# Centered-Residuals-Based Moment Estimator and Test for Stochastic Frontier Models*

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The composed error of a stochastic frontier (SF) model consists of two random variables, and the identification of the model relies heavily on the distribution assumptions for each of these variables. While the literature has put much effort into applying various SF models to a wide range of empirical problems, little has been done to test the distribution assumptions of these two variables. In this paper, by exploiting the specification structures of the SF model, we propose a centered-residuals-based method of moments which can be easily and flexibly applied to testing the distribution assumptions on both of the random variables and to estimating the model parameters. A Monte Carlo simulation is conducted to assess the performance of the proposed method. We also provide two empirical examples to demonstrate the use of the proposed estimator and test using real data.

***Keywords***: stochastic frontier model, centered residuals, method of moments, specification test.

***JEL Codes***: C12, C46, C52.

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## 1 Introduction

The stochastic frontier (SF) model, which was originally proposed by Aigner, Lovell, and Schmidt (1977) and Meeusen and van den Broeck (1977), provides one of the most popular tools for analyzing the production/cost efficiency of individual production units. In addition to the production analysis, the model is also applied to other fields of research in economics and finance. For these applications, the one-sided random variable, which is characteristic of the SF model, represents systematic deviations from the otherwise optimal outcome. Examples include using the SF model to test the underpricing hypothesis of IPOs (Hunt-McCool et al. 1996) and the convergence hypothesis of economic growth (Kumbhakar and Wang 2005), and also using the model to estimate the effects of search cost on observed wage rates (Polachek and Yoon 1996) and the impact of financing constraints on firms' capital investment (Wang 2003).

Unlike the conventional parametric regression that has a zero-mean error term, the SF model features a composed error consisting of a zero-mean noise term ($v_i$) and a (non)negative inefficiency term ($u_i$). Given these two random terms, the composed error of the model has a non-zero mean. While a zero-mean normal random variable is exclusively assumed for $v_i$ in the literature, various distribution assumptions on $u_i$ have been proposed. Representative examples include the half-normal assumption of Aigner, Lovell, and Schmidt (1977), the exponential assumption of Meeusen and van den Broeck (1977), and the truncated-normal model of Stevenson (1980). Together with the normality assumption on $v_i$, closed-form probability density functions (PDFs) can be derived for these $v_i$ and $u_i$ combinations, which is one of the reasons why these combinations are popular in the literature.

In empirical studies, the SF model is often estimated by the maximum likelihood (ML) method and the (in)efficiency index is computed based on the ML estimators (MLEs). Nonetheless, it is known that if the composed error distribution is misspecified, the MLEs are likely to be inconsistent (see, e.g., Schmidt 1986) which would also cause bias in the estimated inefficiency index. Therefore, it is undoubtedly important to test the distribution assumptions of the noise and inefficiency terms in the SF analysis. However, as observed by Kopp and Mullahy (1990, p.168), the distributions of the noise and inefficiency terms "*have been assumed rather than tested*." Their observation may also apply to most empirical studies in the current SF literature.

Some tests regarding the model's error distribution have been proposed in the literature. Schmidt and Lin (1984) suggested a pre-test on the OLS residuals of the corresponding SF model to determine whether they have the implied skewness. The skewness test, however, does not discriminate among different distribution assumptions on $v_i$ and $u_i$. Coelli (1995) proposed a likelihood ratio test on the existence of $u_i$ in the model. The test is not on the particular distribution assumption of $u_i$, and it takes the distribution of $v_i$ as given.

Using Rao's score test principle, Lee (1983) proposed the distribution tests for the normal-half normal model and for the normal-truncated normal models against the Pearson family of truncated distributions based on the ML method. Using Hansen's (1982) over-identifying restriction test, Kopp and Mullahy (1990) introduced the distribution test for the SF model with a *symmetric* noise term and a *one-parameter* inefficiency term based on the generalized method of moments (GMM) method. These tests are designed for a particular type of distributional hypothesis, and non-trivial extensions and derivations are needed if we intend to apply them to wider ranges of distributional assumptions.

The score test principle and the ML method considered by Lee (1983) are theoretically, but not necessarily practically, applicable to the more context. It could be very difficult, if not impossible, to derive a closed-form composed error PDF when we go beyond the context of normal-half normal, -exponential, and -truncated normal models. The normal-Gamma model of Greene (1980) is a well-known example that lacks a closed-form PDF. Recently, Wang et al. (2010) suggested the use of Pearson's chi-squared test and the Kolmogorov-Smirnov test, with the correction for the parameter estimation problem, for checking the SF model. These two tests are based on the quantiles of the composed error distribution being tested. Since this distribution function is unlikely to have a closed from, their testing approach requires simulating the quantiles of the composed error distribution even when the composed error PDF has a closed form. Obviously, for practical and general applications, it is desirable to have certain estimation and testing methods that can be free of computing the composed error PDF (or distribution function). The method of Kopp and Mullahy (1990) is appealing in this respect, but it considers only the one-parameter inefficiency term and assumes the symmetric noise term. In a more general context, the distribution of the inefficiency term can have multiple parameters (e.g., the truncated-normal distribution) and, depending on the applications, the noise term may be asymmetrically distributed.

In this paper, by exploiting the specification structures of the SF model, we propose a centered-residuals-based method of moments which can be applied to testing various distribution assumptions of the noise and inefficiency terms. Unlike the score test, our test does not need to derive the PDF of the compound error and hence is applicable even when the composed error PDF does not have a closed form. Unlike Pearson's chi-squared test and the Kolmogorov-Smirnov test considered in Wang et al. (2010), our test does not need to simulate the quantiles of the composed error distribution. Unlike the GMM test of Kopp and Mullahy (1990), our testing framework does not require the inefficiency term to be a one-parameter random variable and the noise term to be a symmetrically distributed random variable. Therefore, compared to these existing tests, our test can be more easily and flexibly applied to testing the composed error distribution assumptions. Moreover, because of the use of the centered-residuals, our test is invariant to various (correctly specified) regressions and is not restricted to the ML method. This robustness is another important advantage in addition to the simplicity and flexibility of our test.

An added advantage of the moment-based approach is that an estimator, in addition to the test, can be developed in the same framework. It is thus possible to estimate and test the model in an integrated framework. Compared to the ML estimator which is commonly used in the literature, the moment estimator does not require the derivation of the model's PDF and is easier to compute. By the same token, various distribution assumptions are potentially accommodated.

The rest of the paper is organized as follows. Section 2 introduces the framework for the proposed estimator and test, and specific examples of normal-half normal and normal-truncated normal models are given in Section 3. A Monte Carlo analysis of the estimator and the test is conducted in Section 4. Two empirical examples are discussed in Section 5. We conclude the paper in Section 6. The Appendix includes the proof of an asymptotic expansion.

## 2 The Moment-Based Estimator and Test

In this section, we first introduce the general form of the centered-residuals-based moment estimator and test, and then provide a practical form of this estimator and test based on the arithmetic moment (characteristic function) restrictions, which is useful for empirical applications.

### 2.1 The General Form

Let $y_i$ be the logarithmic production of the $i$th firm, and $\mathcal{X}_i$ be the information set available in modeling $y_i$. Following the studies mentioned in Section 1, we consider the SF model:

$$y_i = \alpha + f(x_i, \beta) + \epsilon_i, \qquad \epsilon_i = v_i - u_i, \tag{1}$$

where $x_i$ represents a vector of $\mathcal{X}_i$-measurable explanatory variables, $\alpha + f(x_i, \beta)$ stands for the deterministic production function with the intercept $\alpha$ and the regression coefficient vector $\beta$, and $\epsilon_i$ is the composed error which consists of a zero-mean noise term $v_i$ and a non-negative term $u_i \geq 0$ which usually represents technical inefficiency in production. This model is assumed to have the specification structures:

**A.1** *$\{v_i\}$ and $\{u_i\}$ are two independent sequences of independently and identically distributed (IID) absolutely continuous random variables, and $(v_i, u_i)$ is independent of $\mathcal{X}_i$.*

**A.2** *$\mathbb{E}[y_i|\mathcal{X}_i] = \alpha_o + f(x_i, \beta_o) - \mu_1$, where $\mu_1 := \mathbb{E}[u_{oi}]$ and $u_{oi} := u_i|_{\alpha=\alpha_o,\beta=\beta_o}$, for some $(\alpha_o, \beta_o^\top)^\top$ in the parameter space of $(\alpha, \beta^\top)^\top$.*

Assumption **A.1** is a partial specification of the composed error. Let $g_v(\cdot, \theta_v)$ and $g_u(\cdot, \theta_v)$ be, respectively, the postulated PDFs of $v_i$ and $u_i$ with the parameter vectors $\theta_v$ and $\theta_u$. Denote $\theta := (\theta_v^\top, \theta_u^\top)^\top$ with $\dim(\theta) = p$ for some finite $p$. By combining **A.1** with these two PDFs, we can complete the SF model by setting the postulated PDF for $\epsilon_i$:

$$g_\epsilon(\epsilon, \theta) = \int_0^\infty g_v(\epsilon + u, \theta_v) g_u(u, \theta_u) \mathrm{d}u, \qquad \forall\ \epsilon \in \mathbb{R}. \tag{2}$$

In this model, the postulated conditional PDF for $y_i|\mathcal{X}_i$ is $g_\epsilon(y_i - \alpha - f(x_i, \beta), \theta)$. Assumption **A.2** requires the deterministic production function to be correctly specified. Without this basic assumption, we cannot identify the composed error in a sensible way. Given **A.1** and **A.2**, we can define the null hypothesis of our interest:

$H_o$: *$g_v(\cdot, \theta_{ov})$ and $g_u(\cdot, \theta_{ou})$ are, respectively, the true PDFs of $v_{oi} := v_i|_{\alpha=\alpha_o,\beta=\beta_o}$ and $u_{oi}$ for some $\theta_o := (\theta_{ov}^\top, \theta_{ou}^\top)^\top$ in the parameter space of $\theta$.*

The SF model that comprises (1) and (2) has been widely applied to various empirical studies. The specification correctness of this model requires $H_o$ (and the maintained assumptions: **A.1** and **A.2**).

In the case where $g_v$ is a normal PDF and $g_u$ is an exponential, half-normal, or truncated normal PDF, the PDF $g_\epsilon$ has a closed form. In this scenario, it is common to estimate $\alpha_o$, $\beta_o$, and $\theta_o$ simultaneously using the ML method that maximizes the log-likelihood function:

$$\frac{1}{n}\sum_{i=1}^{n} \ln g_\epsilon(y_i - \alpha - f(x_i, \beta), \theta), \tag{3}$$

where $n$ denotes the sample size. However, if one wants to apply the model in a more general setting where the $(g_v, g_u)$ combination is not confined to the above choices, the closed-form of $g_\epsilon$ for such models may not exist. In the following, by exploiting the specification structures in **A.1** and **A.2**, we propose a centered-residuals-based method of moments for estimating and testing the SF model. The proposed method is useful in this general scenario and is also applicable to the aforementioned particular models.

Let $\hat{\alpha}_n$ be an estimator of $\alpha_o$, and $\hat{\beta}_n$ be a $n^{1/2}$-consistent estimator of $\beta_o$ such that $n^{1/2}(\hat{\beta}_n - \beta_o) = O_p(1)$ under Assumption **A.1**. As we will explain later, our method does not require $\hat{\alpha}_n$ to be a consistent estimator of $\alpha_o$. In practical applications, we may compute $\hat{\alpha}_n$ and $\hat{\beta}_n$ as the least square (LS) estimators of $\alpha_o$ and $\beta_o$, respectively, for simplicity. Given the choice of $\hat{\alpha}_n$ and $\hat{\beta}_n$, we can define the residual $\hat{\epsilon}_i := y_i - \hat{\alpha}_n - f(x_i, \hat{\beta}_n)$. At first sight, it seems natural to base the moment estimator of $\theta_o$ and the moment test for $H_o$ on the $\hat{\epsilon}_i$'s. However, this strategy requires a modification because the $\hat{\epsilon}_i$'s are dependent on the estimator $\hat{\alpha}_n$ which could be inconsistent for $\alpha_o$. In fact, the LS estimator $\hat{\alpha}_n$ is consistent for $\alpha_o - \mu_1$ but not for $\alpha_o$; recall that $\mu_1 := \mathbb{E}[u_{oi}]$. This is due to the fact that the intercept $\alpha_o$ cannot be identified without the information for $\mu_1$, which is determined by the distribution of $u_{io}$ and this distribution is unspecified under Assumption **A.1**. This is an important feature that distinguishes (the moment-based method for) the SF model from (that for) the conventional parametric regression. If a consistent estimator of $\alpha_o$ is desired, we can replace $\hat{\alpha}_n$ with a very simple post-estimation adjustment:

$$\hat{\alpha}_n^* = \hat{\alpha}_n + \mu_1|_{\theta=\hat{\theta}_n}, \tag{4}$$

in which $\hat{\theta}_n$ denotes the centered-residuals-based moment estimator for $\theta_o$ discussed below.

To circumvent the inconsistency of $\hat{\alpha}_n$, we base our moment estimator and test on the centered residual $\hat{\epsilon}_{c,i} := \hat{\epsilon}_i - n^{-1}\sum_{i=1}^{n}\hat{\epsilon}_i$ that cancels out the estimator $\hat{\alpha}_n$. Correspondingly, we define the centered composed error $\epsilon_{c,i} := \epsilon_i - \mathbb{E}[\epsilon_i]$ that eliminates the nuisance parameter $\alpha$. Let $\Delta$ be a compact parameter space of $\delta := (\beta^\top, \theta^\top)^\top \in \Delta$. Denote $\delta_o := (\beta_o^\top, \theta_o^\top)^\top$ and $\epsilon_{c,oi} := \epsilon_{c,i}|_{\delta=\delta_o}$. Suppose that $H_o$ implies two sets of moment restrictions of $\epsilon_{c,oi}$:

$$\mathbb{E}[\phi_j(\epsilon_{c,oi}, \theta_o)] = 0, \qquad j = 1, 2, \tag{5}$$

for some $p$-dimensional estimating function $\phi_1(\epsilon_{c,i}, \theta)$ and some $q$-dimensional testing function $\phi_2(\epsilon_{c,i}, \theta)$ that does not contain the same element. (Practical examples of $\phi_1$ and $\phi_2$ will be given in Section 2.2.) Accordingly, we can estimate $\theta_o$ under $H_o$ using the condition $\mathbb{E}[\phi_1(\epsilon_{c,oi}, \theta)] = 0$, and test $H_o$ using the condition $\mathbb{E}[\phi_2(\epsilon_{c,oi}, \theta_o)] = 0$. Specifically, let $\hat{\theta}_v$ and $\hat{\theta}_u$ be the associated estimators for $\theta_{ov}$ and $\theta_{ou}$, respectively. We solve the moment estimator $\hat{\theta}_n := (\hat{\theta}_v^\top, \hat{\theta}_u^\top)^\top$ from the estimating equation:

$$\frac{1}{n}\sum_{i=1}^{n} \phi_1(\hat{\epsilon}_{c,i}, \hat{\theta}_n) = 0, \tag{6}$$

and base the moment test on the statistic:

$$\hat{M}_n := \frac{1}{n}\sum_{i=1}^{n} \phi_2(\hat{\epsilon}_{c,i}, \hat{\theta}_n). \tag{7}$$

The resulting test is in spirit closer to the conditional moment test of Newey (1985) and Tauchen (1985). However, unlike our method, the latter is focused on the ML method and does not take into account the features of the SF model (and hence is not based on the centered residuals). Alternatively, similar to Kopp and Mullahy (1990), we may also test $H_o$ by applying Hansen's (1982) overidentifying restriction test to the $(p+q)$-dimensional moment function $\phi = (\phi_1^\top, \phi_2^\top)^\top$. An advantage of this alternative method is its conceptual simplicity because Hansen's (1982) asymptotic results are directly applicable. Nonetheless, our setting is useful for estimating $\theta_o$ in an easier way by using only the just-identified moment conditions; recall that $\dim(\phi_1) = \dim(\theta) = p$. This is also useful for allowing us to test $H_o$ using various $\phi_2$'s without the need to re-estimate the model.

Denote $K_j := \mathbb{E}[\nabla_{\theta^\top} \phi_j(\epsilon_{c,i}, \theta)]$ and the transformation:

$$\varphi_{ij} := \phi_j(\epsilon_{c,i}, \theta) - \mathbb{E}\left[\frac{\partial}{\partial \epsilon_{c,i}} \phi_j(\epsilon_{c,i}, \theta)\right] \epsilon_{c,i} \tag{8}$$

that are, respectively, a $p \times p$ ($q \times p$) matrix and a $p \times 1$ ($q \times 1$) vector when $j = 1$ ($j = 2$). We also write $\hat{\delta}_n := (\hat{\beta}_n^\top, \hat{\theta}_n^\top)^\top$, $K_{jo} := K_j|_{\delta=\delta_o}$ $\hat{K}_j := K_j|_{\delta=\hat{\delta}_n}$ $\varphi_{ij,o} := \varphi_{ij}|_{\delta=\delta_o}$, and $\hat{\varphi}_{ij} := \varphi_{ij}|_{\delta=\hat{\delta}_n}$, and define the $q \times q$ covariance matrix: $\Omega_o := \mathbb{E}[(\varphi_{i2,o} - K_{2o}K_{1o}^{-1}\varphi_{i1,o})(\varphi_{i2,o} - K_{2o}K_{1o}^{-1}\varphi_{i1,o})^\top]$ and its sample counterpart:

$$\hat{\Omega}_n := \frac{1}{n}\sum_{i=1}^{n} (\hat{\varphi}_{i2} - \hat{K}_2\hat{K}_1^{-1}\hat{\varphi}_{i1})(\hat{\varphi}_{i2} - \hat{K}_2\hat{K}_1^{-1}\hat{\varphi}_{i1})^\top. \tag{9}$$

We further make the following assumption for our asymptotic analysis:

**A.3** (a) $\phi_j(\epsilon_{c,i}, \theta)$ *is continuously differentiable on* $\Delta$, $\mathbb{E}[\phi_j(\epsilon_{c,i}, \theta)]$ *exists for every* $\delta \in \Delta$, *and* $\mathbb{E}[\sup_{\delta\in\Delta} \|\phi_i(\epsilon_{c,i}, \theta)\|] < \infty$; $j = 1, 2$.

(b) $\mathbb{E}[\phi_1(\epsilon_{c,i}, \theta)] \neq 0$ *when* $\delta \neq \delta_o$, *and* $\delta_o \in \operatorname{int}(\Delta)$ *is unique.*

(c) *The sequences* $\{\frac{\partial}{\partial \epsilon_{c,i}} \phi_j(\epsilon_{c,i}, \theta)\}$, $\{\nabla_{\beta^\top} f(x_i, \beta))\}$, *and* $\{\nabla_{\theta^\top} \phi_j(\epsilon_{c,i}, \theta)\}$ *obey a uniform weak law of large numbers (UWLLN) for both* $j = 1, 2$.

(d) $K_1$ *is positive definite.*

(e) *The statistic* $n^{-1/2} \sum_{i=1}^n \varphi_{ij,o}$ *obeys a central limit theorem such that* $n^{-1/2} \sum_{i=1}^n \varphi_{ij,o} \xrightarrow{\text{d}} N(0, V_{jo})$ *with* $V_{jo} := \mathbb{E}[\varphi_{ij,o} \varphi_{ij,o}^\top]$; $j = 1, 2$.

(f) *The statistic* $\hat{\Omega}_n$ *is consistent for* $\Omega_o$ *and is positive definite.*

This assumption comprises a set of regularity and high-level conditions that are assumed to hold in addition to **A.1**, **A.2**, and $H_o$. It is introduced for showing the consistency of $\hat{\alpha}_n$ and the asymptotic normality of $\hat{\alpha}_n$ and $\hat{M}_n$, as discussed below. The asymptotic arguments underlying these properties are standard in econometrics; see, e.g., Newey and McFadden (1994) or Hall (2005) for related discussions.

Assumption **A.3**(a) is a set of regularity conditions on the moment functions $\phi_1$ and $\phi_2$. Assumption **A.3**(b) further requires that the moment condition $\mathbb{E}[\phi_1(\epsilon_{c,oi}, \theta_o)] = 0$, implied by $H_o$, be uniquely identifiable. The consistency of $\hat{\theta}_n$ can be viewed as a specialization of the consistency of the GMM estimator; see, e.g., Hall (2005, Theorem 3.1) for the latter. Specifically, let $Q_n(\delta)$ be the inner product of the $p \times 1$ vector $n^{-1} \sum_{i=1}^n \phi_1(\epsilon_{c,i}, \theta)$, and $Q_o(\delta)$ be the inner product of $\mathbb{E}[\phi_1(\epsilon_{c,i}, \theta)]$. Given the IIDness of $\{\epsilon_{c,i}\}$ implied by **A.1**, the compactness of $\Delta$, and Assumption **A.3**(a) with $i = 1$, we may follow Hall (2005, Lemma 3.1) to show the uniform convergence: $\sup_{\delta \in \Delta} |Q_n(\delta) - Q_o(\delta)| \xrightarrow{\text{p}} 0$ and obtain the consistency of $\hat{\theta}_n$ for $\theta_o$ from this uniform convergence and the unique identification in Assumption **A.3**(b).

In the Appendix, we further utilize this consistency and Assumptions: **A.1**-**A.3**(c) to derive the following asymptotic expansion under $H_o$:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \phi_j(\hat{\epsilon}_{c,i}, \hat{\theta}_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \varphi_{ij,o} + K_{jo} \sqrt{n}(\hat{\theta}_n - \theta_o) + o_p(1) \tag{10}$$

for both $j = 1, 2$. This expansion is essential for showing the asymptotic normality of $\hat{\theta}_n$ and $\hat{M}_n$, as we will see shortly. An important property of this expansion is that it is free of the estimation effect of $\hat{\beta}_n$. This property is quite appealing because it makes our estimator and test invariantly applicable to various specifications of (1). Chen (2008) also proposed a class of moment functions in the conventional conditional mean-and-variance context, which is free of the estimation effect of the conditional mean-and-variance parameters. However, unlike the latter, the transformation in (10) is a natural by-product of the centered-residuals that we introduce for eliminating the inconsistent estimator $\hat{\alpha}_n$ in the SF analysis. This centralization and the resulting robustness are important features of our method.

By applying (10) to (6) and (7), we can write that, given Assumption **A.3**(d),

$$\sqrt{n}(\hat{\theta}_n - \theta_o) = -K_{1o}^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n \varphi_{i1,o} + o_p(1) \tag{11}$$

and

$$\sqrt{n}\hat{M}_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\varphi_{i2,o} - K_{2o}K_{1o}^{-1}\varphi_{i1,o}) + o_p(1). \tag{12}$$

On the other hand, the central limit theorem in Assumption **A.3**(e) can be justified by the fact that $\{\varphi_{ij,o}\}$ is an IID sequence with $\mathbb{E}[\varphi_{ij,o}|\mathcal{X}_i] = 0$ under **A.1**, **A.2**, and $H_o$, while it also needs certain regularity conditions. From **A.3**(e) and (11), we can obtain the asymptotic normality of $\hat{\alpha}_n$:

$$\sqrt{n}(\hat{\theta}_n - \theta_o) \xrightarrow{\text{d}} N(0, K_{1o}^{-1}V_{1o}(K_{1o}^{-1})^\top); \tag{13}$$

recall that $V_{1o} := \mathbb{E}[\varphi_{i1,o}\varphi_{i1,o}^\top]$. Similarly, from **A.3**(e) and (12), we can obtain the asymptotic normality of $\hat{M}_n$:

$$\sqrt{n}\hat{M}_n \xrightarrow{\text{d}} N(0, \Omega_o). \tag{14}$$

Given the positive definiteness of $\hat{\Omega}_n$ in Assumption **A.3**(f), we can define the test statistic

$$D_n := n\hat{M}_n^\top\hat{\Omega}_n^{-1}\hat{M}_n. \tag{15}$$

From (14) and the consistency of $\hat{\Omega}_n$ for $\Omega_o$ in **A.3**(f), this test statistic has the asymptotic distribution:

$$D_n \xrightarrow{\text{d}} \chi^2(q)$$

under $H_o$ and Assumptions: **A.1-A.3**.

## 2.2 A Practical Form

The estimator $\hat{\theta}_n$ and the test statistic $D_n$ are generally applicable to various $\phi_1$'s and $\phi_2$'s. To facilitate the proposed estimator and test, it is important to give $\phi_1$ and $\phi_2$ certain simple formulas. In this study, we consider the estimating function:

$$\phi_1(\epsilon_{c,i}, \theta) = \left(\epsilon_{c,i}^2 - \mathbb{E}[\epsilon_{c,i}^2], \ldots, \epsilon_{c,i}^{p+1} - \mathbb{E}[\epsilon_{c,i}^{p+1}]\right)^\top \tag{16}$$

and the testing function:

$$\phi_2(\epsilon_{c,i}, \theta) = \begin{bmatrix} \sin(\omega_1\epsilon_{c,i}) - \mathbb{E}[\sin(\omega_1\epsilon_{c,i})] \\ \vdots \\ \sin(\omega_q\epsilon_{c,i}) - \mathbb{E}[\sin(\omega_q\epsilon_{c,i})] \end{bmatrix} \tag{17}$$

or

$$\phi_2(\epsilon_{c,i}, \theta) = \begin{bmatrix} \cos(\omega_1\epsilon_{c,i}) - \mathbb{E}[\cos(\omega_1\epsilon_{c,i})] \\ \vdots \\ \cos(\omega_q\epsilon_{c,i}) - \mathbb{E}[\cos(\omega_q\epsilon_{c,i})] \end{bmatrix}, \tag{18}$$

in which $\omega_k$ is a pre-determined number in $\mathbb{R}^+$ with $k = 1, \ldots, q$. (It is very difficult to determine the optimal choice of the $\omega_k$'s from a theoretical viewpoint. Nonetheless, as shown by the simulations in Section 4, it may be appropriate to choose the $\omega_k$'s around $\omega = 1$.) The choice of these moment functions is motivated by that fact that, by exploiting the specification structures in Assumptions: **A.1** and **A.2**, we can easily derive the arithmetic moment: $\mathbb{E}[\epsilon_{c,i}^k]$ and the characteristic function:

$$\mathbb{E}[\exp(\mathrm{i}\omega\epsilon_{c,oi})] = \mathbb{E}[\cos(\omega\epsilon_{c,oi})] + \mathrm{i}\mathbb{E}[\sin(\omega\epsilon_{c,oi})], \quad \mathrm{i} := \sqrt{-1},$$

from their counterparts of $v_{oi}$ and $u_{oi}$. Unlike the former, the latter are often of closed-form formulas; see, e.g., Balakrishnan and Nevzorov (2003) for arithmetic moments of many popular distributions and Oberhettinger (1973) for characteristic functions of a vast number of distributions. Moreover, this derivation does not involve the computation of the PDF $g_\epsilon$. In the following, we provide the details of this derivation.

To be specific, we rewrite the centered composed error $\epsilon_{c,oi}$ as $\epsilon_{c,oi} = v_{oi} - u_{c,oi}$, in which $u_{c,oi} := u_{oi} - \mu_1$ is the centered inefficiency term. Given the binomial theorem:

$$\epsilon_{c,oi}^k = \sum_{s=0}^{k} (-1)^s \binom{k}{s} v_{oi}^{k-s} u_{c,oi}^s, \tag{19}$$

we can further use the independence between $u_{oi}$ and $v_{oi}$, implied by Assumptions: **A.1** and **A.2**, to write that

$$\mathbb{E}[\epsilon_{c,oi}^k] = \sum_{s=0}^{k} (-1)^s \binom{k}{s} \mathbb{E}[v_{oi}^{k-s}]\mathbb{E}[u_{c,oi}^s], \tag{20}$$

where

$$\mathbb{E}[u_{c,oi}^s] = \sum_{l=0}^{s} (-1)^l \binom{s}{l} \mathbb{E}[u_{oi}^{s-l}]\mu_1^l. \tag{21}$$

By denoting $\nu_k := \mathbb{E}[v_{oi}^k]$ and $\mu_k := \mathbb{E}[u_{oi}^k]$, we can re-express (20) as:

$$\mathbb{E}[\epsilon_{c,oi}^k] = \sum_{s=0}^{k} (-1)^s \binom{k}{s} \nu_{k-s} \left\{ \sum_{l=0}^{s} (-1)^l \binom{s}{l} \mu_{s-l}\mu_1^l \right\}. \tag{22}$$

Using this closed-form formula, we can easily compute $\mathbb{E}[\epsilon_{c,oi}^k]$ from the $\mu_k$'s and $\nu_k$'s.

Similarly, given the addition formulas of the sine and cosine functions:

$$\sin(\omega\epsilon_{c,oi}) = \sin(\omega v_{oi})\cos(\omega u_{c,oi}) - \cos(\omega v_{oi})\sin(\omega u_{c,oi}) \tag{23}$$

and

$$\cos(\omega\epsilon_{c,oi}) = \cos(\omega v_{oi})\cos(\omega u_{c,oi}) + \sin(\omega v_{oi})\sin(\omega u_{c,oi}), \tag{24}$$

we can also utilize the independence between $u_{oi}$ and $v_{oi}$ to write that

$$\mathbb{E}[\sin(\omega\epsilon_{c,oi})] = \mathbb{E}[\sin(\omega v_{oi})]\mathbb{E}[\cos(\omega u_{c,oi})] - \mathbb{E}[\cos(\omega v_{oi})]\mathbb{E}[\sin(\omega u_{c,oi})] \tag{25}$$

and

$$\mathbb{E}[\cos(\omega\epsilon_{c,oi})] = \mathbb{E}[\cos(\omega v_{oi})]\mathbb{E}[\cos(\omega u_{c,oi})] + \mathbb{E}[\sin(\omega v_{oi})]\mathbb{E}[\sin(\omega u_{c,oi})], \tag{26}$$

in which

$$\mathbb{E}[\sin(\omega u_{c,oi})] = \mathbb{E}[\sin(\omega u_{oi})]\cos(\omega\mu_1) - \mathbb{E}[\cos(\omega u_{oi})]\sin(\omega\mu_1) \tag{27}$$

and

$$\mathbb{E}[\cos(\omega u_{c,oi})] = \mathbb{E}[\cos(\omega u_{oi})]\cos(\omega\mu_1) + \mathbb{E}[\sin(\omega u_{oi})]\sin(\omega\mu_1). \tag{28}$$

Using the closed-form formulas in (25) and (26), we can also easily obtain the characteristic function of $\epsilon_{c,oi}$ from the characteristic functions of $v_{oi}$ and $u_{oi}$ (with the real parts: $\mathbb{E}[\cos(\omega u_{oi})]$ and $\mathbb{E}[\cos(\omega v_{oi})]$ and the imaginary parts: $\mathbb{E}[\sin(\omega u_{oi})]$ and $\mathbb{E}[\sin(\omega v_{oi})]$).

In what follows, we refer to the proposed test with (17) as the sine test and the proposed test with (18) as the cosine test. The arithmetic-moments-based estimator and the sine/cosine test can be flexibly applied to various SF models with various $(g_v, g_u)$'s, provided that the arithmetic moments and the characteristic functions of $v_{oi}$ and $u_{oi}$ are of closed forms. Since the normal, half-normal, exponential, Gamma, and truncated-normal distributions are quite popular in the SF literature, we summarize their PDFs and arithmetic moments in Table 1 and show their characteristic functions in Table 2.

# 3 Examples

In this section, we consider the normal-half normal and normal-exponential models as demonstrative examples to show the applicability of the moment estimator and the sine/cosine test introduced in Section 2.2. These two SF models are considered for their popularity in empirical studies and their simplicity in presentation. As addressed before, the proposed estimator and test are also applicable to other SF models.

## 3.1 The Moment Estimator

Note that, in estimating a "two-parameter" model with $p = 2$, the moment condition $\mathbb{E}[\phi_1(\epsilon_{c,oi}, \theta_o)] = 0$, with (16) and (22), degenerates to the following form:

$$\begin{bmatrix} \mathbb{E}[\epsilon_{c,oi}^2] \\ \mathbb{E}[\epsilon_{c,oi}^3] \end{bmatrix} = \begin{bmatrix} \nu_2 + \mu_2 - \mu_1^2 \\ \nu_3 - \mu_3 + 3\mu_2\mu_1 - 2\mu_1^3 \end{bmatrix}. \tag{29}$$

The normal-half normal and normal-exponential models are examples of two-parameter models; see Table 1 for the associated $g_v$, $g_u$'s, and $(\mu_k, \nu_k)$'s. Using these $(\mu_k, \nu_k)$'s, we can further simplify (29) to:

$$\begin{bmatrix} \mathbb{E}[\epsilon_{c,oi}^2] \\ \mathbb{E}[\epsilon_{c,oi}^3] \end{bmatrix} = \begin{bmatrix} \sigma_v^2 + \lambda_2\sigma_u^2 \\ \lambda_3\sigma_u^3 \end{bmatrix}, \tag{30}$$

in which $\lambda_2 = 1 - 2/\pi$ and $\lambda_3 = (1 - 4/\pi)(2/\pi)^{1/2}$ for the normal-half normal model and $\lambda_2 = 1$ and $\lambda_3 = -2$ for the normal-exponential model; the definition of $\theta_o = (\sigma_v^2, \sigma_u^2)^\top$ is given in Table 1.

In this example, the moment estimator $\hat{\theta}_n = (\hat{\sigma}_v^2, \hat{\sigma}_u^2)^\top$ is solved from the second-and-third moments-based estimating equation:

$$\begin{bmatrix} \frac{1}{n}\sum_{i=1}^n \hat{\epsilon}_{c,i}^2 \\ \frac{1}{n}\sum_{i=1}^n \hat{\epsilon}_{c,i}^3 \end{bmatrix} = \begin{bmatrix} \hat{\sigma}_v^2 + \lambda_2\hat{\sigma}_u^2 \\ \lambda_3\hat{\sigma}_u^3 \end{bmatrix}, \tag{31}$$

and is of a simple closed form:

$$\begin{bmatrix} \hat{\sigma}_v^2 \\ \hat{\sigma}_u^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{n}\sum_{i=1}^n \hat{\epsilon}_{c,i}^2 - \lambda_2\hat{\sigma}_u^2 \\ \left\{\left(\lambda_3^{-1}\frac{1}{n}\sum_{i=1}^n \hat{\epsilon}_{c,i}^3\right)^2\right\}^{1/3} \end{bmatrix}. \tag{32}$$

Unlike the MLE for $\theta_o$, this $\hat{\theta}_n$ can be easily computed without using the PDF $g_\epsilon$ and the numerical optimization. The asymptotic distribution of this $\hat{\theta}_n$ follows (13) with the settings:

$$\phi_1(\epsilon_{c,oi}, \theta_o) = \begin{bmatrix} \epsilon_{c,oi}^2 - (\sigma_v^2 + \lambda_2\sigma_u^2) \\ \epsilon_{c,oi}^3 - \lambda_3\sigma_u^3 \end{bmatrix},$$

$$K_{1o} = -\begin{bmatrix} 1 & \lambda_2 \\ 0 & \frac{3}{2}\lambda_3\sigma_u \end{bmatrix}, \tag{33}$$

and

$$\varphi_{i1,o} = \begin{bmatrix} \epsilon_{c,oi}^2 - (\sigma_v^2 + \lambda_2\sigma_u^2) \\ \epsilon_{c,oi}^3 - \lambda_3\sigma_u^3 - 3(\sigma_v^2 + \lambda_2\sigma_u^2)\epsilon_{c,oi} \end{bmatrix}. \tag{34}$$

According to this asymptotic distribution, we can easily evaluate the significance of the moment estimator.

### 3.2 The Sine/Cosine Test

Note that the normal distribution assumption of $v_{oi}$ implies the symmetry and hence the characteristic function restriction: $\mathbb{E}[\sin(\omega v_{oi})] = 0$, $\forall\ \omega \in \mathbb{R}$. We can use this restriction to simplify the sine/cosine test. Specifically, by using this symmetry restriction, we can rewrite (25) as

$$\mathbb{E}[\sin(\omega\epsilon_{c,oi})] = \mathbb{E}[\cos(\omega v_{oi})]h_s(\omega) \tag{35}$$

with

$$h_s(\omega) := \mathbb{E}[\cos(\omega u_{oi})]\sin(\omega\mu_1) - \mathbb{E}[\sin(\omega u_{oi})]\cos(\omega\mu_1),$$

and base the sine test statistic on the following settings:

$$\phi_2(\epsilon_{c,i},\theta) = \begin{bmatrix} \sin(\omega_1\epsilon_{c,i}) - \mathbb{E}[\cos(\omega_1 v_{oi})]h_s(\omega_1) \\ \vdots \\ \sin(\omega_q\epsilon_{c,i}) - \mathbb{E}[\cos(\omega_q v_{oi})]h_s(\omega_q) \end{bmatrix}, \tag{36}$$

$$K_{2o} = -\begin{bmatrix} \frac{\partial}{\partial\sigma_v^2}\mathbb{E}[\cos(\omega_1 v_{oi})]h_s(\omega_1), & \mathbb{E}[\cos(\omega_1 v_{oi})]\frac{\partial}{\partial\sigma_u^2}h_s(\omega_1) \\ \vdots & \vdots \\ \frac{\partial}{\partial\sigma_v^2}\mathbb{E}[\cos(\omega_q v_{oi})]h_s(\omega_q), & \mathbb{E}[\cos(\omega_q v_{oi})]\frac{\partial}{\partial\sigma_u^2}h_s(\omega_q) \end{bmatrix} \tag{37}$$

with

$$\begin{aligned}\frac{\partial}{\partial\sigma_u^2}h_s(\omega) := &\left(\frac{\partial}{\partial\sigma_u^2}\mathbb{E}[\cos(\omega u_{oi})]\right)\sin(\omega\mu_1) - \left(\frac{\partial}{\partial\sigma_u^2}\mathbb{E}[\sin(\omega u_{oi})]\right)\cos(\omega\mu_1) \\ &+ \left(\mathbb{E}[\cos(\omega u_{oi})]\cos(\omega\mu_1) + \mathbb{E}[\sin(\omega u_{oi})]\sin(\omega\mu_1)\right)\omega\frac{\partial}{\partial\sigma_u^2}\mu_1,\end{aligned}$$

and

$$\varphi_{i2,o} = \begin{bmatrix} \sin(\omega_1\epsilon_{c,oi}) - \mathbb{E}[\cos(\omega_1 v_{oi})]h_s(\omega_1) - \mathbb{E}[\cos(\omega_1\epsilon_{c,oi})]\omega_1\epsilon_{c,oi} \\ \vdots \\ \sin(\omega_q\epsilon_{c,oi}) - \mathbb{E}[\cos(\omega_q v_{oi})]h_s(\omega_q) - \mathbb{E}[\cos(\omega_q\epsilon_{c,oi})]\omega_q\epsilon_{c,oi} \end{bmatrix}. \tag{38}$$

The sine test statistic is a particular version of (15) with the statistic $\hat{M}_n$, which is obtained by plugging (36) into (7), and the asymptotic covariance matrix estimator $\hat{\Omega}_n$, which is obtained by introducing (33), (34), (37), and (38) into (9).

Similarly, we can use the same symmetry restriction to simplify (26) as

$$\mathbb{E}[\cos(\omega\epsilon_{c,oi})] = \mathbb{E}[\cos(\omega v_{oi})]h_c(\omega) \tag{39}$$

with

$$h_c(\omega) := \mathbb{E}[\cos(\omega u_{oi})]\cos(\omega\mu_1) + \mathbb{E}[\sin(\omega u_{oi})]\sin(\omega\mu_1),$$

and base the cosine test statistic on the following settings:

$$\phi_2(\epsilon_{c,i},\theta) = \begin{bmatrix} \cos(\omega_1\epsilon_{c,i}) - \mathbb{E}[\cos(\omega_1 v_{oi})]h_c(\omega_1) \\ \vdots \\ \cos(\omega_q\epsilon_{c,i}) - \mathbb{E}[\cos(\omega_q v_{oi})]h_c(\omega_q) \end{bmatrix}, \tag{40}$$

$$K_{2o} = -\begin{bmatrix} \frac{\partial}{\partial \sigma_v^2}\mathbb{E}[\cos(\omega_1 v_{oi})]h_c(\omega_1), & \mathbb{E}[\cos(\omega_1 v_{oi})]\frac{\partial}{\partial \sigma_u^2}h_c(\omega_1) \\ \vdots & \vdots \\ \frac{\partial}{\partial \sigma_v^2}\mathbb{E}[\cos(\omega_q v_{oi})]h_c(\omega_q), & \mathbb{E}[\cos(\omega_q v_{oi})]\frac{\partial}{\partial \sigma_u^2}h_c(\omega_q) \end{bmatrix}, \tag{41}$$

in which

$$\begin{aligned}\frac{\partial}{\partial \sigma_u^2}h_c(\omega) &:= \left(\frac{\partial}{\partial \sigma_u^2}\mathbb{E}[\cos(\omega u_{oi})]\right)\cos(\omega\mu_1) + \left(\frac{\partial}{\partial \sigma_u^2}\mathbb{E}[\sin(\omega u_{oi})]\right)\sin(\omega\mu_1) \\ &\quad - \left(\mathbb{E}[\cos(\omega u_{oi})]\sin(\omega\mu_1) - \mathbb{E}[\sin(\omega u_{oi})]\cos(\omega\mu_1)\right)\omega\frac{\partial}{\partial \sigma_u^2}\mu_1,\end{aligned}$$

and

$$\varphi_{i2,o} = \begin{bmatrix} \cos(\omega_1\epsilon_{c,oi}) - \mathbb{E}[\cos(\omega_1 v_{oi})]h_c(\omega_1) + \mathbb{E}[\sin(\omega_1\epsilon_{c,oi})]\omega_1\epsilon_{c,oi} \\ \vdots \\ \cos(\omega_q\epsilon_{c,oi}) - \mathbb{E}[\cos(\omega_q v_{oi})]h_c(\omega_q) + \mathbb{E}[\sin(\omega_q\epsilon_{c,oi})]\omega_q\epsilon_{c,oi} \end{bmatrix}. \tag{42}$$

The cosine test statistic is another particular version of (15) with the statistic $\hat{M}_n$, which is obtained by plugging (40) into (7), and the asymptotic covariance matrix estimator $\hat{\Omega}_n$, which is obtained by introducing (33), (34), (41), and (42) into (9).

In Table 2, we show the formula of $\mathbb{E}[\cos(\omega v_i)]$ for the normal distribution and the formulas of $\mathbb{E}[\sin(\omega u_i)]$ and $\mathbb{E}[\cos(\omega u_i)]$ for the exponential and half-normal distributions. Given these formulas, it is easy to further show that

$$\frac{\partial}{\partial \sigma_v^2}\mathbb{E}[\cos(\omega v_i)] = -\frac{1}{2}\omega^2\exp\left(-\frac{1}{2}\omega^2\sigma_v^2\right)$$

for the normal distribution,

$$\frac{\partial}{\partial \sigma_u^2}\mathbb{E}[\sin(\omega u_i)] = -\frac{1}{2}\omega^2\mathbb{E}[\sin(\omega u_i)] + \frac{1}{\sqrt{2\pi}\sigma_u}\omega$$

and

$$\frac{\partial}{\partial \sigma_u^2}\mathbb{E}[\cos(\omega u_i)] = -\frac{1}{2}\omega^2\mathbb{E}[\cos(\omega u_i)]$$

for the half-normal distribution, and

$$\frac{\partial}{\partial \sigma_u^2}\mathbb{E}[\sin(\omega u_i)] = \frac{\omega}{2\sigma_u(1+\omega^2\sigma_u^2)} - \frac{\omega^3\sigma_u}{(1+\omega^2\sigma_u^2)^2}$$

and

$$\frac{\partial}{\partial \sigma_u^2}\mathbb{E}[\cos(\omega u_i)] = -\frac{\omega^2}{(1+\omega^2\sigma_u^2)^2}$$

for the exponential distribution. Accordingly, the sine/cosine test statistic can be easily programmed.

# 4 Monte Carlo Simulation

In this section, we conduct a Monte Carlo simulation to assess the finite-sample performance of the moment estimator and the sine/cosine test introduced in Section 3. The model used in the simulation is a practical form of (1):

$$y_i = \alpha_o + \beta_o^\top x_i + \epsilon_i, \qquad \epsilon_i = v_i - u_i.$$

The parameters $\alpha_o$ and $\beta_o$ are estimated by the ordinary LS method. To generate the data in this simulation, we set $\alpha_o = 0$, $\beta_o = (1,1,1)^\top$, and $x_i \sim N(0, I_3)$, and consider the normal-half normal combination and the normal-exponential combination of $(v_i, u_i)$ with various $(\sigma_v^2, \sigma_u^2)$'s. As shown in Table 1, we use $\sigma_u^2$ to indicate both the variance of the exponential distribution and the variance of the normal which is truncated to obtain the half-normal distribution. Specifically, we set $\sigma_v^2 = 1$ for all combinations, $\sigma_u^2 = 0.2, 0.5, 1, 3$ and 6 for the half normal distribution, and $\sigma_u^2$ =1, 1.3, and 1.6 for the exponential combination. Performances of the estimator and the tests are investigated for various sample sizes: $n$ =50, 100, 200, 500, 1,000, and 10,000. The number of replications is 10,000, and we consider a nominal 5% size for the test statistic.

## 4.1 The Estimators

Table 3 shows the results of the moment estimator on $\sigma_u^2$ and $\sigma_v^2$, presented in (32), under different data generating processes (DGPs). We are also interested in seeing what happens to the parameter estimates when an incorrect distribution assumption is used in the estimation, and the results are also reported in the table. For comparison purposes, Table 4 shows the corresponding results for the MLE, which is obtained by maximizing (3) under the same setting.

Table 3 shows that the moment estimator is reasonably accurate and precise when the distribution assumption is correctly specified. It is well known that the MLE is the asymptotically most efficient estimator when the model is correctly specified; nonetheless, our proposed moment estimator fairs pretty well in comparison. It is worthwhile noting that, compared to the MLE, the moment estimator seems to be particularly appealing to cases where $\sigma_u^2$ and $n$ are both small. In these cases, the moment estimates tend to have smaller variances. Small $\sigma_u^2$ and $n$ are known to cause difficulty in the identification of the composed error's distribution for the ML method. The Monte Carlo results indicate that the selected moment equations in (31) are less prone to problems in this scenario. When the distribution of $u_i$ is incorrectly assumed, the moment estimator also performs better than the MLE does for half-normally distributed $u_i$ and small values of $\sigma_u^2$. If $u_i$ is exponentially distributed, the MLE tends to yield smaller MSE of the estimates.

### 4.2 The Sine/Cosine Test

Table 5 reports the empirical sizes and powers of the proposed sine and cosine tests shown in Section 3.2 using $\omega = 1$, 1.5, and 2 and $(\omega_1, \omega_2) = (1, 2)$. The null hypothesis is that the composed error of the model has a normal-half normal distribution.

For the size experiment, we notice that the empirical sizes of the sine and cosine tests with various $\omega$'s systematically approach the 5% level in most of the cases as the sample size increases. The only exception is for the sine test in models with very small values of $\sigma_u^2$ $(= 0.2)$. Generally speaking, compared to the sine test, the cosine test has the empirical sizes that are more consistent across different values of the $\omega$'s (less sensitive to the choice of $\omega$) and over different sample sizes. Even in models with small $\sigma_u^2$ (e.x., 0.2) and small $n$ (e.x., 50), sizes of the cosine tests are still reasonably good. For tests with combined $\omega = (1.0,\ 2.0)$, they may appear to be slightly oversized compared to $\omega = 1.0$ and $\omega = 2.0$, although they are still within reasonable ranges.

The lower half of Table 5 shows the power experiment when the DGP is normal-exponential. Here our focus is on the cosine test because it has better size performance than the sine test. The latter tends to be substantially oversized in some cases. For the cosine test, the empirical power systematically increases with the sample size, and the power is equal or very close to 100% when $n = 10,000$. However, the power is not great when the sample size is equal to or less than 200. For a reasonably large sample size, we find that the cosine test with $\omega = 1$ performs the best.

We also report in Table 6 the empirical sizes (and powers) of the sine/cosine test for testing the normal-exponential model. From Table 6, we can see that the cosine test still has better size performance than the sine test in testing the normal-exponential model. In particular, the cosine test with various values of $\omega$ has the empirical sizes close to the 5% level for all the sample sizes. In comparison, the sine test with $\omega = 1$ is somewhat over-sized when $n$ is small.

Regarding the empirical powers of the tests, the performances shown in Table 6 are not as good as those in Table 5. When the null hypothesis is normal-exponential and the true DGP is normal-half normal, the power of the cosine tests tend to be low for smaller $\sigma_u^2$ $(\leq 1)$. For larger $\sigma_u^2$ and $n$, the power increases with the sample sizes and approaches 100%. Overall, this table indicates that the cosine test with a proper choice of $\omega$, such as $\omega = 1$, performs better.

Wang et al. (2010) also consider similar distributional tests for SF models. They propose the Pearson $\chi^2$ test based on expected and actual numbers of observations in cells defined by values of the composed error, and the Kolmogorov-Smirnov (KS) test based on the maximal difference between the empirical and theoretical cdf. They found using Monte Carlo experiments that the bootstrapped KS test performs the best in general. For a few simulation cases with the same parameter setup[1], we find that the empirical sizes of the KS test and our cosine test (with $\omega = 1.0$) are quite similar and the powers are comparable. In particular, similar to our results in Table 5 and 6, they also show a low power of the KS test, and the lack of power is particularly salient when the DGP is normal-half normal and the null is normal-exponential.

# 5 Empirical Examples

In this section, we provide two empirical examples to demonstrate the use of the moment estimator and the sine/cosine test on model distributions in real data. The first example uses data on fossil fuel-fired steam electric power generation plants in the United States. The same dataset has been used in Rungsuriyawiboon and Stefanou (2003) and Kumbhakar and Wang (2006). In the other example, a female labor dataset from Taiwan is used to estimate a wage equation. The SF model is employed for the estimation where the one-sided error represents the effect of costly job search on the observed wage rates (e.g., Polachek and Yoon, 1987; Hofler and Murphy, 1992; Polachek and Robst, 1998). The example of the female wage rate is chosen because reasonable doubts may be cast *a priori* on the model's distribution assumption for its failure to account for the sample selection bias. We will thus see whether the tests are able to reject the specification.

## 5.1 Power Plants Production

The dataset contains 791 observations on 72 fossil fuel-fired steam electric power generation plants (investor-owned utilities) in the United States for the years 1986-1996. The variables used in the model are as follows (see Rungsuriyawiboon and Stefanou (2003) for the detailed information).

The dependent variable is the net steam electric power generation in megawatt-hours. The inputs are: labor and maintenance, fuel, and capital. Quantities of the first two variables are obtained from their costs divided by the corresponding prices. The quantity of capital stock is measured using the estimates of capital cost discussed in Considine (2000). A time trend variable is also used in the model to capture technical change. The model has a translog specification using all the explanatory variables mentioned above.

Estimation results are reported in Table 7. Since individual coefficient in a translog model is not directly interpretable, we report input elasticities instead. Note that for the moment estimator, slope coefficients are obtained by OLS and so the elasticities are the same for both of the normal-half normal and the normal-exponential models. Here, we are primarily

[1] Given the configuration of ($n, \sigma_u^2$, DGP), the same configurations in the two papers include cases of $(50, 1,$ N-HN), $(50, 1,$ N-Exp), $(100, 1,$ N-HN) and $(100, 1,$ N-Exp) for the size and power of the test, and $(50, 0.5,$ N-HN) and $(100, 0.5,$ N-HN) for the power of the test. The notation N-HN indicates the normal-half normal distribution, and N-Exp indicates the normal-exponential distribution.

interested in the parameters of the error distribution. As shown in the table, estimates of the variance parameters from the moment and the ML estimators are quite different in the normal-exponential model, while the results are closer in the normal-half normal model.

Table 8 shows the sine and cosine test statistics of the model's distribution assumption. The results are clear: the null hypothesis of the normal-half normal error distribution cannot be rejected for the data, while the normal-exponential assumption is decisively rejected by all the test statistics by a large margin. The results are consistent with the fact that the moment and the ML estimators produce similar variance parameter estimates when the distribution is normal-half normal. When the distribution is correctly specified, the results from the two estimators ought to be close.

### 5.2 Female Labor Wage Rate Estimation

The dataset contains 9,509 observations on female wage rates in Taiwan for the year 2004. The data source is Taiwan's Manpower Utilization Survey (Directorate-General of Budget, Accounting and Statistics). Based on search theory, Polachek and Yoon (1987), Hofler and Murphy (1992), Polachek and Robst (1998), and others have argued that the observed wage rates fall below the maximum potential wage offers because of costly job search. The situation gives rise to a SF specification of the wage rate equation in which the one-sided $u_i$ term represents the missing rent due to costly job search.

The model contains the following variables. The dependent variable is the log of monthly earnings, and the explanatory variables include the years of education (edu), years of working experience (exp), the square of working experience (exp2), a marital status dummy variable (DM = 1 if married and 0 otherwise), and a dummy variable indicating whether the worker is in the public sector (DP).

As in the aforementioned studies, we assume a normal distribution for $v_i$ and a half-normal or an exponential distribution for $u_i$. A direct estimation of the SF model for the female wage rate, however, is likely to suffer from the sample selection bias which is a well known issue in the labor literature. In the context of a linear wage equation, the assumption of a symmetric zero-mean error for such a wage equation is erroneous, and a selectivity equation or the Heckman correction is often used to rectify the problem. In the current context of a SF model, the same issue persists, making the normality assumption on $v_i$ particularly suspicious. Without proper treatment, the assumption of a normal-half normal or a normal-exponential error is likely to be faulty.

Table 9 shows the estimated coefficients and variance parameters. For the moment estimator, the coefficients are based on OLS estimates and are invariant to distribution assumptions. The only exception is the intercept which is adjusted according to (4). The test statistics are presented in Table 10, which shows that both of the normal-half normal and the normal-exponential distribution assumptions are overwhelmingly rejected by all the test statistics. The results are consistent with our prior expectation that the wage equation's error distribution is erroneous because of the sample selection problem.

## 6 Conclusion

The composed error of a stochastic frontier model consists of two random variables, and the identification of the model relies heavily on the distribution assumptions of the variables. Evidently, the correct distribution assumptions on both of the error components are crucial for the empirical analysis. In this paper, we propose a centered-residuals-based method of moments in which we may estimate the model parameters as well as testing the distribution assumptions on both of the error components. Algebraic formulas of the estimator and the test statistics for the normal-half normal and the normal-exponential models are fully worked out in the paper. Formulas for other distribution assumptions may be similarly derived.

Monte Carlo simulations show that the size of the cosine test performs quite well, and the power is also satisfactory for large samples. We use two empirical examples for the purpose of illustration. For the example of power plants in the U.S., the normal-exponential assumption is rejected while the normal-half normal assumption is not. As for the example of the female wage equation, both of the normal-half normal and the normal-exponential assumptions are rejected. The latter result is consistent with our expectation since the wage equation is known to suffer from sample selection bias, rendering the normal assumption regarding $v_i$ erroneous.

## Appendix: Derivation of (10)

By the definition of the residual:

$$\hat{\epsilon}_i := y_i - \hat{\alpha}_n - f(x_i, \hat{\beta}_n),$$

we can rewrite the centered residual as:

$$\begin{aligned}\hat{\epsilon}_{c,i} &= \left(y_i - \frac{1}{n}\sum_{i=1}^{n} y_i\right) - \left(f(x_i, \hat{\beta}_n) - \frac{1}{n}\sum_{i=1}^{n} f(x_i, \hat{\beta}_n)\right), \\ &= (\epsilon_{oi} - \bar{\epsilon}_{o,n}) + \left(f(x_i, \beta_o) - \bar{f}_n(\beta_o)\right) - \left(f(x_i, \hat{\beta}_n) - \bar{f}_n(\hat{\beta}_n)\right),\end{aligned}$$

with $\epsilon_{oi} := \epsilon_i|_{\beta=\beta_o}$, $\bar{\epsilon}_{o,n} := n^{-1}\sum_{i=1}^{n} \epsilon_{oi}$, and $\bar{f}_n(\beta) := n^{-1}\sum_{i=1}^{n} f(x_i, \beta)$; the second equality is due to Assumption A.2. Using the relationship:

$$\epsilon_{oi} - \bar{\epsilon}_{o,n} = \epsilon_{c,oi} - (\bar{\epsilon}_{o,n} - \mathbb{E}[\epsilon_{oi}]),$$

we can further re-express the above result as:

$$\hat{\epsilon}_{c,i} = \epsilon_{c,oi} - (\bar{\epsilon}_{o,n} - \mathbb{E}[\epsilon_{oi}]) - \left(f(x_i, \hat{\beta}_n) - \bar{f}_n(\hat{\beta}_n)\right) + \left(f(x_i, \beta_o) - \bar{f}_n(\beta_o)\right). \tag{A1}$$

Given (A1), we follow a standard first-order asymptotic method to derive (10); see, e.g., Newey and McFadden (1994) for this method. Denote the derivatives:

$$\xi_{\epsilon_c,ij} := \frac{\partial}{\partial \epsilon_{c,i}} \phi_j(\epsilon_{c,i}, \theta),$$

$$\xi_{\beta,ij} := \xi_{\epsilon_c,ij} \nabla_{\beta^\top} \left(f(x_i, \beta) - \bar{f}_n(\beta)\right),$$

and

$$\xi_{\theta,ij} := \nabla_{\theta^\top} \phi_j\left(\epsilon_{c,i}, \theta\right),$$

and their sample averages: $\bar{\xi}_{\epsilon_c,jn} := n^{-1}\sum_{i=1}^{n} \xi_{\epsilon_c,ij}$, $\bar{\xi}_{\beta,jn} := n^{-1}\sum_{i=1}^{n} \xi_{\beta,ij}$, and $\bar{\xi}_{\theta,jn} := n^{-1}\sum_{i=1}^{n} \xi_{\theta,ij}$. Given (A1) and Assumption **A.3**(a), we can take the mean-value expansion of $n^{-1/2}\sum_{i=1}^{n} \phi_j(\hat{\epsilon}_{c,i}, \hat{\theta}_n)$ with respect to $\bar{\epsilon}_{o,n}$, $\beta_o$, and $\theta_o$, and obtain the result:

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \phi_j(\hat{\epsilon}_{c,i}, \hat{\theta}_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \phi_j(\epsilon_{c,oi}, \theta_o) - \bar{\xi}_{\epsilon_c,jn}|_{\delta=\delta_n^*} \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \epsilon_{c,oi} \\ - \bar{\xi}_{\beta,jn}|_{\delta=\delta_n^*} \sqrt{n}(\hat{\beta}_n - \beta_o) + \bar{\xi}_{\theta,jn}|_{\delta=\delta_n^*} \sqrt{n}(\hat{\theta}_n - \theta_o),\end{aligned} \tag{A2}$$

where $\delta_n^*$ is a mean value between $\hat{\delta}_n$ and $\delta_o$ such that $\|\delta_n^* - \delta_o\| \leq \|\hat{\delta}_n - \delta_o\|$. Using the triangle inequality, we can write that

$$\|\bar{\xi}_{\epsilon_c,jn}|_{\delta=\delta_n^*} - \mathbb{E}[\xi_{\epsilon_c,ij}]_{\delta=\delta_o}\| \leq \|\bar{\xi}_{\epsilon_c,jn} - \mathbb{E}[\xi_{\epsilon_c,ij}]\|_{\delta=\delta_n^*} + \|\mathbb{E}[\xi_{\epsilon_c,ij}]_{\delta=\delta_n^*} - \mathbb{E}[\xi_{\epsilon_c,ij}]_{\delta=\delta_o}\|.$$

Note that the UWLLN in **A.3**(c) implies that $\sup_{\delta\in\Delta}\|\bar{\xi}_{\epsilon_c,jn} - \mathbb{E}[\xi_{\epsilon_c,ij}]\| \xrightarrow{\text{p}} 0$ and hence $\|\bar{\xi}_{\epsilon_c,jn} - \mathbb{E}[\xi_{\epsilon_c,ij}]\|_{\delta=\delta_n^*} \xrightarrow{\text{p}} 0$. In addition, $\hat{\delta}_n$ is consistent for $\delta_o$, and **A.3**(a) implies that $\xi_{\epsilon_c,ij}$ is continuous at $\delta_o$; thus, Slutsky's theorem implies that $\mathbb{E}[\xi_{\epsilon_c,ij}]_{\delta=\delta_n^*} \xrightarrow{\text{p}} \mathbb{E}[\xi_{\epsilon_c,ij}]_{\delta=\delta_o}$. Consequently, we have

$$\bar{\xi}_{\epsilon_c,jn}|_{\delta=\delta_n^*} \xrightarrow{\text{p}} \mathbb{E}[\xi_{\epsilon_c,ij}]_{\delta=\delta_o} := \mathbb{E}\left[\frac{\partial}{\partial\epsilon_{c,i}}\phi_j(\epsilon_{c,oi},\theta_o)\right]. \tag{A3}$$

Similarly, it is also standard to show that

$$\bar{\xi}_{\beta,jn}|_{\delta=\delta_n^*} \xrightarrow{\text{p}} \mathbb{E}[\xi_{\beta,ij}]_{\delta=\delta_o} = \mathbb{E}\left[\frac{\partial}{\partial\epsilon_{c,i}}\phi_j(\epsilon_{c,oi},\theta_o)\right] \times \mathbb{E}\left[\nabla_{\beta^\intercal} f(x_i,\beta_o) - \mathbb{E}[\nabla_{\beta^\intercal} f(x_i,\beta_o)]\right] = 0, \tag{A4}$$

where the first equality is due to the independence between $\epsilon_{c,i}$ and $\mathcal{X}_i$ under Assumption **A.1**, and

$$\bar{\xi}_{\theta,jn}|_{\delta=\delta_n^*} \xrightarrow{\text{p}} \mathbb{E}[\xi_{\theta,j}]_{\delta=\delta_o} = \mathbb{E}[\nabla_{\theta^\intercal}\phi_j(\epsilon_{c,oi},\theta_o)]. \tag{A5}$$

The result in (10) is derived by introducing (A3), (A4), (A5) into (A2). $\square$

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Table 1: PDFs and Arithmetic Moments of Various Distributions

| distribution | PDF | the $k$-th moment |
|---|---|---|
| normal | $g_v(v,\theta_v) = \frac{1}{\sqrt{2\pi}\sigma_v}\exp\left(-\frac{v^2}{2\sigma_v^2}\right)$ | $\nu_k = \sigma_v^k \prod_{j=1}^{k/2}(2j-1)$, if $k$ is even |
| half-normal | $g_u(u,\theta_u) = \frac{2}{\sqrt{2\pi}\sigma_u}\exp\left(-\frac{u^2}{2\sigma_u^2}\right)$ | $\mu_k = 2^{k/2}(\sigma_u^k/\sqrt{\pi})\Gamma\left(\frac{k+1}{2}\right), \quad k=1,2,3,\ldots$ |
| exponential | $g_u(u,\theta_u) = \frac{1}{\sigma_u}\exp\left(-\frac{u}{\sigma_u}\right)$ | $\mu_k = \sigma_u^k k!, \quad k=1,2,3,\ldots$ |
| Gamma | $g_u(u,\theta_u) = \frac{1}{\Gamma(m)\sigma_u^m}\exp\left(-\frac{u}{\sigma_u}\right)u^{m-1}$ | $\mu_k = \sigma_u^k \frac{\Gamma(m+k)}{\Gamma(m)}, \quad k=1,2,3,\ldots$ |
| truncated-normal | $g_u(u,\theta_u) = \frac{1}{\sqrt{2\pi}\sigma_u}\exp\left(-\frac{(u-m)^2}{2\sigma_u^2}\right)\Big/\Phi\left(\frac{m}{\sigma_u}\right)$ | $\mu_{k+1} = k\sigma_u^2\mu_{k-1} + m\mu_k, \quad k=1,2,3,\ldots$ |

Notes: The parameters $\sigma_v$, $\sigma_u$, and $m$ are all positive. For the normal distribution, $\theta_v = \sigma_v^2$ and $\nu_k = 0$ if $k$ is an odd number. For the half-normal and exponential distributions, $\theta_u = \sigma_u^2$. For the Gamma and truncated-normal distributions, $\theta_u = (\sigma_u^2, m)^\top$ for some $m \geq 0$. The arithmetic moments of the exponential, gamma, and normal distributions can be found in Balakrishnan and Nevzorov (2003, p.159, p.181, and p.215); the case of the half-normal distribution is due to Elandt (1961, Equation 15), subject to a simple transformation. For the truncated-normal distribution, $\mu_0 := 1$, and $\mu_1 = m + \sigma_u\phi(-m/\sigma_u)/\Phi(m/\sigma_u)$, with $\phi(\cdot)$ denoting the PDF and $\Phi(\cdot)$ denoting the distribution function of $N(0,1)$; the formula of $\mu_{k+1}$ follows Equations (4.13)-(4.14) of Lee (1983).

Table 2: Characteristic Functions of Various Distributions

| distribution | real part | imaginary part |
|---|---|---|
| normal | $\exp\left(-\frac{1}{2}(\omega\sigma_v)^2\right)$ | 0 |
| half-normal | $\exp\left(-\frac{1}{2}(\omega\sigma_u)^2\right)$ | $\exp\left(-\frac{1}{2}(\omega\sigma_u)^2\right)\operatorname{erfi}\left(\frac{1}{\sqrt{2}}\omega\sigma_u\right)$ |
| exponential | $\left(1+(\omega\sigma_u)^2\right)^{-1}$ | $\left(1+(\omega\sigma_u)^2\right)^{-1}\omega\sigma_u$ |
| Gamma | $\left(1+(\omega\sigma_u)^2\right)^{-m/2}\cos(m\arctan(\omega\sigma_u))$ | $\left(1+(\omega\sigma_u)^2\right)^{-m/2}\sin(m\arctan(\omega\sigma_u))$ |
| truncated-normal | $\exp\left(-\frac{1}{2}(\omega\sigma_u)^2\right)\left(2\Phi(\frac{m}{\sigma_u})\right)^{-1}$ $\times\Big\{\exp(\mathrm{i}\omega m)\Phi\left(\frac{m}{\sigma_u}+\mathrm{i}\omega\sigma_u\right)$ $+\exp(-\mathrm{i}\omega m)\Phi\left(\frac{m}{\sigma_u}-\mathrm{i}\omega\sigma_u\right)\Big\}$ | $-\mathrm{i}\exp\left(-\frac{1}{2}(\omega\sigma_u)^2\right)\left(2\Phi(\frac{m}{\sigma_u})\right)^{-1}$ $\times\Big\{\exp(\mathrm{i}\omega m)\Phi\left(\frac{m}{\sigma_u}+\mathrm{i}\omega\sigma_u\right)$ $-\exp(-\mathrm{i}\omega m)\Phi\left(\frac{m}{\sigma_u}-\mathrm{i}\omega\sigma_u\right)\Big\}$ |

Notes: Given a characteristic function, the real part is $\mathbf{E}[\cos(\omega v_i)]$ for $v_i$ and $\mathbf{E}[\cos(\omega u_i)]$ for $u_i$; the imaginary part is $\mathbf{E}[\sin(\omega v_i)]$ for $v_i$ and $\mathbf{E}[\sin(\omega u_i)]$ for $u_i$; $\omega \in \mathbb{R}$. For the normal distribution, its characteristic function is well known. For the half-normal, exponential, Gamma, and truncated-normal distributions, the real (imaginary) parts are, respectively, rewritten from Formula #73 on page 24, #60 on page 23, #65 on page 23, #79 on page 24 (Formula #73 on page 80, #60 on page 79, #65 on page 79, #79 on page 80) of Oberhettinger (1973). Recall that $\Phi(\cdot)$ is the distribution function of $N(0,1)$, and denote the error function $\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\mathrm{d}t$, the complementary error function $\operatorname{erfc}(x) := 1 - \operatorname{erf}(x)$, and the imaginary error function $\operatorname{erfi}(x) := -\mathrm{i}\times\operatorname{erf}(\mathrm{i}x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(t^2)\mathrm{d}t$. In showing the imaginary part of the half-normal distribution's characteristic function, we use the definition of $\operatorname{erfi}(x)$. In showing the characteristic function of the truncated-normal distribution, we use the identity: $\operatorname{erfc}(x) = 2\Phi(-x\sqrt{2})$.

Table 3: The Mean and Variance (MSE) of the Moment Estimator

| | | Normal-Half Normal | | | | Normal-Exponential | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | mean | | Var/MSE | | mean | | Var/MSE | |
| DGP | $n$ | $\hat{\sigma}_v^2$ | $\hat{\sigma}_u^2$ | $\hat{\sigma}_v^2$ | $\hat{\sigma}_u^2$ | $\hat{\sigma}_v^2$ | $\hat{\sigma}_u^2$ | $\hat{\sigma}_v^2$ | $\hat{\sigma}_u^2$ |
| | 50 | 0.614 | 1.029 | 0.061 | 0.392 | 0.753 | 0.235 | 0.103 | 0.022 |
| normal- | 100 | 0.712 | 0.874 | 0.042 | 0.252 | 0.831 | 0.199 | 0.055 | 0.013 |
| half-normal | 200 | 0.788 | 0.725 | 0.026 | 0.160 | 0.886 | 0.165 | 0.029 | 0.010 |
| $\sigma_v^2 = 1$ | 1,000 | 0.910 | 0.436 | 0.009 | 0.056 | 0.969 | 0.099 | 0.006 | 0.013 |
| $\sigma_u^2 = 0.2$ | 5,000 | 0.973 | 0.273 | 0.003 | 0.021 | 1.010 | 0.062 | 0.002 | 0.020 |
| | 10,000 | 0.988 | 0.232 | 0.002 | 0.015 | 1.019 | 0.053 | 0.001 | 0.022 |
| | 50 | 0.673 | 1.143 | 0.075 | 0.495 | 0.828 | 0.261 | 0.081 | 0.083 |
| normal- | 100 | 0.778 | 0.982 | 0.052 | 0.319 | 0.910 | 0.224 | 0.041 | 0.093 |
| half-normal | 200 | 0.859 | 0.825 | 0.033 | 0.212 | 0.970 | 0.188 | 0.020 | 0.108 |
| $\sigma_v^2 = 1$ | 1,000 | 0.972 | 0.562 | 0.013 | 0.086 | 1.048 | 0.128 | 0.009 | 0.143 |
| $\sigma_u^2 = 0.5$ | 5,000 | 1.004 | 0.488 | 0.005 | 0.036 | 1.070 | 0.111 | 0.007 | 0.153 |
| | 10,000 | 1.003 | 0.490 | 0.003 | 0.020 | 1.070 | 0.112 | 0.006 | 0.152 |
| | 50 | 0.758 | 1.370 | 0.103 | 0.733 | 0.943 | 0.313 | 0.072 | 0.511 |
| normal- | 100 | 0.867 | 1.217 | 0.072 | 0.496 | 1.031 | 0.278 | 0.045 | 0.548 |
| half-normal | 200 | 0.943 | 1.081 | 0.054 | 0.372 | 1.090 | 0.247 | 0.035 | 0.587 |
| $\sigma_v^2 = 1$ | 1,000 | 1.007 | 0.963 | 0.020 | 0.149 | 1.137 | 0.220 | 0.028 | 0.616 |
| $\sigma_u^2 = 1$ | 5,000 | 1.002 | 0.991 | 0.004 | 0.032 | 1.136 | 0.226 | 0.021 | 0.601 |
| | 10,000 | 1.001 | 0.996 | 0.002 | 0.016 | 1.136 | 0.227 | 0.019 | 0.598 |
| | 50 | 0.985 | 2.588 | 0.269 | 2.667 | 1.335 | 0.591 | 0.265 | 5.944 |
| normal- | 100 | 1.044 | 2.648 | 0.196 | 1.906 | 1.402 | 0.604 | 0.261 | 5.839 |
| half-normal | 200 | 1.041 | 2.771 | 0.126 | 1.288 | 1.415 | 0.632 | 0.230 | 5.670 |
| $\sigma_v^2 = 1$ | 1,000 | 1.009 | 2.948 | 0.025 | 0.255 | 1.407 | 0.673 | 0.178 | 5.430 |
| $\sigma_u^2 = 3$ | 5,000 | 1.002 | 2.991 | 0.005 | 0.051 | 1.406 | 0.683 | 0.167 | 5.373 |
| | 10,000 | 1.000 | 2.997 | 0.002 | 0.025 | 1.405 | 0.684 | 0.166 | 5.366 |
| | 50 | 1.179 | 4.820 | 0.642 | 7.932 | 1.829 | 1.100 | 1.011 | 24.425 |
| normal- | 100 | 1.137 | 5.273 | 0.409 | 5.012 | 1.850 | 1.203 | 0.910 | 23.270 |
| half-normal | 200 | 1.071 | 5.625 | 0.226 | 2.935 | 1.831 | 1.284 | 0.788 | 22.389 |
| $\sigma_v^2 = 1$ | 1,000 | 1.015 | 5.916 | 0.045 | 0.571 | 1.814 | 1.350 | 0.683 | 21.653 |
| $\sigma_u^2 = 6$ | 5,000 | 1.003 | 5.986 | 0.009 | 0.115 | 1.812 | 1.366 | 0.663 | 21.481 |
| | 10,000 | 1.000 | 5.995 | 0.004 | 0.057 | 1.811 | 1.368 | 0.659 | 21.459 |
| | 50 | 0.596 | 3.428 | 0.607 | 12.757 | 1.060 | 0.782 | 0.158 | 0.341 |
| normal- | 100 | 0.542 | 3.783 | 0.517 | 12.107 | 1.053 | 0.863 | 0.111 | 0.254 |
| exponential | 200 | 0.492 | 4.036 | 0.460 | 12.007 | 1.038 | 0.921 | 0.068 | 0.151 |
| $\sigma_v^2 = 1$ | 1,000 | 0.429 | 4.296 | 0.373 | 11.481 | 1.009 | 0.980 | 0.015 | 0.033 |
| $\sigma_u^2 = 1$ | 5,000 | 0.412 | 4.367 | 0.356 | 11.466 | 1.002 | 0.996 | 0.003 | 0.007 |
| | 10,000 | 0.409 | 4.376 | 0.354 | 11.461 | 1.001 | 0.998 | 0.002 | 0.003 |
| | 50 | 0.518 | 4.408 | 0.889 | 20.549 | 1.113 | 1.006 | 0.206 | 0.541 |
| normal- | 100 | 0.412 | 4.931 | 0.785 | 19.944 | 1.078 | 1.125 | 0.147 | 0.393 |
| exponential | 200 | 0.337 | 5.272 | 0.726 | 20.001 | 1.050 | 1.203 | 0.089 | 0.229 |
| $\sigma_v^2 = 1$ | 1,000 | 0.257 | 5.589 | 0.620 | 19.347 | 1.012 | 1.275 | 0.021 | 0.050 |
| $\sigma_u^2 = 1.3$ | 5,000 | 0.235 | 5.677 | 0.560 | 19.364 | 1.003 | 1.296 | 0.004 | 0.010 |
| | 10,000 | 0.232 | 5.688 | 0.597 | 19.358 | 1.001 | 1.298 | 0.002 | 0.005 |
| | 50 | 0.430 | 5.411 | 1.230 | 30.290 | 1.162 | 1.235 | 0.261 | 0.783 |
| normal- | 100 | 0.278 | 6.088 | 1.116 | 29.799 | 1.101 | 1.389 | 0.190 | 0.561 |
| exponential | 200 | 0.182 | 6.505 | 1.059 | 30.059 | 1.062 | 1.484 | 0.117 | 0.326 |
| $\sigma_v^2 = 1$ | 1,000 | 0.085 | 6.882 | 0.931 | 29.259 | 1.015 | 1.570 | 0.028 | 0.072 |
| $\sigma_u^2 = 1.6$ | 5,000 | 0.585 | 6.988 | 0.907 | 29.320 | 1.003 | 1.595 | 0.006 | 0.015 |
| | 10,000 | 0.055 | 7.001 | 0.904 | 29.314 | 1.001 | 1.598 | 0.003 | 0.008 |

Note: The first column indicates the true DGP, and results for the normal-half normal and the normal-exponential estimators are reported in the table. The var/MSE column reports the sample variance (MSE) of the estimator in this simulation when the model is correctly (incorrectly) specified.

Table 4: The Mean and Variance (MSE) of the MLE

| DGP | $n$ | Normal-Half Normal mean $\hat{\sigma}_v^2$ | Normal-Half Normal mean $\hat{\sigma}_u^2$ | Normal-Half Normal Var/MSE $\hat{\sigma}_v^2$ | Normal-Half Normal Var/MSE $\hat{\sigma}_u^2$ | Normal-Exponential mean $\hat{\sigma}_v^2$ | Normal-Exponential mean $\hat{\sigma}_u^2$ | Normal-Exponential Var/MSE $\hat{\sigma}_v^2$ | Normal-Exponential Var/MSE $\hat{\sigma}_u^2$ |
|---|---|---|---|---|---|---|---|---|---|
| | 50 | 0.768 | 0.624 | 0.112 | 0.717 | 0.839 | 0.165 | 0.107 | 0.070 |
| normal- | 100 | 0.840 | 0.539 | 0.073 | 0.475 | 0.919 | 0.119 | 0.049 | 0.035 |
| half-normal | 200 | 0.906 | 0.405 | 0.041 | 0.252 | 0.963 | 0.091 | 0.024 | 0.025 |
| $\sigma_v^2 = 1$ | 1,000 | 0.971 | 0.269 | 0.013 | 0.088 | 1.006 | 0.063 | 0.006 | 0.023 |
| $\sigma_u^2 = 0.2$ | 5,000 | 0.996 | 0.208 | 0.005 | 0.035 | 1.019 | 0.053 | 0.002 | 0.023 |
| | 10,000 | 1.001 | 0.195 | 0.003 | 0.023 | 1.022 | 0.050 | 0.002 | 0.023 |
| | 50 | 0.814 | 0.775 | 0.138 | 0.950 | 0.902 | 0.207 | 0.110 | 0.179 |
| normal- | 100 | 0.886 | 0.705 | 0.094 | 0.653 | 0.985 | 0.160 | 0.054 | 0.156 |
| half-normal | 200 | 0.953 | 0.570 | 0.054 | 0.360 | 1.033 | 0.129 | 0.030 | 0.158 |
| $\sigma_v^2 = 1$ | 1,000 | 1.005 | 0.474 | 0.019 | 0.136 | 1.073 | 0.106 | 0.014 | 0.162 |
| $\sigma_u^2 = 0.5$ | 5,000 | 1.006 | 0.481 | 0.006 | 0.042 | 1.078 | 0.103 | 0.008 | 0.159 |
| | 10,000 | 1.004 | 0.488 | 0.003 | 0.022 | 1.077 | 0.104 | 0.007 | 0.158 |
| | 50 | 0.880 | 1.052 | 0.184 | 1.382 | 0.944 | 0.352 | 0.143 | 0.586 |
| normal- | 100 | 0.934 | 1.056 | 0.131 | 1.014 | 1.040 | 0.291 | 0.077 | 0.576 |
| half-normal | 200 | 0.994 | 0.950 | 0.078 | 0.581 | 1.099 | 0.247 | 0.050 | 0.601 |
| $\sigma_v^2 = 1$ | 1,000 | 1.014 | 0.948 | 0.023 | 0.175 | 1.148 | 0.212 | 0.032 | 0.630 |
| $\sigma_u^2 = 1$ | 5,000 | 1.002 | 0.993 | 0.004 | 0.032 | 1.148 | 0.215 | 0.024 | 0.618 |
| | 10,000 | 1.001 | 0.996 | 0.002 | 0.016 | 1.148 | 0.216 | 0.023 | 0.616 |
| | 50 | 1.011 | 2.520 | 0.381 | 3.851 | 1.201 | 0.823 | 0.364 | 5.296 |
| normal- | 100 | 0.983 | 2.871 | 0.252 | 2.617 | 1.302 | 0.766 | 0.279 | 5.275 |
| half-normal | 200 | 1.003 | 2.896 | 0.127 | 1.291 | 1.356 | 0.726 | 0.226 | 5.307 |
| $\sigma_v^2 = 1$ | 1,000 | 1.003 | 2.971 | 0.021 | 0.215 | 1.365 | 0.737 | 0.152 | 5.148 |
| $\sigma_u^2 = 3$ | 5,000 | 0.999 | 2.999 | 0.004 | 0.041 | 1.363 | 0.745 | 0.135 | 5.089 |
| | 10,000 | 1.000 | 2.998 | 0.002 | 0.021 | 1.363 | 0.745 | 0.134 | 5.088 |
| | 50 | 1.124 | 4.906 | 0.649 | 8.309 | 1.459 | 1.692 | 0.897 | 20.188 |
| normal- | 100 | 0.968 | 5.835 | 0.340 | 4.887 | 1.535 | 1.688 | 0.682 | 19.460 |
| half-normal | 200 | 0.977 | 5.931 | 0.145 | 2.196 | 1.569 | 1.672 | 0.511 | 19.129 |
| $\sigma_v^2 = 1$ | 1,000 | 0.998 | 5.974 | 0.024 | 0.375 | 1.582 | 1.684 | 0.372 | 18.695 |
| $\sigma_u^2 = 6$ | 5,000 | 0.998 | 5.999 | 0.005 | 0.073 | 1.581 | 1.695 | 0.343 | 18.544 |
| | 10,000 | 0.999 | 5.998 | 0.002 | 0.038 | 1.581 | 1.695 | 0.341 | 18.544 |
| | 50 | 0.769 | 2.836 | 0.297 | 6.828 | 0.920 | 0.998 | 0.221 | 0.549 |
| normal- | 100 | 0.707 | 3.315 | 0.220 | 7.470 | 0.967 | 0.995 | 0.116 | 0.278 |
| exponential | 200 | 0.719 | 3.360 | 0.138 | 6.546 | 0.992 | 0.982 | 0.054 | 0.126 |
| $\sigma_v^2 = 1$ | 1,000 | 0.730 | 3.402 | 0.083 | 5.938 | 0.999 | 0.994 | 0.009 | 0.023 |
| $\sigma_u^2 = 1$ | 5,000 | 0.729 | 3.421 | 0.075 | 5.896 | 0.999 | 1.000 | 0.002 | 0.004 |
| | 10,000 | 0.730 | 3.420 | 0.074 | 5.872 | 1.000 | 0.999 | 0.001 | 0.002 |
| | 50 | 0.751 | 3.553 | 0.318 | 10.805 | 0.929 | 1.269 | 0.251 | 0.746 |
| normal- | 100 | 0.669 | 4.169 | 0.239 | 12.592 | 0.965 | 1.293 | 0.129 | 0.382 |
| exponential | 200 | 0.685 | 4.217 | 0.155 | 11.528 | 0.987 | 1.286 | 0.057 | 0.170 |
| $\sigma_v^2 = 1$ | 1,000 | 0.702 | 4.246 | 0.098 | 10.745 | 0.999 | 1.293 | 0.010 | 0.031 |
| $\sigma_u^2 = 1.3$ | 5,000 | 0.703 | 4.265 | 0.090 | 10.700 | 0.999 | 1.300 | 0.002 | 0.006 |
| | 10,000 | 0.704 | 4.263 | 0.089 | 10.667 | 1.000 | 1.299 | 0.001 | 0.003 |
| | 50 | 0.738 | 4.240 | 0.334 | 15.663 | 0.932 | 1.553 | 0.278 | 0.962 |
| normal- | 100 | 0.639 | 4.991 | 0.253 | 18.929 | 0.960 | 1.596 | 0.139 | 0.496 |
| exponential | 200 | 0.659 | 5.039 | 0.170 | 17.727 | 0.983 | 1.589 | 0.060 | 0.222 |
| $\sigma_v^2 = 1$ | 1,000 | 0.681 | 5.060 | 0.111 | 16.744 | 0.998 | 1.593 | 0.011 | 0.040 |
| $\sigma_u^2 = 1.6$ | 5,000 | 0.683 | 5.079 | 0.103 | 16.689 | 0.999 | 1.600 | 0.002 | 0.008 |
| | 10,000 | 0.683 | 5.077 | 0.101 | 16.646 | 1.000 | 1.599 | 0.001 | 0.004 |

Note: The first column indicates the true DGP, and results for the normal-half normal and the normal-exponential estimators are reported in the table. The var/MSE column reports the sample variance (MSE) of the estimator in this simulation when the model is correctly (incorrectly) specified.

Table 5: The Sine and Cosine Tests for the Normal-Half Normal Model

| | | The Sine Test | | | | The Cosine Test | | | |
|---|---|---|---|---|---|---|---|---|---|
| DGP | $n$ | $\omega$=1.0 | $\omega$=1.5 | $\omega$=2.0 | $\omega$=(1.0, 2.0) | $\omega$=1.0 | $\omega$=1.5 | $\omega$=2.0 | $\omega$=(1.0, 2.0) |
| | 50 | 42.79 | 33.22 | 19.76 | 46.93 | 7.10 | 6.96 | 6.13 | 4.05 |
| normal- | 100 | 42.42 | 31.89 | 18.55 | 46.38 | 8.66 | 7.24 | 5.74 | 6.15 |
| half-normal | 200 | 41.44 | 31.22 | 18.06 | 45.01 | 8.37 | 6.60 | 5.13 | 9.26 |
| $\sigma_v^2 = 1$ | 1,000 | 35.48 | 26.10 | 15.08 | 38.53 | 6.49 | 5.58 | 5.08 | 9.71 |
| $\sigma_u^2 = 0.2$ | 5,000 | 26.43 | 18.42 | 11.07 | 28.55 | 5.74 | 5.30 | 4.90 | 6.53 |
| | 10,000 | 20.94 | 14.65 | 8.77 | 23.45 | 5.37 | 5.00 | 4.97 | 5.76 |
| | 50 | 38.36 | 27.69 | 15.84 | 42.70 | 6.85 | 6.68 | 5.86 | 3.88 |
| normal- | 100 | 35.65 | 25.44 | 14.37 | 39.39 | 8.50 | 7.06 | 5.74 | 6.02 |
| half-normal | 200 | 32.48 | 23.14 | 12.48 | 35.64 | 8.11 | 6.27 | 4.97 | 9.02 |
| $\sigma_v^2 = 1$ | 1,000 | 20.00 | 13.63 | 8.45 | 22.09 | 6.48 | 5.55 | 5.02 | 8.93 |
| $\sigma_u^2 = 0.5$ | 5,000 | 7.60 | 6.30 | 5.66 | 8.23 | 5.61 | 5.26 | 5.13 | 6.33 |
| | 10,000 | 4.72 | 4.90 | 5.01 | 5.22 | 5.27 | 5.04 | 4.69 | 5.64 |
| | 50 | 30.93 | 20.94 | 11.63 | 34.89 | 6.67 | 6.45 | 5.23 | 3.75 |
| normal- | 100 | 25.46 | 17.33 | 9.57 | 28.85 | 7.84 | 6.44 | 5.14 | 5.73 |
| half-normal | 200 | 19.46 | 12.70 | 7.56 | 21.73 | 7.43 | 5.85 | 5.00 | 8.56 |
| $\sigma_v^2 = 1$ | 1,000 | 7.78 | 6.14 | 5.34 | 9.04 | 6.08 | 5.52 | 5.30 | 7.84 |
| $\sigma_u^2 = 1$ | 5,000 | 5.48 | 5.07 | 5.39 | 6.99 | 5.35 | 5.10 | 5.46 | 5.70 |
| | 10,000 | 5.00 | 4.66 | 4.53 | 5.82 | 5.38 | 5.17 | 4.72 | 5.66 |
| | 50 | 12.51 | 8.15 | 6.17 | 14.97 | 6.23 | 5.41 | 4.34 | 3.47 |
| normal- | 100 | 7.75 | 5.94 | 5.03 | 8.12 | 6.75 | 5.07 | 4.59 | 6.04 |
| half-normal | 200 | 6.27 | 5.37 | 5.54 | 5.45 | 6.20 | 4.55 | 4.61 | 6.32 |
| $\sigma_v^2 = 1$ | 1,000 | 8.13 | 5.43 | 5.08 | 8.77 | 5.39 | 4.98 | 5.05 | 5.67 |
| $\sigma_u^2 = 3$ | 5,000 | 6.17 | 5.33 | 5.52 | 6.79 | 5.18 | 4.96 | 4.70 | 5.06 |
| | 10,000 | 5.56 | 4.81 | 4.76 | 5.60 | 5.11 | 4.72 | 4.87 | 5.38 |
| | 50 | 5.72 | 5.61 | 4.65 | 6.94 | 5.86 | 4.44 | 3.74 | 4.23 |
| normal- | 100 | 5.46 | 5.19 | 4.85 | 4.78 | 5.84 | 4.47 | 4.26 | 5.32 |
| half-normal | 200 | 6.99 | 5.48 | 4.71 | 6.20 | 5.06 | 4.45 | 4.29 | 4.92 |
| $\sigma_v^2 = 1$ | 1,000 | 7.25 | 4.89 | 5.10 | 7.28 | 4.63 | 4.53 | 4.58 | 4.85 |
| $\sigma_u^2 = 6$ | 5,000 | 5.37 | 5.16 | 5.30 | 5.67 | 5.01 | 4.98 | 4.74 | 5.06 |
| | 10,000 | 5.09 | 4.66 | 4.72 | 5.07 | 5.01 | 5.07 | 4.89 | 4.90 |
| | 50 | 9.21 | 7.17 | 5.80 | 13.11 | 4.10 | 4.03 | 3.53 | 3.69 |
| normal- | 100 | 4.89 | 6.14 | 5.69 | 7.87 | 4.69 | 4.46 | 4.11 | 5.35 |
| exponential | 200 | 4.06 | 7.11 | 7.19 | 7.50 | 8.20 | 7.30 | 4.79 | 8.44 |
| $\sigma_v^2 = 1$ | 1,000 | 28.03 | 38.04 | 25.85 | 19.67 | 63.74 | 44.51 | 24.47 | 51.21 |
| $\sigma_u^2 = 1$ | 5,000 | 98.87 | 99.11 | 89.24 | 96.37 | 99.98 | 99.63 | 96.46 | 99.98 |
| | 10,000 | 99.95 | 99.97 | 99.62 | 99.91 | 100.00 | 99.98 | 99.93 | 100.00 |
| | 50 | 6.79 | 6.13 | 4.82 | 10.95 | 3.91 | 3.96 | 3.34 | 4.51 |
| normal- | 100 | 4.35 | 6.44 | 4.71 | 8.85 | 4.90 | 4.67 | 3.69 | 5.97 |
| exponential | 200 | 4.67 | 9.69 | 5.43 | 10.57 | 10.20 | 7.56 | 4.19 | 9.57 |
| $\sigma_v^2 = 1$ | 1,000 | 41.32 | 49.06 | 21.59 | 32.75 | 77.21 | 54.71 | 29.00 | 65.68 |
| $\sigma_u^2 = 1.3$ | 5,000 | 99.64 | 99.68 | 87.69 | 99.00 | 100.00 | 99.85 | 98.73 | 100.00 |
| | 10,000 | 99.98 | 99.97 | 99.46 | 99.99 | 100.00 | 99.96 | 99.93 | 100.00 |
| | 50 | 5.94 | 5.29 | 4.20 | 10.40 | 3.77 | 3.72 | 3.06 | 5.91 |
| normal- | 100 | 4.12 | 6.40 | 3.78 | 10.15 | 5.03 | 4.57 | 3.24 | 6.69 |
| exponential | 200 | 5.65 | 11.04 | 3.90 | 14.66 | 11.60 | 7.39 | 3.68 | 10.46 |
| $\sigma_v^2 = 1$ | 1,000 | 51.89 | 56.63 | 7.36 | 46.25 | 83.85 | 59.10 | 31.56 | 74.83 |
| $\sigma_u^2 = 1.6$ | 5,000 | 99.79 | 99.83 | 27.46 | 99.76 | 100.00 | 99.74 | 98.46 | 100.00 |
| | 10,000 | 99.99 | 99.99 | 48.12 | 99.99 | 100.00 | 99.96 | 99.93 | 100.00 |

Note: The null hypothesis is that the composed error has a normal-half normal distribution. The entries are rejection frequencies in percentages for the various tests.

Table 6: The Sine and Cosine Tests for the Normal-Exponential Model

| DGP | $n$ | The Sine Test $\omega$=1.0 | The Sine Test $\omega$=1.5 | The Sine Test $\omega$=2.0 | The Sine Test $\omega$=(1.0, 2.0) | The Cosine Test $\omega$=1.0 | The Cosine Test $\omega$=1.5 | The Cosine Test $\omega$=2.0 | The Cosine Test $\omega$=(1.0, 2.0) |
|---|---|---|---|---|---|---|---|---|---|
| | 50 | 40.20 | 28.17 | 16.28 | 45.44 | 9.76 | 9.68 | 7.45 | 5.97 |
| normal- | 100 | 40.26 | 28.05 | 15.26 | 45.47 | 11.24 | 9.48 | 6.64 | 7.33 |
| half-normal | 200 | 39.83 | 27.75 | 15.03 | 44.37 | 10.24 | 8.16 | 5.75 | 8.33 |
| $\sigma_v^2 = 1$ | 1,000 | 34.82 | 24.58 | 13.16 | 38.04 | 7.52 | 6.22 | 5.28 | 9.63 |
| $\sigma_u^2 = 0.2$ | 5,000 | 26.06 | 18.01 | 10.40 | 28.17 | 6.40 | 5.53 | 5.12 | 7.14 |
| | 10,000 | 20.82 | 14.31 | 8.54 | 23.45 | 5.78 | 5.29 | 5.02 | 6.73 |
| | 50 | 35.53 | 23.52 | 12.99 | 40.51 | 10.02 | 9.24 | 6.77 | 5.66 |
| normal- | 100 | 33.61 | 22.01 | 11.68 | 38.13 | 11.39 | 8.93 | 6.54 | 7.39 |
| half-normal | 200 | 30.84 | 20.10 | 10.46 | 34.64 | 10.62 | 7.87 | 5.43 | 8.95 |
| $\sigma_v^2 = 1$ | 1,000 | 19.63 | 13.13 | 7.80 | 21.25 | 8.33 | 6.56 | 5.40 | 10.24 |
| $\sigma_u^2 = 0.5$ | 5,000 | 9.52 | 7.58 | 6.33 | 9.74 | 9.38 | 7.69 | 6.02 | 9.98 |
| | 10,000 | 9.07 | 7.70 | 6.43 | 8.70 | 10.49 | 8.14 | 6.34 | 10.61 |
| | 50 | 28.05 | 17.35 | 9.84 | 32.57 | 10.01 | 8.62 | 6.23 | 5.36 |
| normal- | 100 | 23.56 | 14.78 | 8.12 | 27.29 | 11.67 | 8.55 | 6.07 | 7.71 |
| half-normal | 200 | 18.34 | 11.59 | 6.96 | 20.77 | 11.55 | 7.84 | 5.64 | 10.49 |
| $\sigma_v^2 = 1$ | 1,000 | 11.11 | 8.71 | 6.42 | 9.85 | 11.98 | 8.14 | 5.89 | 13.64 |
| $\sigma_u^2 = 1$ | 5,000 | 25.70 | 17.95 | 10.91 | 23.74 | 23.96 | 15.08 | 8.30 | 23.01 |
| | 10,000 | 41.77 | 29.12 | 15.91 | 38.52 | 37.39 | 22.78 | 11.03 | 35.19 |
| | 50 | 11.07 | 7.71 | 5.75 | 12.85 | 11.12 | 7.11 | 5.26 | 5.84 |
| normal- | 100 | 8.65 | 7.39 | 5.06 | 7.63 | 13.02 | 7.26 | 5.52 | 10.42 |
| half-normal | 200 | 12.55 | 9.29 | 6.04 | 8.61 | 16.19 | 8.31 | 6.24 | 13.94 |
| $\sigma_v^2 = 1$ | 1,000 | 50.95 | 28.46 | 10.16 | 43.30 | 38.36 | 17.04 | 8.93 | 32.54 |
| $\sigma_u^2 = 3$ | 5,000 | 97.54 | 86.96 | 32.73 | 95.75 | 93.98 | 60.06 | 25.39 | 90.41 |
| | 10,000 | 100.00 | 99.39 | 56.20 | 99.94 | 99.83 | 86.72 | 44.41 | 99.67 |
| | 50 | 7.50 | 6.26 | 5.19 | 6.70 | 10.54 | 6.35 | 4.98 | 6.92 |
| normal- | 100 | 13.24 | 7.54 | 5.34 | 8.86 | 12.92 | 7.66 | 5.82 | 10.39 |
| half-normal | 200 | 26.66 | 10.74 | 5.38 | 19.32 | 19.67 | 11.32 | 6.72 | 15.21 |
| $\sigma_v^2 = 1$ | 1,000 | 79.72 | 37.54 | 6.59 | 72.52 | 62.78 | 35.27 | 14.81 | 52.18 |
| $\sigma_u^2 = 6$ | 5,000 | 100.00 | 96.03 | 11.08 | 99.99 | 99.92 | 94.73 | 52.22 | 99.72 |
| | 10,000 | 100.00 | 99.97 | 16.89 | 100.00 | 100.00 | 99.80 | 81.41 | 100.00 |
| | 50 | 9.13 | 7.00 | 5.88 | 10.77 | 7.09 | 5.26 | 4.80 | 4.24 |
| normal- | 100 | 5.74 | 6.25 | 5.20 | 5.32 | 6.51 | 5.09 | 4.87 | 5.98 |
| exponential | 200 | 7.93 | 6.05 | 5.22 | 6.25 | 5.45 | 4.73 | 4.87 | 5.64 |
| $\sigma_v^2 = 1$ | 1,000 | 11.84 | 6.07 | 5.20 | 10.79 | 5.01 | 5.25 | 4.96 | 5.02 |
| $\sigma_u^2 = 1$ | 5,000 | 7.31 | 5.61 | 5.32 | 7.64 | 5.09 | 5.28 | 5.02 | 4.90 |
| | 10,000 | 6.76 | 5.17 | 4.52 | 6.29 | 5.09 | 4.90 | 4.76 | 5.11 |
| | 50 | 7.37 | 6.54 | 5.45 | 8.02 | 6.49 | 4.87 | 4.61 | 4.20 |
| normal- | 100 | 6.59 | 5.96 | 5.08 | 5.15 | 5.71 | 5.14 | 4.96 | 5.66 |
| exponential | 200 | 9.70 | 6.32 | 5.03 | 7.34 | 5.08 | 4.65 | 4.65 | 4.79 |
| $\sigma_v^2 = 1$ | 1,000 | 11.11 | 6.29 | 4.87 | 10.53 | 4.67 | 5.13 | 4.77 | 4.91 |
| $\sigma_u^2 = 1.3$ | 5,000 | 6.89 | 5.63 | 5.10 | 7.29 | 5.18 | 5.27 | 4.82 | 4.79 |
| | 10,000 | 6.66 | 5.21 | 4.46 | 5.98 | 4.88 | 4.91 | 4.75 | 5.03 |
| | 50 | 6.42 | 6.15 | 5.24 | 6.96 | 5.99 | 4.53 | 4.62 | 4.34 |
| normal- | 100 | 7.76 | 6.02 | 4.61 | 5.49 | 5.31 | 4.94 | 4.95 | 5.32 |
| exponential | 200 | 11.02 | 6.36 | 4.88 | 8.45 | 4.91 | 4.70 | 4.44 | 4.50 |
| $\sigma_v^2 = 1$ | 1,000 | 10.67 | 6.42 | 4.97 | 9.88 | 4.53 | 4.93 | 4.64 | 4.76 |
| $\sigma_u^2 = 1.6$ | 5,000 | 6.76 | 5.78 | 5.09 | 7.00 | 5.06 | 5.37 | 4.63 | 4.68 |
| | 10,000 | 6.48 | 5.37 | 4.71 | 5.90 | 5.04 | 5.23 | 4.80 | 5.04 |

Note: The null hypothesis is that the composed error has a normal-exponential distribution. The entries are rejection frequencies in percentages for the various tests.

Table 7: The Power Plants Model: Estimated Input Elasticities and Variance Parameters

| | The Moment Estimator | | The MLE | |
|---|---|---|---|---|
| | N-HN[2] | N-Exp | N-HN | N-Exp |
| *input elasticities*[1] | | | | |
| labor | 0.126 | 0.126 | 0.150 | 0.162 |
| | (0.025) | (0.025) | (0.163) | (0.018) |
| fuel | 0.586 | 0.586 | 0.558 | 0.543 |
| | (0.028) | (0.028) | (0.018) | (0.020) |
| capital | 0.292 | 0.292 | 0.227 | 0.246 |
| | (0.036) | (0.036) | (0.025) | (0.025) |
| trend | 0.236 | 0.236 | 0.016 | 0.060 |
| | (0.075) | (0.075) | (0.044) | (0.049) |
| *variance parameters* | | | | |
| $\hat{\sigma}_v^2$ | 0.0005 | 0.023 | 0.0018 | 0.006 |
| | (0.004) | (0.002) | (0.0007) | (0.001) |
| $\hat{\sigma}_u^2$ | 0.168 | 0.038 | 0.165 | 0.067 |
| | (0.016) | (0.004) | (0.011) | (0.007) |

Note 1: Input elasticities are evaluated at the sample means of the variables. The standard errors are in the parentheses and are computed using the Delta method.
Note 2: N-HN: normal-half normal; N-Exp: normal-exponential.
Note 3: Standard errors are in the parentheses.

Table 8: The Sine/Cosine Test Statistics for the Power Plants Model

| | The Sine Test | | | | The Cosine Test | | | |
|---|---|---|---|---|---|---|---|---|
| $H_o$ | $\omega$=1.0 | $\omega$=1.5 | $\omega$=2.0 | $\omega$=(1.0, 2.0) | $\omega$=1.0 | $\omega$=1.5 | $\omega$=2.0 | $\omega$=(1.0, 2.0) |
| normal-half-normal | 0.069 | 0.052 | 0.032 | 3.677 | 1.075 | 1.193 | 1.362 | 5.284 |
| normal-exponential | 14.359 | 14.924 | 15.676 | 21.650 | 22.152 | 22.839 | 23.289 | 23.305 |

Note: The entries are the test statistics. The critical values of $\chi^2(1)$ and $\chi^2(2)$ at the 5% level are 3.8415 and 5.9915, respectively.

Table 9: The Wage Rate Model: Estimated Coefficients and Variance Parameters

| | The Moment Estimator | | The MLE | |
|---|---|---|---|---|
| | N-HN[1] | N-Exp | N-HN | N-Exp |
| *frontier variables* | | | | |
| edu | 0.080 | 0.080 | 0.081 | 0.082 |
| | (0.002) | (0.002) | (0.001) | (0.001) |
| exp | 0.021 | 0.021 | 0.021 | 0.020 |
| | (0.001) | (0.001) | (0.001) | (0.001) |
| exp2 | -0.026 | -0.026 | -0.023 | -0.021 |
| | (0.003) | (0.003) | (0.003) | (0.003) |
| DM | 0.007 | 0.007 | 0.008 | 0.007 |
| | (0.009) | (0.009) | (0.008) | (0.008) |
| DP | 0.204 | 0.204 | 0.204 | 0.211 |
| | (0.011) | (0.011) | (0.010) | (0.010) |
| constant | 9.328 | 9.160 | 9.189 | 9.099 |
| | (0.022) | (0.022) | (0.021) | (0.020) |
| *variance parameters* | | | | |
| $\hat{\sigma}_v^2$ | 0.024 | 0.061 | 0.065 | 0.069 |
| | (0.006) | (0.004) | (0.002) | (0.002) |
| $\hat{\sigma}_u^2$ | 0.277 | 0.063 | 0.157 | 0.050 |
| | (0.021) | (0.005) | (0.007) | (0.002) |

Note 1: N-HN: normal-half normal; N-Exp: normal-exponential.

Note 2: Standard errors are in the parentheses.

Table 10: The Sine/Cosine Test Statistics for the Wage Equation

| | The Sine Test | | | | The Cosine Test | | | |
|---|---|---|---|---|---|---|---|---|
| $H_o$ | $\omega$=1.0 | $\omega$=1.5 | $\omega$=2.0 | $\omega$=(1.0, 2.0) | $\omega$=1.0 | $\omega$=1.5 | $\omega$=2.0 | $\omega$=(1.0, 2.0) |
| normal-half-normal | 45.036 | 46.063 | 47.474 | 57.764 | 142.053 | 151.587 | 165.699 | 243.907 |
| normal-exponential | 29.832 | 33.315 | 37.005 | 42.087 | 99.457 | 107.583 | 117.956 | 153.083 |

Note: The entries are the test statistics. The critical values of $\chi^2(1)$ and $\chi^2(2)$ at the 5% level are 3.8415 and 5.9915, respectively.