Finite Elements for Fluids
Prof. Sheu

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http://homepage.ntu.edu.tw/~ydeleuze/ff2016/

Taipei, March 28, 2016

1 Homework from March 10, 2016

Write a short report

1. compare the P0, P1, P2, P3, ... basis function and give their formulation in barycentric coordinates;
2. how many triangles (elements), vertices, degree of freedoms in a mesh ?
3. how many vertices, degree of freedoms in a triangle (element) ?
4. plot the projection of continuous functions in the P0, P1, P2, P3,... spaces;
5. Compute $\int_{T_K} \lambda_i(M)\,\lambda_j(M)dy$, $i \neq j$.

2 Correction

1-4. Refer to the PDF for the formulation of the basis function and the website (http://homepage.ntu.edu.tw/~ydeleuze/ff2016/) for the plots. Number of triangles, vertices, degrees of freedoms, and plots can be computed with the following FreeFem++ code

```
load "Element_P3"
load "Element_P4"

mesh Th=square(2,2);   // mesh generation
fespace Vh(Th,P1b); // declaration of the finite element space and the basis function (ex. P0)
Vh u; // declaration of the finite element function u
int n=u.n; // declaration of the number of degree of freedom (dof)
int nbtriangles = Th.nt; // get the number of triangles in the mesh
int nbvertices = Th.nv; // get the number of vertices in the mesh

cout << " number of triangles =" + n << endl;
cout << " number of vertices =" + n << endl;
cout << " number of degree of freedom =" + n << endl;

u=0; // set u=0 on every dof

for (int k =0;k<n;k++)      // loop on all degree of freedom
{
        u[][k]=1;                       // the basic function on k
        plot(u,wait=1,ps="basis"+k+".eps",value=1,dim=3,fill=1); // plot the basic function and save the plot image
        u[][k]=0;                 // reset
}

//projection on the P0 space
Vh v = sin(pi *x)*cos(pi*y);
plot(v,wait=1,ps="projection-P"+0+".eps",value=1,dim=3,fill=1);
```

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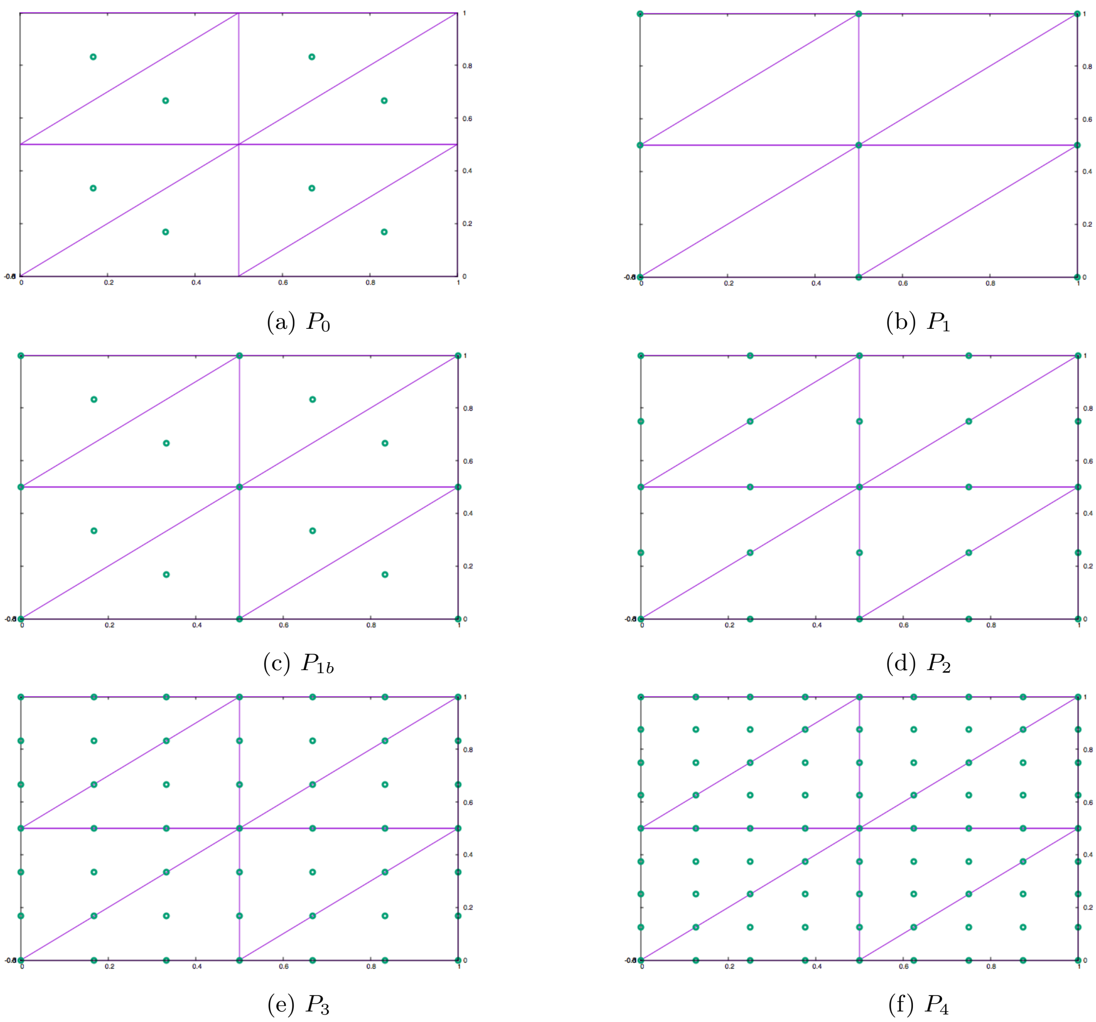
```
//get the vertices coordinates
ofstream file1("triangles.dat"); // export to a file

for (int k =0;k<nbtriangles;k++)      // loop on all the triangles of the mesh
        {
        for (int j =0;j<3;j++)      // loop on all vertices of the triangle
                file1 << Th[k][j].x << " " << Th[k][j].y << " " << 0. << endl; //
                    export in gnuplot format for exemple
        file1 <<endl<<endl;
        }

//get the coordinates and the value of the finite element function of each d.o.f.
Vh xx = x; //get the x-coordinate of eaf dof
Vh yy = y; //get the y-coordinate of eaf dof
ofstream file2("dof.dat");
for (int k =0;k<n;k++)      // loop on all degree of freedom
        file2 << xx[][k] << " " << yy[][k] << " " << 0. << endl; // export in gnuplot
            format for exemple
```

Using the files `triangles.dat` and `dof.dat` produced by the FreeFem++ script, one can use GNUplot to exhibit the degrees of freedoms with this command

```
splot 'triangles.dat' w l notitle, 'dof.dat' t 'Deegrees of freedom' pt 6 lw 3
```



(a) P_0 (b) P_1

(c) P_{1b} (d) P_2

(e) P_3 (f) P_4

Figure 1: Positions of the degrees of freedom in the elements (triangles).

5. In the variational formulation, one needs to compute integrals on a given element (triangle) T_K of the polynomial functions such as

$$\int_{T_K} \lambda_i(M)\, \lambda_j(M) dy,\ i \neq j.$$

To this end we use the reference element $\hat{T}$ defined by the three points $A^1(0,0)$, $A^2(1,0)$, and $A^3(0,1)$. Then, in this triangle $\hat{T}$ the basis polynomials are

$$\hat{\lambda}_1(M) = (1 - x_1 - x_2), \quad \hat{\lambda}_2(M) = x_1, \quad \hat{\lambda}_3(M) = x_2.$$

Using the affine transformation G_K^{-1}, that is to say $y = G_K(x)$ with $x \in \hat{T}$,

$$\int_{T_K} \lambda_i(M)\, \lambda_j(M) dy = \int_{\hat{T}} \hat{\lambda}_i(M)\, \hat{\lambda}_j(M)\, J_K dx,\ i \neq j.$$

where J_K is the Jacobian of the change of variable. Then, using the change of variable formula with the function 1 yields

$$\text{area}(T_K) = \int_{T_K} 1\, dy = \int_{\hat{T}} J_K\, dx$$

where $J_K = |det\, \nabla G_K|$ with G_k affine so that ∇G_K is constant and thus J_K is constant. Then

$$\text{area}(T_K) = J_K \int_{\hat{T}} dx = J_K\, \text{area}(\hat{T}).$$

So

$$J_K = \frac{\text{area}(T_K)}{\text{area}(\hat{T})} = \frac{\text{area}(T_K)}{\frac{1}{2}} = 2\, \text{area}(T_K)$$

and

$$\int_{T_K} \lambda_i(M)\, \lambda_j(M) dy = 2\, \text{area}(T_K) \int_{\hat{T}} \hat{\lambda}_i(M)\, \hat{\lambda}_j(M) dx,\ i \neq j.$$

(i=1,j=2)

$$\begin{aligned}\int_{T_K} \lambda_1(M)\, \lambda_2(M) dy &= 2\, \text{area}(T_K) \int_{\hat{T}} \hat{\lambda}_1(M)\, \hat{\lambda}_2(M) dx \\ &= 2\, \text{area}(T_K) \int_{\hat{T}} (1 - x_1 - x_2)\, x_1\, dx1\, dx2\end{aligned}$$

By Fubini's theorem on the triangle $\hat{T}$, that is to say if $x_1 \in [1,0]$ then $x_2 \in [0, 1 - x_1]$,

$$\begin{aligned}\int_{T_K} \lambda_1(M)\, \lambda_2(M) dy &= 2\, \text{area}(T_K) \int_0^1 x_1 \int_0^{1-x_1} (1 - x_1 - x_2)\, dx2 \;\; dx1 \\ &= 2\, \text{area}(T_K) \int_0^1 x_1 \left[(1 - x_1)\, x_2 - \frac{(x_2)^2}{2} \right]_0^{1-x_1} \; dx1 \\ &= 2\, \text{area}(T_K) \int_0^1 x_1((1 - x_1)^2 - \frac{(1 - x_1)^2}{2}) dx1 \\ &= 2\, \text{area}(T_K) \int_0^1 \frac{x_1}{2}\, (1 - x_1)^2 dx1 \\ &= 2\, \text{area}(T_K) \int_0^1 \frac{x_1}{2} - x_1^2 + \frac{x_1^3}{2} dx1 \\ &= 2\, \text{area}(T_K) \left[\frac{x_1^2}{4} - \frac{x_1^3}{3} + \frac{x_1^4}{8} \right]_0^1 \\ &= \frac{\text{area}(T_K)}{12}\end{aligned}$$

(i=1,j=3)

$$
\begin{aligned}
\int_{T_K} \lambda_1(M)\,\lambda_3(M)dy &= 2\,\mathrm{area}(T_K)\,\int_{\hat{T}} \hat{\lambda_1}(M)\,\hat{\lambda_3}(M)dx \\
&= 2\,\mathrm{area}(T_K)\,\int_{\hat{T}}(1-x_1-x_2)\,x_2\,dx1\,dx2 \\
&= 2\,\mathrm{area}(T_K)\,\int_0^1 \int_0^{1-x_1} (1-x_1-x_2)\,x_2\,dx2 \quad dx1 \\
&= 2\,\mathrm{area}(T_K)\,\int_0^1 \left[(1-x_1)\frac{x_2^2}{2} - \frac{x_2^3}{3}\right]_0^{1-x_1} \quad dx1 \\
&= 2\,\mathrm{area}(T_K)\,\int_0^1 \frac{1}{6}(1-x_1)^3\,dx1 \\
&= 2\,\mathrm{area}(T_K)\,\frac{1}{6}\left[-\frac{1}{4}(1-x_1)^4\right]_0^1 \\
&= \frac{\mathrm{area}(T_K)}{12}
\end{aligned}
$$

(i=2,j=3)

$$
\begin{aligned}
\int_{T_K} \lambda_2(M)\,\lambda_3(M)dy &= 2\,\mathrm{area}(T_K)\,\int_{\hat{T}} \hat{\lambda_2}(M)\,\hat{\lambda_3}(M)dx \\
&= 2\,\mathrm{area}(T_K)\,\int_{\hat{T}} x1\,x_2\,dx1\,dx2 \\
&= 2\,\mathrm{area}(T_K)\,\int_0^1 x_1 \int_0^{1-x_1} x_2\,dx2 \quad dx1 \\
&= 2\,\mathrm{area}(T_K)\,\int_0^1 \frac{1}{2}\,x_1\,(1-x_1)^2 dx1 \\
&= 2\,\mathrm{area}(T_K)\,\frac{1}{2}\left[\frac{x_1^2}{2} - \frac{2}{3}x_1^3 + \frac{x_1^4}{4}\right]_0^1 \\
&= \frac{\mathrm{area}(T_K)}{12}
\end{aligned}
$$