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NOTE

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A note on Stokes' hypothesis

Abstract The so-called Stokes' hypothesis for a Newtonian fluid is reconsidered, and a possible explanation is given of the fact that, in spite of its evidently weak physical justification, it permits to obtain good results for the description of most compressible flows. The explanation stands upon a closer analysis of the effect of the terms of the complete stress tensor in which the viscosity coefficients appear. An alternative formulation of the hypothesis is proposed, which also permits to clearly identify the very particular flow conditions in which it cannot justify the experimental evidence.

1 Introduction

The momentum balance equation of a fluid may be written in the following form:

$$\rho \frac{\mathrm{D}V}{\mathrm{D}t} = \rho f + \mathrm{div}\mathbf{T}, \tag{1}$$

where $\mathrm{D}/\mathrm{D}t$ is the material derivative operator, ρ the density, V the velocity, f the body force per unit mass, and $\mathbf{T}$ the stress tensor. In order to specify the behavior of the considered fluid, it is customary to decompose the stress tensor as follows:

$$\mathbf{T} = -p_e\mathbf{I} + \boldsymbol{\tau}_v, \tag{2}$$

where $\mathbf{I}$ is the identity tensor and p_e is the *equilibrium* (or *thermodynamic*) pressure, which may be expressed as a function of density and temperature through a suitable thermodynamic relation. Thus, the tensor $-p_e\mathbf{I}$ coincides with the stress tensor that would act on a fluid element if it belonged to a fluid at rest with the same values of density and temperature that it experiences when it is moving. Consequently, it depends on motion only through the values of density and temperature. Conversely, the *viscous stress tensor* $\boldsymbol{\tau}_v$ is completely dependent on motion and derives from the momentum transport mechanisms at the molecular level that are at the basis of the origin of viscosity. It must be emphasized that, while the tensor $-p_e\mathbf{I}$ is clearly isotropic by construction, there is no reason to assume, at this stage, that $\boldsymbol{\tau}_v$ is a deviatoric tensor, i.e., a tensor whose trace is zero.

The constitutive equations for the viscous stress tensor stand upon certain assumptions on the dependence of the tensor $\boldsymbol{\tau}_v$ on the local velocity of deformation of an elementary volume of fluid, which is completely determined, at first order, by the tensor $\mathrm{grad}V$. To express this dependence, it is convenient to decompose $\mathrm{grad}V$ into its symmetric and antisymmetric parts, say $\mathbf{E}$ and $\boldsymbol{\Omega}$, respectively. The tensor $\mathbf{E}$, usually denoted as the *rate of strain tensor*, characterizes the changes in shape and volume of an elementary volume of fluid, whereas the tensor $\boldsymbol{\Omega}$ describes the rigid rotation of that volume in an elementary time interval.

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We will consider only *Newtonian* fluids, which satisfy the conditions that $\boldsymbol{\tau}_\mathrm{v}$ is independent of $\boldsymbol{\Omega}$ and depends linearly on $\mathbf{E}$. It may be demonstrated (see, e.g., [1]) that the most general form of the viscous stress tensor of a homogeneous and isotropic Newtonian fluid is the following one:

$$\boldsymbol{\tau}_\mathrm{v} = (\lambda \mathrm{div}V)\mathbf{I} + 2\mu \mathbf{E}. \tag{3}$$

In this relation, λ and μ are two coefficients which depend, in principle, on the thermodynamic state of the fluid. The coefficient μ is the *dynamic viscosity coefficient*, while the coefficient λ is usually called *second viscosity coefficient*. It may now be useful to express the tensor $\mathbf{E}$ as the sum of its isotropic and deviatoric parts, say $\mathbf{A}$ and $\mathbf{D}$:

$$\mathbf{A} = (\frac{1}{3}tr\mathbf{E})\mathbf{I} = (\frac{1}{3}\mathrm{div}V)\mathbf{I}; \quad \mathbf{D} = \mathbf{E} - \mathbf{A}, \tag{4}$$

where $tr\mathbf{E} = \mathrm{div}V$ is the trace of tensor $\mathbf{E}$. We thus get:

$$\boldsymbol{\tau}_\mathrm{v} = (\lambda \mathrm{div}V)\mathbf{I} + 2\mu(\mathbf{A} + \mathbf{D}) = (\lambda + \frac{2}{3}\mu)(\mathrm{div}V)\mathbf{I} + 2\mu\mathbf{D} = (\kappa \mathrm{div}V)\mathbf{I} + 2\mu\mathbf{D}, \tag{5}$$

where we introduced the *bulk viscosity coefficient* κ, defined as $\kappa = (\lambda + 2\mu/3)$.

As can be seen, the viscous stress tensor has, in general, both an isotropic part and a deviatoric part. By recalling decomposition (2), we thus get the following expression for the complete stress tensor:

$$\mathbf{T} = (-p_\mathrm{e} + \kappa \mathrm{div}V)\mathbf{I} + 2\mu\mathbf{D}, \tag{6}$$

or, in terms of components and introducing the Kronecker delta function δ_{ik},

$$T_{ik} = (-p_\mathrm{e} + \kappa \mathrm{div}V)\delta_{ik} + \mu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\mathrm{div}V\delta_{ik} \right). \tag{7}$$

Relations (6) and (7) show that, in general, the isotropic part of the complete stress tensor contains a viscous term that is additive to the pressure term. We may then interpret $\kappa \mathrm{div}V$ as the difference between the thermodynamic pressure and the opposite of the average of the normal stresses acting on any three orthogonal planes passing through a point in the fluid, which is usually referred to as the *mechanical pressure*. This difference is generally considered to be due to the time lag with which the thermodynamic equilibrium condition is reached in a motion that implies an isotropic dilatation of a fluid element.

2 Stokes' hypothesis revisited

Following a suggestion by Stokes [2], it is customary to assume that κ is negligible or, in other words, that the two coefficients of viscosity appearing in (3) are linked by the relation

$$\lambda + \frac{2}{3}\mu = 0. \tag{8}$$

This is known as *Stokes' hypothesis*, and its use has become common practice in the analysis of the motion of compressible fluids. As may be clearly seen from relation (6), it implies that the thermodynamic pressure coincides with the mechanical pressure and characterizes the isotropic part of the complete stress tensor; furthermore, the viscous stress tensor becomes a purely deviatoric tensor and corresponds to the deviatoric part of $\mathbf{T}$. In other words, assuming the validity of relation (8) is equivalent to state that isotropic dilatations of an elementary volume of fluid do not produce viscous stresses.

The use of Stokes' hypothesis renders the mathematical treatment of compressible flows considerably easier, but it has been the object of long-lasting discussions (see, e.g., the papers presented at the workshop chaired by Rosenhead [3], or the considerations in [4]). Indeed, for polyatomic gases, the available data for κ—though not numerous due to the complexity of its experimental evaluation—show that it is certainly not zero and actually often far from negligible. For instance, κ is of the same order as μ for nitrogen and oxygen, but other gases, such as carbon dioxide, are characterized by much larger values of κ, of the order of $10^3 \mu$. On the other hand, it is usually claimed that a fundamental result of the kinetic theory of gases is that $\kappa = 0$ for monatomic gases. This would be in agreement with the physical interpretation that the bulk viscosity is connected with the time lag for reaching local equilibrium conditions. In fact, this time would obviously be

much shorter for monatomic gases, due to the absence of the rotational and vibrational modes characterizing polyatomic gases. Furthermore, the rather scarce available experimental evidence (see, e.g., [5]) seems to suggest that the bulk viscosity of monatomic gases is indeed practically negligible. However, already more than sixty years ago, Truesdell [6] observed that a zero value of the bulk viscosity is not derived from but implicitly assumed in Maxwell's kinetic theory of gases. More recently, Rajagopal [7], starting from a different approach to the constitutive equations of a viscous fluid—in which the rate of strain is expressed as a linear function of the stress tensor rather than the opposite—concluded that the bulk viscosity can never be zero for physically realistic fluids, including monatomic gases. Finally, the bulk viscosity is significantly higher than the dynamic viscosity also for many liquids (see, e.g., [3,8]).

In summary, one may consider as amply demonstrated that for most fluids of practical interest the bulk viscosity is not zero, and that the ratio κ/μ may vary from almost negligible to very large values. Therefore, such evaluations of the order of magnitude of the bulk viscosity seem to be in contradiction with the fact that, excluding very particular conditions, in most applications concerning compressible flows good results may indeed be obtained by using the usual formulation of Stokes' hypothesis, namely that $\kappa = 0$. Thus, we are seemingly left with the paradox that, in order to get adequate descriptions of most flow fields, we must retain the viscosity coefficient μ in relation (5), which defines the viscous stress tensor, but we are allowed to neglect the other one, κ, which may be of the same order of magnitude or even much larger.

A reasonable explanation of this circumstance may be obtained from a deeper analysis of the contribution of the stress tensor term in which the bulk viscosity appears—namely $\kappa \operatorname{div} V$—and from a reflection on the admissibility of the operation that is done when a term is neglected in a relation or in an equation (see also [9]). To this end, it is useful to rewrite the expressions of the tangential and normal components of the complete stress tensor separately. For a generic tangential stress ($i \neq k$), we have

$$T_{ik} = \mu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) = 2\mu D_{ik}, \tag{9}$$

whereas the expression for a generic normal stress ($i = k$) is:

$$T_{ii} = (-p_e + \kappa \operatorname{div} V) + 2\mu \left(\frac{\partial u_i}{\partial x_i} - \frac{1}{3} \operatorname{div} V \right) = (-p_e + 3\kappa A_{ii}) + 2\mu D_{ii}. \tag{10}$$

The essential point to be observed now is that the terms A_{ii} are all equal, whereas this cannot happen for the terms D_{ii} because the trace of the deviatoric tensor **D** is zero. The effect of $\kappa \operatorname{div} V$ is thus perfectly additive to that of the thermodynamic pressure; in other words, this term is associated with the same deformation—isotropic dilatation of a fluid element—that is connected with the thermodynamic pressure, which, however, is generally larger than $\kappa \operatorname{div} V$ (in absolute value) by several orders of magnitude. Conversely, the coefficient μ, however small it may be, is associated with normal and tangential stresses causing deformations that cannot be justified without taking viscosity into account, namely pure strain (i.e., non-isotropic normal deformation) and pure distortion.

The above analysis suggests that we might formulate a different condition, which in practical applications is equivalent to the original Stokes' hypothesis but is much more satisfactory from the physical point of view and does not require to disregard the available experimental evidence. Indeed, rather than putting $\kappa = 0$, we may simply assume that the absolute value of $\kappa \operatorname{div} V$ is negligible compared to the thermodynamic pressure, i.e., that the following relation holds:

$$|\kappa \operatorname{div} \mathbf{V}| \ll p_e. \tag{11}$$

In other words, with this assumption, we are neglecting any difference between the mechanical and the thermodynamic pressures. The rationale for this different approach and for its applicability stands upon the obvious fact that a term appearing in a relation or equation cannot be neglected just because it is small, but only if it is small compared to a qualitatively similar one, i.e., to one that has the same effect. If this different point of view is adopted, there are indeed good reasons for the modified formulation (11) of Stokes' hypothesis to be a largely acceptable approximation. As a matter of fact, only in very particular conditions will the term $\kappa \operatorname{div} V$ be comparable to the thermodynamic pressure. This may happen, for instance, when the fluid is characterized by large values of κ (e.g., CO_2), and the motion is such that extremely large values of $\operatorname{div} V$ occur, as happens in hypersonic flows or in flows through shock waves (see [10,11]).

There is at least another situation in which, although the above conditions do not occur, neglecting the term $\kappa \operatorname{div} V$ leads to results that are in contradiction with experimental evidence. In fact, as may be derived from the internal energy balance, the damping of acoustic waves at very high frequencies cannot be justified

without reintroducing the viscous stresses connected with an isotropic dilatation. Indeed, an acoustic wave is associated with an oscillatory isotropic change in volume between opposite values. If we assume the mechanical and thermodynamic pressures to coincide, the corresponding variation in internal energy would be given by $p_e \mathrm{div} V$; consequently, in an entire cycle of volume fluctuation—which produces a variation in $\mathrm{div} V$ between opposite values—the internal energy would remain unaltered. Conversely, if we renounce to any form of Stokes' hypothesis, it is easily seen that a further term equal to $\kappa (\mathrm{div} V)^2$ would appear in the internal energy balance. Considering that κ cannot be negative (as may be shown from the entropy balance), this additional term is always dissipative. Obviously, if $\kappa \mathrm{div} V$ is small (as is the case in normal conditions), only acoustic waves at very high frequency would involve a sufficient number of cycles for this term to sum up, in an acceptable time interval, to an increase in internal energy with appreciable effects. Indeed, measuring the attenuation of high-frequency waves (ultrasounds) is one of the few methods to determine experimentally the bulk viscosity (see, e.g., [8]).

In summary, we may say that assuming the coincidence between thermodynamic and mechanical pressures gives rise to a simplified model, which permits an accurate description of the dynamics of most fluids in many flow conditions and whose limits of applicability may be clearly identified and physically justified. The main point raised in this note is that to derive this model, it is not necessary to assume the bulk viscosity of the fluid to be negligible—which we have seen to be in contradiction with experimental and theoretical findings—but only that condition (11) applies, which is a perfectly reasonable assumption considering the role of the term $\kappa \mathrm{div} V$ in the dynamic contribution of the complete stress tensor. Accepting this point of view, we immediately realize that the derived model may be seen as one of the most widely applicable of the many models of different complexity that are profitably used in fluid dynamics to obtain reasonably accurate predictions with affordable effort. To devise such models, it is of paramount importance to formulate physically and mathematically sound assumptions on the relative importance of the various terms playing a particular role in a certain flow condition. In any case, what is really essential is to be able to identify the limits of applicability of the models as a function of the desired level of accuracy of their results.

As an example of practical application of these ideas, we may refer to the incompressible flow model. In effect, we perfectly know that no incompressible fluid exists, but this does not preclude the possibility of using the incompressible flow equations to predict the fluid dynamic loads acting on bodies moving in a fluid at low Mach numbers. Furthermore, the limits of applicability of the model are also perfectly clear, and for instance, the very small variations in density caused by limited variations in temperature cannot be disregarded if one is interested in convection problems in liquids. Although in that case it is still possible to use the incompressible form of the continuity equation, one cannot neglect the driving term deriving from the multiplication of the density variations by the acceleration of gravity. A possible solution is then to apply the Boussinesq approximations, in which the variations with temperature of all fluid properties are ignored completely and those of density are considered only as regards their contribution to the gravitational force; this gives rise to another simplified model that may be successfully used in many situations (see, e.g., [12]). Nonetheless, when thermally stratified turbulent flows are examined, one may find conditions in which, even if the velocity field may still be considered solenoidal, it is necessary to also take the variations of viscosity and thermal expansion into account in order to accurately predict the flow evolution (see, e.g., [13]).

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