[Not before published. (See page 229.)]

SUPPLEMENT TO A PAPER ON THE THEORY OF OSCILLATORY WAVES.

THE labour of the approximation in proceeding to a high order, when conducted according to the method of the former paper whether we employ the function ϕ or ψ, depends in great measure upon the circumstance that the two equations which have to be satisfied simultaneously at the free surface are both composed in a rather complicated manner of the independent variables, and in the elimination of y the length of the process is still further increased by the necessity of expanding the exponentials in y according to series of powers, giving for each exponential a whole set of terms. This depends upon the circumstance that of the limits of y belonging to the boundaries of the fluid, one instead of being a constant is a function of x, and that too a function which is only known from the solution of the problem.

If we convert the wave motion into steady motion, and refer the fluid to two independent variables of which one is the parameter of the stream lines or a function of the parameter, and the other is x or a quantity which extends with x from $-\infty$ to $+\infty$, we shall ensure constancy of each independent variable at both its limits, but in general the equations obtained will be of great complexity. It occurred to me however that if from among the infinite number of systems of independent variables possessing the above character we were to take the functions ϕ, ψ, where

$$\phi = \int(udx + vdy), \qquad \psi = \int(udy - vdx),$$

simplicity might be gained in consequence of the immediate relation of these functions to the problem.

We know that ϕ, ψ are conjugate solutions of the equation

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0 \quad\text{.........................(1)},$$

satisfying the equations

$$\frac{d\phi}{dx}=\frac{d\psi}{dy},\qquad \frac{d\phi}{dy}=-\frac{d\psi}{dx} \quad\text{..................(2)},$$

so that if the form of either be assigned, satisfying of course the equation (1), the other may be deemed known, since it can be obtained by the integration of a perfect differential. If now we take ϕ, ψ for the independent variables, of which x and y are regarded as functions, we get by changing the independent variables in differentiation

$$\frac{d\phi}{dx}=\frac{1}{S}\frac{dy}{d\psi},\quad \frac{d\phi}{dy}=-\frac{1}{S}\frac{dx}{d\psi},\quad \frac{d\psi}{dx}=-\frac{1}{S}\frac{dy}{d\phi},\quad \frac{d\psi}{dy}=\frac{1}{S}\frac{dx}{d\phi} \quad\text{...(3)},$$

where

$$S=\frac{dx}{d\phi}\frac{dy}{d\psi}-\frac{dx}{d\psi}\frac{dy}{d\phi},$$

whence from (2)

$$\frac{dx}{d\phi}=\frac{dy}{d\psi},\qquad \frac{dx}{d\psi}=-\frac{dy}{d\phi} \quad\text{...................(4)},$$

so that x, y are conjugate solutions of the equation

$$\frac{d^2x}{d\phi^2}+\frac{d^2x}{d\psi^2}=0 \quad\text{...........................(5)}.$$

We have also from (4)

$$S=\left(\frac{dx}{d\phi}\right)^2+\left(\frac{dx}{d\psi}\right)^2 \quad\text{........................(6)}.$$

We get from (3), (4) and (6)

$$u^2+v^2=\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\psi}{dx}\right)^2=\frac{1}{S^2}\left\{\left(\frac{dx}{d\phi}\right)^2+\left(\frac{dx}{d\psi}\right)^2\right\}=\frac{1}{S} \quad\text{......(7)},$$

whence

$$\frac{p}{\rho}=g(y+C)-\frac{1}{2S} \quad\text{......................(8)},$$

where C is an arbitrary constant.

The mode of proceeding is the same in principle whether the depth of the fluid be finite or infinite; but as the formulæ are simpler in the latter case, it may be well to consider it separately in the first instance.

If c be the velocity of propagation, c will be the horizontal velocity at a great depth when the wave motion is converted into steady motion. The difference between ϕ and $-cx$ will be a periodic function of x or of ϕ. We may therefore assume in accordance with equation (5)

$$x = -\frac{\phi}{c} + \Sigma_1^\infty (A_i e^{im\psi/c} + B_i e^{-im\psi/c}) \sin im\phi/c \text{.........(9)}.$$

No cosines are inserted in this equation because if we take, as we may, the origins of x and of ϕ at a trough or a crest (suppose a trough), x will be an odd function of ϕ, in accordance with what has already been shown at page 212. Corresponding to the above value of x we have

$$y = -\frac{\psi}{c} + \Sigma_1^\infty (A_i e^{im\psi/c} - B_i e^{-im\psi/c}) \cos im\phi/c \text{.....(10)},$$

the arbitrary constant being omitted, as may be done provided we leave open the origin of y.

The origin of ψ being arbitrary, we may take, as it will be convenient to do, $\psi = 0$ at the free surface. We see from (10) that ψ increases negatively downwards; and therefore of the two exponentials that with $-im\psi/c$ for index is the one which must be omitted, as expressing a disturbance that increases indefinitely in descending.

We may without loss of generality shorten the formulæ during a rather long approximation by writing 1 for any two of the constants which depend differently on the units of space and time. These constants can easily be reintroduced in the end by rendering the equations homogeneous. We may accordingly put $m = 1$ and $c = 1$. The expressions for x and y as thus shortened become, on retaining only the exponential which decreases downwards,

$$x = -\phi + \Sigma_1^\infty A_i e^{i\psi} \sin i\phi \text{.................(11)},$$

$$y = -\psi + \Sigma_1^\infty A_i e^{i\psi} \cos i\phi \text{.................(12)}.$$

At the free surface $\psi = 0$, and we must therefore have for $\psi = 0$

$$g(y + C) S - \tfrac{1}{2} = 0,$$

which gives

$$(C + \Sigma A_i \cos i\phi)\{1 - 2\Sigma i A_i \cos i\phi + \Sigma i^2 A_i^2 + 2\Sigma ij A_i A_j \cos[(i-j)\phi]\} - \frac{1}{2g} = 0 \text{.................(13)},$$

where in the last term within parentheses each different combination of unequal integers i, j is to be taken once.

On account of the complicated form of this equation, we can proceed further only by adopting some system of approximation. The most obvious is that adopted in the former paper, namely to proceed according to powers of the coefficient of the term of the first order. If we multiply out in equation (13), and replace products of cosines by cosines of sums and differences, we may arrange the equation in the form

$$B_0 + B_1 \cos\phi + B_2 \cos 2\phi + \ldots = 0,$$

where the several B's are series of terms involving the coefficients A. And as the equation has to be satisfied independently of ϕ, we must have separately

$$B_0 = 0, \quad B_1 = 0, \quad B_2 = 0, \text{ \&c.}$$

A slight examination of the process will show that A_i is of the order i, and that consequently the product of any number of the A's is of the order marked by the sum of the suffixes, and that B_i is of the order i. In proceeding therefore to any desired order we can see at once what terms need not be written down, as being of a superior order.

Thus in proceeding to the fifth order we must take the six equations $B_0 = 0$, $B_1 = 0, \ldots B_5 = 0$, which when written at length are

$$C(1 + A_1^2 + 4A_2^2) - A_1^2 + 2A_1^2A_2 - 2A_2^2 - \tfrac{1}{2}g^{-1} = 0,$$
$$C(-2A_1 + 4A_1A_2 + 12A_2A_3) + A_1 + A_1^3 - 3A_1A_2 + 6A_1A_2^2 + 3A_1^2A_3 - 5A_2A_3 = 0,$$
$$C(-4A_2 + 6A_1A_3) + A_2 - A_1^2 + 3A_1^2A_2 - 4A_1A_3 = 0,$$
$$C(-6A_3 + 8A_1A_4) + A_3 - 3A_1A_2 + 4A_1^2A_3 + 2A_1A_2^2 - 5A_1A_4 = 0,$$
$$C(-8A_4) + A_4 - 4A_1A_3 - 2A_2^2 = 0,$$
$$C(-10A_5) + A_5 - 5A_1A_4 - 5A_2A_3 = 0.$$

These equations may be looked on as giving, the first, the arbitrary constant C, the second, the velocity of propagation, and the succeeding ones taken in order the values of the constants A_2, A_3, A_4, A_5, respectively. I say "may be looked on as giving", for it is only when we restrict ourselves to the terms of the lowest order in each equation that those quantities are actually given in succession; the equations contain also terms of higher orders; and to get the complete values of the quantities true to the order to which we are working, we must use the method of successive substitutions. As to the second equation, if we take the terms of lowest order in the first two we get $C = \frac{1}{2}g^{-1}$, and then by substitution in the second equation $1 = g$, the constant A_1 dividing out. The equation $1 = g$ becomes on generalizing the units of space and time $c^2 = g/m$, and accordingly gives the velocity of propagation to the lowest order of approximation.

On eliminating the arbitrary constant in the above equations, and writing b for A_1, the results become

$$1 = g\,(1 + b^2 + \tfrac{7}{2}b^4) \quad\text{.......................(14)},$$

$$x = -\phi + be^{\psi}\sin\phi - (b^2 + \tfrac{1}{2}b^4)e^{2\psi}\sin 2\phi + (\tfrac{3}{2}b^3 + \tfrac{19}{12}b^5)e^{3\psi}\sin 3\phi$$
$$- \tfrac{8}{3}b^4e^{4\psi}\sin 4\phi + \tfrac{125}{24}b^5e^{5\psi}\sin 5\phi \quad\text{...........(15)},$$

$$y = -\psi + be^{\psi}\cos\phi - (b^2 + \tfrac{1}{2}b^4)e^{2\psi}\cos 2\phi + (\tfrac{3}{2}b^3 + \tfrac{19}{12}b^5)e^{3\psi}\cos 3\phi$$
$$- \tfrac{8}{3}b^4e^{4\psi}\cos 4\phi + \tfrac{125}{24}b^5e^{5\psi}\cos 5\phi \quad\text{...........(16)}.$$

The equation (14) gives to the fifth order the square of the velocity of propagation in the wave motion; and (15), (16) give the point where the parameters ϕ, ψ have given values, and also, by the aid of the formulæ previously given, the components of the velocity, and the pressure, in the steady motion. These same equations (15), (16), if we suppose ψ constant give implicitly the equation of the corresponding stream line, or if we suppose ϕ constant the equation of one of the orthogonal trajectories.

To find implicitly the equation of the surface, we have only to put $\psi = 0$ in (15), (16), which gives

$$x = -\phi + b\sin\phi - (b^2 + \tfrac{1}{2}b^4)\sin 2\phi + (\tfrac{3}{2}b^3 + \tfrac{19}{12}b^5)\sin 3\phi$$
$$- \tfrac{8}{3}b^4\sin 4\phi + \tfrac{125}{24}b^5\sin 5\phi \quad\text{..............(17)},$$

$$y = \qquad b\cos\phi - (b^2 + \tfrac{1}{2}b^4)\cos 2\phi + (\tfrac{3}{2}b^3 + \tfrac{19}{12}b^5)\cos 3\phi$$
$$- \tfrac{8}{3}b^4\cos 4\phi + \tfrac{125}{24}b^5\cos 5\phi \quad\text{..............(18)}.$$

It is not necessary to form the explicit equation, but we can do so if we please, most conveniently by the aid of Lagrange's theorem. The result, carried to the fourth order only, which will suffice for the object more immediately in view, is

$$y + \tfrac{1}{2}b^2 + b^4 = (b + \tfrac{9}{8}b^3)\cos x - (\tfrac{1}{2}b^2 + \tfrac{11}{6}b^4)\cos 2x$$
$$+ \tfrac{3}{8}b^3\cos 3x - \tfrac{1}{3}b^4\cos 4x \quad\text{...(19)}.$$

If we put $b + \frac{9}{8} b^3 = a$, we have to the fourth order

$$b = a - \frac{9}{8} a^3,$$

and substituting in (19) we get

$$y + \frac{1}{2} a^2 - \frac{1}{8} a^4 = a \cos x - (\frac{1}{2} a^2 + \frac{17}{24} a^4) \cos 2x + \frac{3}{8} a^3 \cos 3x - \frac{1}{3} a^4 \cos 4x \ldots\ldots (20).$$

The expression (14) for the square of the velocity of propagation, and the equation of the surface (20), agree with the results previously obtained by the former method (see p. 221) to the degree of approximation to which the latter were carried, as will be seen when we remember that the origins of y are not the same in the two cases; but it would have been much more laborious to obtain the approximation true to the fifth order by the old method.

It has already been remarked (p. 211) that the equation of the profile in deep water agrees with a trochoid to the third order, which is as far as the approximation there proceeded. This is no longer true when we proceed to the fourth order. On shifting the origin of y so as to get rid of the constant term, the equation (20) of the profile becomes

$$y = a \cos x - (\frac{1}{2} a^2 + \frac{17}{24} a^4) \cos 2x + \frac{3}{8} a^3 \cos 3x - \frac{1}{3} a^4 \cos 4x \ldots (21).$$

On the other hand, the equation of a trochoid is given implicitly by the pair of equations

$$x = \alpha\theta + \beta \sin \theta, \qquad y = \beta \cos \theta + \gamma.$$

In order that x may have the same period in the trochoid as in the profile of the wave, we must have $\alpha = 1$. We get then by development to the fourth order, choosing γ so as to make the constant term vanish,

$$y = (\beta - \frac{3}{8}\beta^3) \cos x - (\frac{1}{2} \beta^2 - \frac{1}{3}\beta^4) \cos 2x + \frac{3}{8}\beta^3 \cos 3x - \frac{1}{3}\beta^4 \cos 4x,$$

and putting

$$\beta - \frac{3}{8} \beta^3 = a,$$

we get to the fourth order

$$y = a \cos x - (\frac{1}{2} a^2 + \frac{1}{24} a^4) \cos 2x + \frac{3}{8} a^3 \cos 3x - \frac{1}{3} a^4 \cos 4x \ldots (22).$$

Hence if y_w, y_t denote the ordinates for the wave and trochoid respectively, we have to the fourth order

$$y_w - y_t = -\frac{2}{3} a^4 \cos 2x.$$

Hence the wave lies a little above the trochoid at the trough and crest, and a little below it in the shoulders.

This result agrees well with what might have been expected. It has been shown (p. 227) that the limiting form for a series of uniformly propagated irrotational waves is one presenting edges of 120°, and that the inclination in this limiting form is in all probability restricted to 30°, whereas in the trochoidal waves investigated by Gerstner and Rankine the limiting form is the cycloid, presenting accordingly cusps, and an inclination increasing to 90°. Hence the limiting form must be reached with a much smaller value of the parameter a in the former case than in the latter. Hence when a is just large enough to make the difference of form of the irrotational and trochoidal waves begin to tell, since the limiting form is more nearly approached in the former case than in the latter, we should expect the curvature at the summit to be greater, while at the same time as the general inclination is probably rather less, and the inclination begins by increasing more rapidly as we recede from the summit, the troughs must be shallower and flatter for an equal mean height of wave.

Let us proceed now to the case of a finite depth. As before we may choose the units of space and time so that c and m shall each be 1, and we may choose 0 for the value of the parameter ψ at the surface. Let $-k$ be its value at the bottom. Then since $d\phi/dy=0$ at the bottom we have from (3) and (4) $dy/d\phi=0$ when $\psi+k=0$, and consequently

$$A_i e^{-ik} = B_i e^{ik},$$

whence writing $A_i e^{ik}$ for A_i we have

$$x = -\phi + \Sigma A_i \{e^{i(\psi+k)} + e^{-i(\psi+k)}\} \sin i\phi \ldots\ldots\ldots\ldots(23),$$

$$y = -\psi + \Sigma A_i \{e^{i(\psi+k)} - e^{-i(\psi+k)}\} \cos i\phi \ldots\ldots\ldots\ldots(24).$$

Putting for shortness

$$e^{ik} + e^{-ik} = S_i, \quad e^{ik} - e^{-ik} = D_i,$$

we have by the condition at the free surface

$$(C + \Sigma A_i D_i \cos i\phi)\{1 - 2\Sigma A_i i S_i \cos i\phi + (\Sigma i A_i S_i \cos i\phi)^2 + (\Sigma i A_i D_i \sin i\phi)^2\} - \frac{1}{2g} = 0 \ldots\ldots(25).$$

As the expressions are longer than in the case of an infinite depth, and the problem itself of rather less interest, I shall content myself with proceeding to the third order. We have to this order from (25), on taking account of the relations

$$S_iS_j = S_{i+j} + S_{i-j}, \quad D_iD_j = S_{i+j} - S_{i-j}, \quad D_iS_j = D_{i+j} + D_{i-j},$$

$$(C + A_1D_1 \cos\phi + A_2D_2 \cos 2\phi + A_3D_3 \cos 3\phi)$$

$$\times \begin{Bmatrix} 1 & -2A_1S_1 \cos\phi & -4A_2S_2 \cos 2\phi - 6A_3S_3 \cos 3\phi \\ + A_1^2S_2 & + 4A_1A_2S_3 \cos\phi + 2A_1^2 \cos 2\phi & + 4A_1A_2S_1 \cos 3\phi \end{Bmatrix}$$

$$- \frac{1}{2g} = 0.$$

Multiplying out, retaining terms up to the third order only, arranging the terms according to cosines of multiples of ϕ, and equating to zero the coefficients of the cosines of the same multiple, we get the four equations

$$C(1 + A_1^2S_2) - A_1^2S_1D_1 - \frac{1}{2g} = 0,$$

$$C(-2A_1S_1 + 4A_1A_2S_3) + A_1D_1 + A_1^3S_2D_1 - 2A_1A_2S_2D_1 + A_1^3D_1 - A_1A_2S_1D_2 = 0,$$

$$C(-4A_2S_2 + 2A_1^2) - A_1^2S_1D_1 + A_2D_2 = 0,$$

$$C(-6A_3S_3 + 4A_1A_2S_1) - 2A_1A_2S_2D_1 + A_1^3D_1 - A_1A_2S_1D_2 + A_3D_3 = 0.$$

A slight examination of the process of approximation will show that whatever be the order to which we proceed, C, and the coefficients A_2, A_4, ... with even suffixes, will contain only even powers, and the coefficients A_3, A_5, ... with odd suffixes only odd powers, of the first coefficient A_1. Writing b for A_1, we may therefore assume, in proceeding to the third order only,

$$C = \alpha + \beta b^2,$$

$$A_2 = \gamma b^2,$$

$$A_3 = \delta b^3.$$

Substituting in the last three equations of the preceding group, which after the substitution may be divided by b, b^2, b^3 respectively, arranging, and equating coefficients of like powers of b, we get

$$-2S_1\alpha + D_1 = 0,$$

$$(4S_3\alpha - 2S_2D_1 - S_1D_2)\gamma - 2S_1\beta + S_2D_1 + D_1 = 0,$$

$$2\alpha - S_1D_1 + (D_2 - 4S_2\alpha)\gamma = 0,$$

$$(4S_1\alpha - 2S_2D_1 - S_1D_2)\gamma + D_1 + (D_3 - 6S_3\alpha)\delta = 0.$$

The substitution for C and the coefficients A_2, A_3,... of series according to even or odd powers of b with indeterminate coefficients was hardly worth making in proceeding to the third order only, but seems advantageous when we want to proceed to a rather high order. In proceeding to the n^{th} order it is to be noted that the coefficients of C in the group of $n+1$ equations got by equating to zero the coefficients of cosines of multiples of ϕ (including the zero multiple, or constant term), are of the orders 0, 1, 2,... n in b, so that C being determined only to b^{n-1} in the equations after the first, the terms of the order n in the first equation (which could only occur when n is even) are not required, but this first equation need only be carried as far as to $n-1$. In fact, in proceeding to the orders 1, 2, 3, 4, 5, 6,..., the velocity of propagation is given to an order not higher than 0, 1, 2, 3, 4, 5,... in b, and therefore actually to 0, 0, 2, 2, 4, 4,... since it involves only even powers of b.

The last equations give in succession

$$\alpha = \frac{D_1}{2S_1} \qquad\qquad (26),$$

$$\gamma = -\frac{1}{D_1^2}(S_2+1) \qquad\qquad (27),$$

$$\delta = \frac{1}{2D_1^4}(3S_4+4S_2+4) \qquad\qquad (28),$$

$$\beta = \frac{1}{S_1 D_1}(S_2+1)^2 \qquad\qquad (29),$$

and then by substituting in the first equation of the group on the middle of p. 321, we get

$$\frac{1}{g} = \frac{D_1}{S_1} + \frac{1}{S_1 D_1}(S_4+2S_2+12)\,b^2 \qquad\qquad (30).$$

We get now from (23), (24), after rendering the equations homogeneous,

$$x = -\frac{\phi}{c} + b\left(e^{m(\psi+k)/c} + e^{-m(\psi+k)/c}\right)\sin m\phi/c$$
$$-\frac{S_2+1}{D_1^2} mb^2\left(e^{2m(\psi+k)/c} + e^{-2m(\psi+k)/c}\right)\sin 2m\phi/c$$
$$+\frac{1}{2D_1^4}(3S_4+4S_2+4)m^2b^3\left(e^{3m(\psi+k)/c}+e^{-3m(\psi+k)/c}\right)\sin 3m\phi/c \qquad (31),$$

$$y = -\frac{\psi}{c} + b\left(e^{m(\psi+k)/c} - e^{m(\psi+k)/c}\right)\cos m\phi/c + \&c. \qquad (32),$$

the expression for y after the first term differing from that for x only in having a *minus* sign before the second exponential in each term, and cosines in place of sines. We have also from (30)

$$\frac{mc^2}{g} = \frac{D_1}{S_1} + \frac{1}{S_1 D_1}(S_4 + 2S_2 + 12)\, b^2 \qquad \text{(33)},$$

which gives the velocity of propagation according to one of its possible definitions (see Art. 3, p. 202). In these expressions it is to be observed that

$$S_i = e^{imk/c} + e^{-imk/c}, \quad D_i = e^{imk/c} - e^{-imk/c}.$$

We might of course in the numerators of the coefficients have used expressions proceeding according to powers of S_1 instead of according to the functions $S_1, S_2, S_3 \ldots$

Let h be the value of y at the bottom, which is a stream line for which $\psi = -k$, then we have from (24) generalized as to units

$$k = ch \qquad \text{(34)},$$

so that it remains only to specify the origin of y and the meaning of c. To the first order of small quantities we have

$$x = -\frac{\phi}{c} + b\left(e^{m(\psi+k)/c} + e^{-m(\psi+k)/c}\right)\sin m\phi/c \qquad \text{(35)},$$

$$y = -\frac{\psi}{c} + b\left(e^{m(\psi+k)/c} - e^{-m(\psi+k)/c}\right)\cos m\phi/c \qquad \text{(36)},$$

and at the surface

$$x = -\frac{\phi}{c} + bS_1 \sin m\phi/c \qquad \text{(37)},$$

$$y = bD_1 \cos m\phi/c \qquad \text{(38)}.$$

Since y in (38) is a small quantity of the first order, we may replace ϕ/c in its expression by x, in accordance with (37), which gives for the equation of the surface

$$y = bD_1 \cos mx,$$

so that to this order of approximation the origin is in the plane of mean level, and therefore h denotes the mean depth of the fluid. Also since $u = d\phi/dx$ we have to the first order from (3), (4), (6)

$$u = \left(\frac{dx}{d\phi}\right)^{-1} = \left\{-\frac{1}{c} + \frac{mb}{c}\left(e^{m(\psi+k)/c} + e^{-m(\psi+k)/c}\right)\cos m\phi/c\right\}^{-1}$$

$$= -c - c\,.\,mb\left(e^{m(h-y)} + e^{-m(h-y)}\right)\cos mx,$$

and consists therefore of two parts, one representing a uniform flow in the negative direction with a velocity c, and the other a motion of periodic oscillation. To this order therefore there can be no question that c should be the horizontal velocity in a positive direction which we must superpose on the whole mass of fluid in order to pass to pure wave motion without current. In passing to the higher orders it will be convenient still to regard this constant as the velocity of propagation, and accordingly as representing the velocity which we must superpose, in the positive direction, on the steady motion in order to arrive at the wave motion; but what, in accordance with this definition, may be the mean horizontal velocity of the whole mass of fluid in the residual wave motion, or what may be the mean horizontal velocity at the bottom, &c., or again what is the distance of the origin from the plane of mean level, are questions which we could only answer by working out the approximation, and which it would be of very little interest to answer, as we may just as well suppose the constant h defined by (34) given as suppose the mean depth given, and similarly as regards the flow.

Putting $\psi = 0$ in (31) and (32), we have implicitly for the equation of the surface the pair of equations

$$x = -\frac{\phi}{c} + S_1 b \sin m\phi/c - \frac{1}{D_1^2}(S_2+1) S_2 m b^2 \sin 2m\phi/c$$
$$+ \frac{1}{2D_1^4}(3S_4 + 4S_2 + 4) S_3 m^2 b^3 \sin 3m\phi/c,$$

$$y = D_1 b \cos m\phi/c - \frac{1}{D_1^2}(S_2+1) D_2 m b^2 \cos 2m\phi/c$$
$$+ \frac{1}{2D_1^4}(3S_4 + 4S_2 + 4) D_3 m^2 b^3 \cos 3m\phi/c.$$

The ratios of the coefficients of the successive cosines in y or sines in x to what they would have been for an infinite depth, supposing that of $\cos m\phi/c$ the same in the two cases, are

$$1, \quad \frac{1}{D_1^2}(S_2+1), \quad \frac{1}{D_1^4}(S_4 + \tfrac{4}{3}S_2 + \tfrac{4}{3}),$$

multiplied respectively by

$$1, \quad \frac{D_2}{D_1^2}, \quad \frac{D_3}{D_1^3}$$

for the cosines in y, and by

$$\frac{S_1}{D_1}, \qquad \frac{S_2}{D_1^2}, \qquad \frac{S_3}{D_1^3}$$

for the sines in x. Expressed in terms of D_1, the first three ratios become

$$1, \quad 1+3D_1^{-2}, \quad 1+\tfrac{16}{3}D_1^{-2}+6D_1^{-4},$$

and increase therefore as the depth diminishes, and consequently D_1 diminishes. The same is the case with the multipliers D_2/D_1^2, D_3/D_1^3, S_1/D_1, &c., and on both accounts therefore the series converge more slowly as the depth diminishes. Thus for $D_1^2=3$ the first three ratios are 1, 2, $3\frac{4}{9}$. $D_1^2=3$ corresponds to $h/\lambda=0{\cdot}125$, nearly, so that the average depth is about the one-eighth of the length of a wave.

The disadvantage of the approximation for the case of a finite as compared with that of an infinite depth is not however quite so great as might at first sight appear. There can be little doubt that in both cases alike the series cease to be convergent when the limiting wave, presenting an edge of 120°, is reached. In the case of an infinite depth, the limit is reached for some determinate ratio of the height of a wave to the length, but clearly the same proportion could not be preserved when the depth is much diminished. In fact, high oscillatory waves in shallow water tend to assume the character of a series of disconnected solitary waves, and the greatest possible height depends mainly on the depth of the fluid, being but little influenced by the length of the waves, that is, the distance from crest to crest. To make the comparison fair therefore between the convergency of the series in the cases of a finite and of an infinite depth, we must not suppose the coefficient of $\cos m\phi/c$ the same in the two cases for the same length of wave, but take it decidedly smaller in the case of the finite depth, such for example as to bear the same proportion to the greatest possible value in the two cases.

But with all due allowance to this consideration, it must be confessed that the approximation is slower in the case of a finite depth. That it must be so is seen by considering the character of the developments, in the two cases, of the ordinate of the profile in a harmonic series in terms of the abscissa, or of a quantity having the same period and the same mean value as the abscissa. The flowing outline of the profile in deep water lends itself readily to expansion in such a series. But the approximately isolated and widely separated elevations that represent the profile in very shallow water would require a comparatively large number of terms in their expression in harmonic series in order that the form should be represented with sufficient accuracy. In extreme cases the fact of the waves being in series at all has little to do with the character of the motion in the neighbourhood of the elevations, where alone the motion is considerable, and it is not therefore to be wondered at if an analysis essentially involving the length of a wave should prove inconvenient.