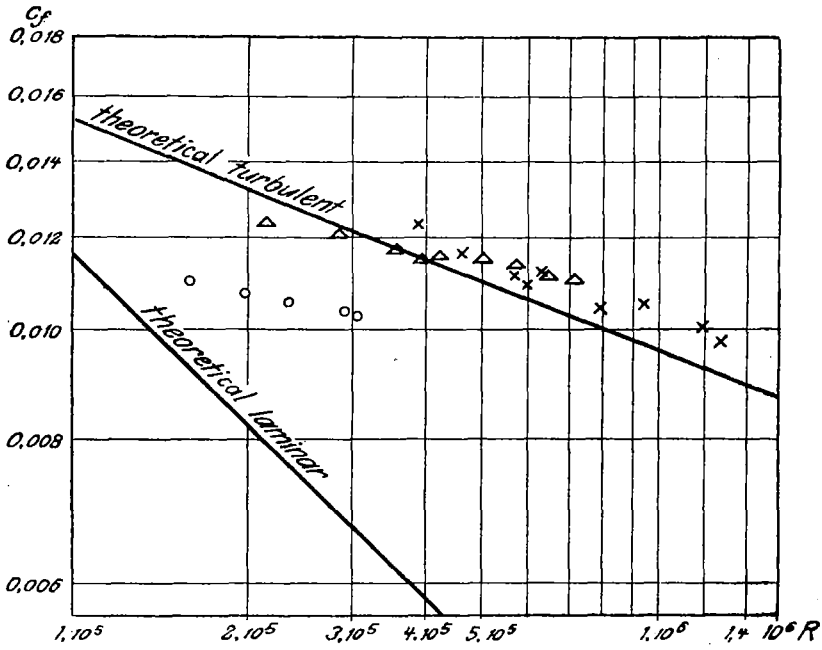

Figure 7.