Wave Equation

One dimensional second-order hyperbolic wave equation(Classical wave equation)

$$u_{tt} = c^2 u_{xx}$$

One dimensional first-order hyperbolic linear convection equation

$$u_t + c u_x = 0$$

it describes a wave propagating in x direction with velocity C.

Initial condition $u(x,0) = F(x) \quad ,(-\infty < x < \infty)$

The solution is u(x,t)=F(x-ct)

(一)Euler explicit methods

(i) $$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^n - u_j^n}{\Delta x} = 0$$

truncation error $o(\Delta t, \Delta x)$ 1st order accuracy

if c>0, for stable solution,backward differencing is used

if c<0, forward differencing is used

(ii) $$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

truncation error $o(\Delta t, (\Delta x)^2)$ 1st order accuracy

Von Neumann analysis shows unconditional unstable(Homework)

(二)upstream differencing method

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0 \quad ; \quad c > 0$$

Truncation error $o(\Delta t, \Delta x)$

$$-\frac{\Delta t}{2}u_{tt} + \frac{c\Delta x}{2}u_{xx} - \frac{(\Delta t)^2}{6}u_{ttt} - c\frac{(\Delta x)^2}{6}u_{xxx} + ...$$

(Homework)Von Neumann analysis shows for stability

$$0 \le \upsilon \equiv c \Delta t / \Delta x \le 1$$

if $\upsilon = 1 \rightarrow$ the unstream scheme reduces to $u_j^{n+1} = u_{j-1}^n$ from the modified equation

→This differencing scheme satisfies the shift condition

(三)Lax Method (1954)

$$\frac{u_j^{n+1} - \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right)}{\Delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

・Explicit one-step method

- First order accuracy $o(\Delta t, {(\Delta x)^2}/{(\Delta t)})$
- Stability condition $|\upsilon| \le 1$
- Not uniformly consistence since ${(\Delta x)^2}/{\Delta t}$ may not appraoch to zero as $\Delta x, \Delta t \to 0$
- modified equation is

$$u_t + cu_x = \frac{c\Delta x}{2}\left(\frac{1}{\upsilon} - \upsilon\right)u_{xx} + \frac{c(\Delta x)^2}{3}\left(1-\upsilon^2\right)u_{xxx} + ...$$

- Large disspiation error where $\upsilon \neq 1$,it can be seen by comparing the upstream differencing scheme.
- Satify shift condition
- Amplification factor $G = \cos\beta - i\upsilon\sin\beta$
- Relative phase error ${\phi}/{\phi_e} = \frac{\tan^{-1}(-\upsilon\tan\beta)}{-\beta\upsilon}$

(四)Euler Implicit Method

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} = 0$$

- Implicit
- First-order accuracy with truncation error $o(\Delta t, (\Delta x)^2)$
- Unconditional stable
- A system of tridiagonal matrix to be solved by Thomas algorithm
- Capable of permitting large time step which will produce large truncation error
- Modified equation is

$$u_t + cu_x = \left(\frac{1}{2}c^2\Delta t\right)u_{xx} - \left[\frac{1}{6}c(\Delta x)^2 + \frac{1}{3}c^3(\Delta t)^2\right]u_{xxx} + ...$$

- Not satisfy shift condition
- Amplication factor $G = \frac{1 - i\upsilon\sin\beta}{1 + \upsilon^2\sin^2\beta}$
- Relative phase error ${\phi}/{\phi_e} = \frac{\tan^{-1}(-\upsilon\sin\beta)}{-\beta\upsilon}$
- High dissipation for intermediate wave number
- Large lagging phase error for high wave number

(五)Leap Frog Method

$$\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t}+c\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}=0$$

- explicit
- Three-time level scheme
- Second-order accuracy with truncation error $o((\Delta t)^2,(\Delta x)^2)$
- Stability requirement $|\upsilon|\le 1$
- Modified equation is

$$u_t+cu_x=\frac{1}{6}c(\Delta x)^2(\upsilon^2-1)u_{xxx}-\frac{1}{120}c(\Delta x)^4(9\upsilon^4-10\upsilon^2+1)u_{xxxxx}....$$

- Predominantly exhibit dispersive errors which is typical in second order accurate method
- No dissipative truncation terms such that the algorithm is neutrally stable and errors caused by improper B.C. or computer round-off error won't be damped.
- Amplication factor $G=\pm(1-\upsilon^2\sin^2\beta)^{\frac{1}{2}}-i\upsilon\sin\beta$
- Relative phase error $\phi/\phi_e=\dfrac{\tan^{-1}\left[-\upsilon\sin\beta\Big/\pm(1-\upsilon^2\sin^2\beta)^{1/2}\right]}{-\beta\upsilon}$
- Disadvantages

  (1)Initial conditions must be specified at two-time levels

  (2)Leap frog nature of differencing $\left(u_j^{n+1}\ne f\left(u_j^n\right)\right)$ such that two independent solutions develop as the calculation proceeds.

  (3)Additional storage may be required due to the three-time level scheme

(六)Lax-Wendroff Method

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+c\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}=\frac{c^2(\Delta t)}{2(\Delta x)^2}\left(u_{j+1}^n-2u_j^n+u_{j-1}^n\right)$$

∵ Employing wave equations

$u_t=-cu_x$

$u_{tt}=c^2u_{xx}$

in the Taylor expansion of u_j^{n+1}

$$u_j^{n+1}=u_j^n+u_t\Delta t+\frac{1}{2}u_{tt}(\Delta t)^2+o\left[(\Delta t)^3\right]$$

and employing second-order central difference expression for u_x and u_{xx}

- Explicit one-step scheme

- Second order accuracy with truncation error $o((\Delta t)^2,(\Delta x)^2)$
- Stability requirement $|\upsilon| \le 1$
- Modified equation is
  $$u_t + cu_x = -\frac{1}{6}c(\Delta x)^2(1-\upsilon^2)u_{xxx} - \frac{1}{8}c(\Delta x)^3\upsilon(1-\upsilon^2)u_{xxxx} + \ldots$$
- Satisfy shift condition
- Amplication factor G
  $$G = 1-\upsilon^2(1-\cos\beta) - i\upsilon\sin\beta$$
- Relative phase error $\phi/\phi_e = \dfrac{\tan^{-1}\left[-\upsilon\sin\beta \big/ \left[1-\upsilon^2(1-\cos\beta)\right]\right]}{-\beta\upsilon}$
- Predominantly lagging phase error except for large wave number $(0.5)^{0.5} < \upsilon < 1$

(七)Two-Step Lax-Wendroff Method

Step 1: $$\frac{u_{j+1/2}^{n+1/2} - \frac{1}{2}\left(u_{j+1}^n + u_j^n\right)}{\Delta t/2} + c\frac{u_{j+1}^n - u_j^n}{\Delta x} = 0$$

Step 2: $$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}}{\Delta x} = 0$$

- Applied for nonlinear equations such as inviscid flow equations
- Two step method
- Three-time level method
- Second-order accuracy with truncation error $o((\Delta t)^2,(\Delta x)^2)$
- Stability requirement $|\upsilon| \le 1$
- Step 2 is leap frog method for the latter half time step
- When applied to linear wave equation, two-Step Lax-Wendroff method ≡original Lax-Wendroff scheme.(Homework)
- Modified equation and amplification factor are the same as original Lax-Wendroff method.

(八)MacCormack Method (1969)

Predictor step : $$\overline{u_j^{n+1}} = u_j^n - c\frac{\Delta t}{\Delta x}\left(u_{j+1}^n - u_j^n\right)$$

Correct step : $$u_j^{n+1} = \frac{1}{2}\left[u_j^n + \left[\overline{u_j^{n+1}} - \frac{c\Delta t}{\Delta x}\left(\overline{u_j^{n+1}} - \overline{u_{j-1}^{n+1}}\right)\right]\right]$$

- Widely used for solving fluid flow equations
- A variation of two-step Lax-Wendroff scheme which removes the necessity of computing unknowns at grid points j+1/2,j-1/2.

- Partically useful when solving nonlinear P.D.E.
- Explicit,two step method
- In predictor step,forward differencing is employed for u_x
  In correct step,backward differencing is employing for u_x
- In moving discontinuities problems,the differencing can be reversed.
- MacCorwack scheme is equivalent to the original Lax-Wendroff scheme for the present linear wave equation
- Truncation error
  Stability limit
  modified equation
  amplification factor = those of Lax-Wendroff scheme

(九)Upwind Method (Warming and Beam 1975)

Predictor step : $\overline{u_j^{n+1}} = u_j^n - c\dfrac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$; c>0

Correct step :

$$u_j^{n+1} = \frac{1}{2}\left[u_j^n + \overline{u_j^{n+1}} - \frac{c\Delta t}{\Delta x}\left(\overline{u_j^{n+1}} - \overline{u_{j-1}^{n+1}}\right) - \frac{c\Delta t}{\Delta x}\left(u_j^n - 2u_{j-1}^n + u_{j-2}^n\right)\right] \qquad c>0$$

- A variation of MacCormack method
- Backward differencing is applied to both predictor and corrector steps
- second-order accurate with truncation error
  $o((\Delta t)^2, (\Delta t)(\Delta x), (\Delta x)^2)$
- Substitute predictor equation in corrector equation to obtain one-step algorithm
  $$u_j^{n+1} = u_j^n - \upsilon\left(u_j^n - u_{j-1}^n\right) + \frac{1}{2}\upsilon(\upsilon - 1)\left(u_j^n - 2u_{j-1}^n + u_{j-2}^n\right)$$
- Modified equation
  $$u_t + cu_x = \frac{1}{6}c(\Delta x)^2(1-\upsilon)(2-\upsilon)u_{xxx}$$
  $$-\frac{(\Delta x)^4}{8\Delta t}\upsilon(1-\upsilon)^2(2-\upsilon)u_{xxxx} + \ldots.$$
- Satisfing shift condition at $\upsilon = 1, \upsilon = 2$
- Amplication factor G
  $$G = 1 - 2\upsilon\left[\upsilon + 2(1-\upsilon)\sin^2\frac{\beta}{2}\right]\sin^2\frac{\beta}{2} - i\upsilon\sin\beta\left[1 + 2(1-\upsilon)\sin^2\frac{\beta}{2}\right]$$
- Stability limit $0 \le \upsilon \le 2$
- Predominantly leading phase error for $0 < \upsilon < 1$; lagging phase error for $0 < \upsilon < 1$
- upwind and Lax-Wendroff method have opposite errors

for $0 < \upsilon < 1$
→Considerable reduction of dispersive error will occur if a linear combination of two methods are used.

(十)Time-centered implicit method (Trapezoidal differencing method)

$$u_j^{n+1} = u_j^n - \frac{\upsilon}{4}\left(u_{j+1}^{n+1} + u_{j+1}^n - u_{j-1}^{n+1} - u_{j-1}^n\right) \qquad (4\text{-}60)$$

- Implicit method
- Second-order accurate with truncation error $o((\Delta t)^2, (\Delta x)^2)$
- Unconditional Stable
- Modified equation

$$u_t + cu_x = -\left[\frac{c^3(\Delta t)^2}{12} + \frac{c(\Delta x)^2}{6}\right]u_{xxx}$$
$$-\left[\frac{c(\Delta x)^4}{120} + \frac{c^3(\Delta t)^2(\Delta x)^2}{24} + \frac{c^4(\Delta t)^4}{80}\right]u_{xxxx} + \ldots.$$

- No implicit artificial viscosity
- Explicity artificial viscosity may be necessary to add to prevent the solution from blowing up
- Amplication factor $G = \dfrac{1-\left(i\upsilon/2\right)\sin\beta}{1+\left(i\upsilon/2\right)\sin\beta}$
- Modified equation and phase error can be found from Beam and Warming(1976)
- Tridiagonal coefficient matrix must be solved at each new time step

(十一)Rusanov (Burstein-Mirin)method (1970)

Step 1: $u_{j+1/2}^{(1)} = \frac{1}{2}\left(u_{j+1}^n + u_j^n\right) - \frac{1}{3}\upsilon\left(u_{j+1}^n - u_j^n\right)$

Step 2: $u_j^{(2)} = u_j^n - \frac{2}{3}\upsilon\left(u_{j+1/2}^{(1)} - u_{j-1/2}^{(1)}\right)$

$$u_j^{n+1} = u_j^n - \frac{\upsilon}{24}\left(-2u_{j+2}^n + 7u_{j+1}^n - 7u_{j-1}^n + 2u_{j-2}^n\right)$$

Step 3:
$$-\frac{3}{8}\upsilon\left(u_{j+1}^{(2)} - u_{j-1}^{(2)}\right)$$
$$-\frac{\omega}{24}\left(u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n\right)$$

- Explicit,three-step method
- ω is added to stabilize the scheme since the stability limits are

$$|\upsilon| \le 1$$

and $4\upsilon^2 - \upsilon^4 \le \omega < 3$

- Third-order accurate
- Modified equation

$$u_t + cu_x = -\frac{c(\Delta x)^3}{24}\left(\frac{\omega}{\upsilon} - 4\upsilon + \upsilon^3\right)u_{xxxx}$$

$$+\frac{c(\Delta x)^4}{120}\left(-5\omega + 4 + 15\upsilon^2 - 4\upsilon^4\right)u_{xxxxx} + \dots$$

- Reducing dissipation $\rightarrow \omega = 4\upsilon^2 - \upsilon^4$

  Reducing dispersion $\rightarrow \omega = \left(4\upsilon^2 + 1\right)\left(4 - \upsilon^2\right)\big/5$
- Amplication factor G

$$G = 1 - \frac{\upsilon^2}{2}\sin^2\beta - \frac{2\omega}{3}\sin^4\frac{\beta}{2} - i\upsilon\sin\beta\left[1 + \frac{2}{3}\left(1 - \upsilon^2\right)\sin^2\frac{\beta}{2}\right]$$

- leading or lagging phase error depending on the free parameter ω.

(十二)Warming-Kulter-Lomax (WKL)method (1973)

Step 1: $u_j^{(1)} = u_j^n - \frac{2}{3}\upsilon\left(u_{j+1}^n - u_j^n\right)$

Step 2: $u_j^{(2)} = \frac{1}{2}\left[u_j^n + u_j^{(1)} - \frac{2}{3}\upsilon\left(u_j^{(1)} - u_{j-1}^{(1)}\right)\right]$

Step 3:

$$u_j^{n+1} = u_j^n - \frac{\upsilon}{24}\left(-2u_{j+2}^n + 7u_{j+1}^n - 7u_{j-1}^n + 2u_{j-2}^n\right)$$
$$-\frac{3}{8}\upsilon\left(u_{j+1}^{(2)} - u_{j-1}^{(2)}\right)$$
$$-\frac{\omega}{24}\left(u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n\right)$$

- MacCormack methods for first two steps; Rusanov method for the third step
- Same stability limit bound and modified equation as Rusanov method
- Third-order accurate method at the expense of additional computing complexity
- Explicit
- WKL method has same advantage over Rusanov method that the MacCormack method has over the two-step Lax-Wendroff method

Conculsion:

- Second-order accurate explicit schemes(Lax-Wendroff,upwind schemes) give excellent results with a min of computational effort

- Implicit scheme is probably not the optimum choice.

- Explicit schemes seem to provide a more natural F.D. approximation for hyperbolic P.D.E. which possess limited zones of influence.

- Implicit methods are more appropriate for solving a parabolic P.D.E. since it normally assimilates information from all grid points located on or below the characteristics t=const.

A.1-D Heat Equation

Parabolic one-dimensional heat equation(diffusion equation)

$u_t = \alpha u_{xx}$

with I.C.u(x,0)=f(x)

B.C. u(0,t)=u(1,t)=0

is used as the model equation

(一)Simple explicit method

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}$$

- Explicit,one step method
- First order accurate with truncation error $o\left[\Delta t, (\Delta x)^2\right]$
- Stability limit $0 \le \gamma \equiv \frac{\alpha \Delta t}{(\Delta x)^2} \le 1/2$
- Modified equation is

$$u_t - \alpha u_{xx} = \left[-\frac{1}{2}\alpha^2 \Delta t + \frac{\alpha (\Delta x)^2}{12}\right] u_{xxxx}$$
$$+\left[\frac{1}{3}\alpha^3 (\Delta t)^2 - \frac{1}{12}\alpha^2 \Delta t (\Delta x)^2 + \frac{1}{360}\alpha (\Delta x)^4\right] u_{xxxxxx}$$
$$+ \ldots\ldots\ldots\ldots$$

- No dispersive error
  It is usually the behavior of other schemes for the heat equation
- Amplification factor G
  G=1+2 (cos -1)
  it has no imaginary part and hence no phase shift
- Exact amplification factor Ge
  Ge=e ^{2}
  where =$k_m \triangle x$
- highly dissipative for large when =1/2
- Not properly model the physical behavior of a parabolic PDE since the interior solution at point P can be calculated without the knowledge at the boundary.
  Howerer,for parabolic equation,the solution should depend on the B.C..Since parabolic heat equation has the characteristic t=const such that the solution at t=const depends on everything which occured in the physical domain at all earlier times.

(二)Richardson's Method

$$\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t}=\alpha\frac{u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}}{(\Delta x)^2}$$

- Explicit,one step method
- Three-time level scheme
- Second-order accurate with truncation error $\mathrm{o}\left[\Delta t^2,\Delta x^2\right]$
- Unconditional unstable

(三)Laasonen method (1949)

$$\frac{u_j^{n+1}-u_j^{n}}{\Delta t}=\alpha\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{(\Delta x)^2}$$

- Implicit
- First-order accurate with truncation error $o\left[\Delta t,(\Delta x)^2\right]$
- Unconditional stable
- Modified equation

$$u_t-\alpha u_{xx}=\left[\frac{1}{2}\alpha^2\Delta t+\frac{\alpha(\Delta x)^2}{12}\right]u_{xxxx}$$
$$+\left[\frac{1}{3}\alpha^3(\Delta t)^2+\frac{1}{12}\alpha^2\Delta t(\Delta x)^2+\frac{1}{360}\alpha(\Delta x)^4\right]+.......$$

- Amplitication factor G

$$\mathrm{G}=\left[1+2\gamma(1-\cos\beta)\right]^{-1}$$

(四)Crank-Nicolson method (1947)

Defining central differencing scheme

$$\delta_x^2 u_j^n=u_{j+1}^n-2u_j^n+u_{j-1}^n$$

$$\frac{u_j^{n+1}-u_j^{n}}{\Delta t}=\frac{\alpha}{\Delta x^2}\frac{1}{2}\left(\underbrace{\delta_x^2u_j^n}_{u_{j-1}^n-2u_j^n+u_{j+1}^n}+\overbrace{\delta_x^2u_j^{n+1}}^{u_{j-1}^{n+1}-2u_j^{n+1}+u_{j+1}^{n+1}}\right)$$

- Implicit method
- Unconditional stable
- Second-order accuracy with truncation error $o\left[(\Delta t)^2,(\Delta x)^2\right]$
- Modified equation is

$$u_t - \alpha u_{xx} = \frac{\alpha\left(\Delta x\right)^2}{12} u_{xxxx} + \left[\frac{1}{12}\alpha^3\left(\Delta t\right)^2 + \frac{1}{360}\alpha\left(\Delta x\right)^4\right] u_{xxxxxx} + ...$$

- Amplification factor G

$$G = \frac{1-\gamma(1-\cos\beta)}{1+\gamma(1-\cos\beta)}$$

(五)Generalized explicit,Laasonen and Crank-Nicolson method

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\alpha}{\left(\Delta x\right)^2}\left[\theta \delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n\right]$$

where

$$\begin{cases} \theta = 0 & \text{explicit method(1)} \\ \theta = 1 & \text{Laasonen method(3)} \\ \theta = 1/2 & \text{Crank-Nicolson method(4)} \end{cases}$$

- Usually, it is first-order accurate with the truncation error $o\left[\Delta t, \left(\Delta x\right)^2\right]$
- Modified equation

$$u_t - \alpha u_{xx} = \left[\left(\theta - 1/2\right)\alpha^2 \Delta t + \frac{\alpha\left(\Delta x\right)^2}{12}\right] u_{xxxx}$$

$$+\left[\left(\theta^2 - \theta + 1/3\right)\alpha^3\left(\Delta t\right)^2 + \frac{1}{6}\left(\theta - \frac{1}{2}\right)\alpha^2 \Delta t\left(\Delta x\right)^2 + \frac{1}{360}\alpha\left(\Delta x\right)^4\right] u_{xxxxxx} + ...$$

(六)Richtymer and Morton (1967)combined method

$$\left(1+\theta\right)\frac{u_j^{n+1} - u_j^n}{\Delta t} - \theta\frac{u_j^n - u_j^{n-1}}{\Delta t} = \alpha\frac{\delta_x^2 u_j^{n+1}}{\left(\Delta x\right)^2}$$

- First order accurate with truncation error $o\left[\Delta t, \left(\Delta x\right)^2\right]$
- Modified equation

$$u_t - \alpha u_{xx} = \left[-\left(\theta - \frac{1}{2}\right)\alpha^2 \Delta t + \frac{\alpha}{12}\left(\Delta x\right)^2\right] u_{xxxx} + ...$$

(七)DuFort-Frankel method

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{\alpha}{\left(\Delta x\right)^2}\left(u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n\right)$$

- Replacing $u_j^n = \frac{1}{2}\left(u_j^{n+1} + u_j^{n-1}\right)$ in Richardson's method
- Explicit method
- Three-time level scheme

- $o\left[(\Delta t)^2, (\Delta x)^2, \left(\Delta t / \Delta x \right)^2 \right]$
- To be a consistent scheme $\Delta t / \Delta x \to 0$ as $\Delta t, \Delta x \to 0$
- Modified equation

$$u_t - \alpha u_{xx} = \left[\frac{1}{12} \alpha (\Delta t)^2 - \frac{\alpha^3 (\Delta t)^2}{(\Delta x)^2} \right] u_{xxxx}$$

$$+ \left[\frac{1}{360} \alpha (\Delta x)^4 - \frac{1}{3} \alpha^3 (\Delta t)^2 + 2\alpha^5 \left(\frac{\Delta t}{\Delta x} \right)^4 \right] u_{xxxxxx} + \ldots$$

- Amplification factor G

$$G = \frac{2\gamma \cos\beta \pm \left(1 - 4\gamma^2 \sin^2\beta\right)^{1/2}}{1 + 2\gamma}$$

- Unconditional stable

(八)Alternating-Directional Explicit (ADE) method

(1)Saulyev,V.K. (1957)

Step 1: $$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha \left(\frac{u_{j+1}^n - u_j^{n+1} - u_j^n + u_{j-1}^{n+1}}{(\Delta x)^2} \right)$$

Marching the solution from the left boundary to the right boundary u_j^{n+1} is determined explicitly from known u_{j-1}^{n+1}

Step 2: $$\frac{u_j^{n+2} - u_j^{n+1}}{\Delta t} = \alpha \left(\frac{u_{j+1}^{n+2} - u_j^{n+2} - u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2} \right)$$

Marching the solution from the right boundary to the left boundary u_j^{n+2} is determined explicitly from known u_{j-1}^{n+2}

- Three-time level
- Truncation error $o\left[(\Delta t)^2, (\Delta x)^2, \left(\Delta t / \Delta x \right)^2 \right]$
- Unconditional stable

(2)Barakat and Clark (1966)

$$\frac{P_j^{n+1} - P_j^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2} \left(P_{j-1}^{n+1} - P_j^{n+1} - P_j^n + P_{j+1}^n \right)$$

$$\frac{q_j^{n+1} - q_j^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2} \left(q_{j-1}^n - q_j^n - q_j^{n+1} + q_{j+1}^{n+1} \right)$$

The calculation procedure is simultaneously marched in both directions,the solution

$$u_j^{n+1} = 1/2 \left(p_j^{n+1} + q_j^{n+1} \right)$$

- Unconditional stable
- Truncation error $o\left[(\Delta t)^2, (\Delta x)^2 \right]$

(3)Larkin (9164)

$$\frac{P_j^{n+1}-u_j^n}{\Delta t}=\frac{\alpha}{(\Delta x)^2}\left(P_{j-1}^{n+1}-P_j^{n+1}-u_j^n+u_{j+1}^n\right)$$

$$\frac{q_j^{n+1}-u_j^n}{\Delta t}=\frac{\alpha}{(\Delta x)^2}\left(u_{j-1}^n-u_j^n-q_j^{n+1}+q_{j+1}^{n+1}\right)$$

$$u_j^{n+1}=\frac{1}{2}\left(P_j^{n+1}+q_j^{n+1}\right)$$

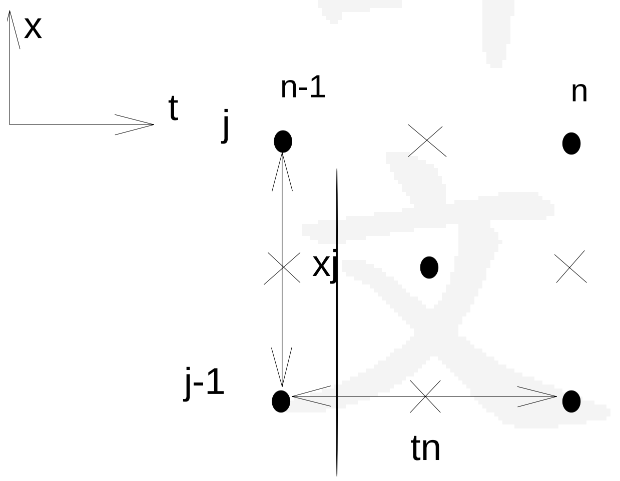
(九)Keller Box and modified Box method

(1)Keller Box (keller 1970)

$u_t=\alpha u_{xx}$

Define $v=u_x$

$$\rightarrow\begin{cases}u_x=v\\ u_t=\alpha v_x\end{cases}$$



$$\begin{cases}\dfrac{u_j^n-u_{j-1}^n}{\Delta x_j}=v_{j-\frac{1}{2}}^n=\dfrac{1}{2}\left(v_j^n+v_{j-1}^n\right) & \text{(7-33)}\\[2ex] \dfrac{u_{j-\frac{1}{2}}^n-u_{j-\frac{1}{2}}^n}{\Delta t_n}=\dfrac{\alpha}{\Delta x_j}\left(v_j^{n-\frac{1}{2}}-v_{j-1}^{n-\frac{1}{2}}\right)=\dfrac{\alpha}{2\Delta x_j}\left(v_j^n+v_j^{n-1}-v_{j-1}^n-v_{j-1}^{n-1}\right) & \text{(7-34)}\end{cases}$$

Where

$$u_{j-\frac{1}{2}}^n=\frac{1}{2}\left(u_j^n+u_{j-1}^n\right)$$

$$v_j^{n-\frac{1}{2}}=\frac{1}{2}\left(v_j^n+v_j^{n-1}\right)$$

- The system of (7-33),(7-34) can be written in block tridiagonal form with 2x2 blocks
- Solved by block elimination scheme
- Implicit,second order in accuracy

(二)Modified Keller Box method

$$\frac{u_j^{n+1} - u_{j-1}^{n+1}}{\Delta x_j} = v_{j-\frac{1}{2}}^{n+1} = \frac{1}{2}\left(v_j^{n+1} + v_{j-1}^{n+1}\right) \qquad (7\text{-}35)$$

$$\frac{u_{j-\frac{1}{2}}^{n+1} - u_{j-\frac{1}{2}}^{n}}{\Delta t_{n+1}} = \frac{\alpha}{\Delta x_j}\left(v_j^{n+\frac{1}{2}} - v_{j-1}^{n+\frac{1}{2}}\right)$$

$$= \frac{\alpha}{2\Delta x_j}\left(v_j^n + v_j^{n+1} - v_{j-1}^n - v_{j-1}^{n+1}\right) \qquad (7\text{-}37)$$

$\underbrace{v_{j-1}^{n+1}, v_{j-1}^{n},}_{\text{can be eliminated from(7-37) from(7-35) at(n+1),(n) respectively}}$ $\underbrace{v_j^{n+1}, v_j^{n}}_{\text{are written in terms of u's}}$ similar as (7-35),(7-37) by advancing j by 1

$$\rightarrow B_j u_{j-1}^{n+1} + D_j u_j^{n+1} + A_j u_{j+1}^{n+1} = c_j$$

where $B_j = \frac{\Delta x_j}{\Delta t_{n+1}} - \frac{2\alpha}{\Delta x_j}$

$$A_j = \frac{\Delta x_{j+1}}{\Delta t_{n+1}} - \frac{2\alpha}{\Delta x_{j+1}}$$

$$D_j = \frac{\Delta x_j}{\Delta t_{n+1}} + \frac{\Delta x_{j+1}}{\Delta t_{n+1}} + \frac{2\alpha}{\Delta x_j} + \frac{2\alpha}{\Delta x_{j+1}}$$

$$C_j = 2\alpha\frac{u_{j-1}^n - u_j^n}{\Delta x_j} + 2\alpha\frac{u_{j+1}^n - u_j^n}{\Delta x_{j+1}}$$

$$+\left(u_j^n - u_{j-1}^n\right)\frac{\Delta x_j}{\Delta t_{n+1}} + \left(u_{j+1}^n + u_j^n\right)\frac{\Delta x_{j+1}}{\Delta t_{n+1}}$$

- Second-order in accuracy even in nonuniform spacing(Homework)
- Crank-Nicolson is second-order in accuracy in uniform spacing

B.2-D heat equation

$$u_t = \alpha\left(u_{xx} + u_{yy}\right)$$

The direct extension of 1-D numerical scheme to 2-D problems has the following difficulties.

(1)For explicit method,the stability limit becomes more restrictive,thus it is more impractical
(2)For implicit method,the coefficient matrix is no longer tridiagonal,the equation solver requires substantially more computing time.

(一)Alternating-Direction-Implicit (ADI) method

$$\begin{cases} Peaceman, Rachford \quad (1955) \\ Douglas \quad (1955) \end{cases}$$

step 1: $$\frac{u_{i,j}^{n+1/2} - u_{i,j}^{n}}{\frac{\Delta t}{2}} = \alpha\left(\hat{\delta}_x^2 u_{i,j}^{n+1/2} + \hat{\delta}_y^2 u_{i,j}^{n}\right)$$

step 2: $$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+1/2}}{\frac{\Delta t}{2}} = \alpha\left(\hat{\delta}_x^2 u_{i,j}^{n+1/2} + \hat{\delta}_y^2 u_{i,j}^{n+1}\right)$$

- Two-step splitting scheme
- step 1:tridiagonal matrix is solved for each j row of grid points (i.e. for each j, i=0→i_{max})
- step 2:tridiagonal matrix is solved for each i row of grid points (i.e. $\forall$i, j=0→j_{max})
- Second-order accurate with truncation error $o\left[(\Delta t)^2, (\Delta x)^2, (\Delta y)^2\right]$ (Homeowrk)
- Amplification factor G

$$G = \frac{\left[1 - r_x\left(1 - \cos\beta_x\right)\right]\left[1 - r_y\left(1 - \cos\beta_y\right)\right]}{\left[1 + r_x\left(1 - \cos\beta_x\right)\right]\left[1 + r_y\left(1 - \cos\beta_y\right)\right]}$$

- Unconditional stable

(二)Splitting or Fractional-Step method (Yanenko,N.N.,1971)

step 1: $$\frac{u_{i,j}^{n+1/2} - u_{i,j}^{n}}{\Delta t/2} = \alpha \hat{\delta}_x^2 u_{i,j}^{n+1/2}$$

step 2: $$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+1/2}}{\Delta t/2} = \alpha \hat{\delta}_x^2 u_{i,j}^{n+1}$$

- First-Order accurate with truncation error $o\left[(\Delta t),(\Delta x)^2,(\Delta y)^2\right]$

(三)Hopscotch method

1st step:at each point which (i+j+n=even)

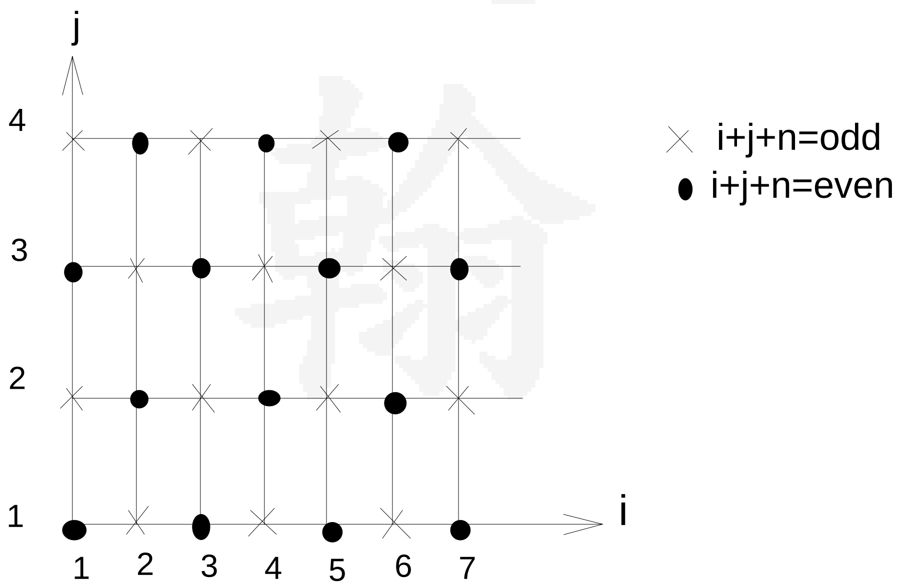
$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \alpha\left[\hat{\delta}_x^2 u_{i,j}^{n} + \hat{\delta}_y^2 u_{i,j}^{n}\right]$$

$u_{i,j}^{n+1}$ is calculated explicitly

2nd step:at each point which (i+j+n=odd)

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \alpha\left[\hat{\delta}_x^2 u_{i,j}^{n+1} + \hat{\delta}_y^2 u_{i,j}^{n+1}\right]$$

- $u_{i,j}^{n+1}$ appears to be implicit,but no simulation equations are to be solved,since $u_{i+1,j}^{n+1}, u_{i-1,j}^{n+1}, u_{i,j+1}^{n+1}, u_{i,j-1}^{n+1}$ are known in 1st sweep
- Explicit method
- Truncation error $o\left[(\Delta t),(\Delta x)^2,(\Delta y)^2\right]$ (Homework)
- Unconditional stable



n=0 for example

Conclusions:

- In general,implicit methods are more suitable than explicit methods
- For 1-D heat equation,Crank-Nicolson method is recommended.
- For 2-D,3-D heat equation,ADI scheme of Douglas and Gum and Keller box and modified box methods give excellent results.

Inviscid Burgers‘ Equation

The model nonlinear equation is hyperbolic equation

$$u_t + uu_x = 0 \qquad (4\text{-}129)$$

or

$$u_t + F_x = 0$$

or

$$u_t + Au_x = 0 \qquad (4\text{-}131)$$

where $A = A(u) = dF/du$ is the Jacobian matrix and the eigenvalues of A are all real.

Inviscid Burgers‘ equation is the simplified form of parabolic viscous Burgers' equation

$$u_t + uu_x = \mu u_{xx}$$

without considering the viscous effect.

- (4-129) can be viewed as a nonlinear wave equation where each point on the wave front can propagate with a different speed.
- Jenuine solution of (4-131) is one in which u is continuous but the bounded discontinuity in the derivatives of u may occur.
- Shocks and rarefactions are frequently encountered in high speed flow governed by nonlinear Burgers‘ equation of hyperbolic type.
- Weak solution of (4-131) is a solution which is genuine except along a surface (x,t) space across which the function u may be discontinuous.

  The existence of shock waves in inviscid supersonic flow is an example of a weak solution.

Aim: Develop the requirements for a weak solution (or the requirements necessary for the existence of a solution with a discontinuity)

- The spaced-centered algorithms for inviscid Burgers' equations (Euler equation) were historically important.
- All centered second order accurate schemes refer to the Lax-Wendroff algorithm (It is the unique second order explicit scheme for the linear convection equation on a three point support)
  - -It plays the essential role as guideline for all schemes attempting to improve certain of its deficiencies.
  - -The generation of oscillations at discontinuities is the weakness.
- Model equation

  (a)Conservative form-$U_t + F_x = 0$

(b)Quasi-linear form-$U_t + AU_x = 0$; $A = \partial F / \partial U$

- Lax-Friedrichs (1954) scheme is the first numerical discretization of Euler equations.
- Numerical flux

  -An essential property of the discretized schemes

  -For $u_t + F_x = 0$

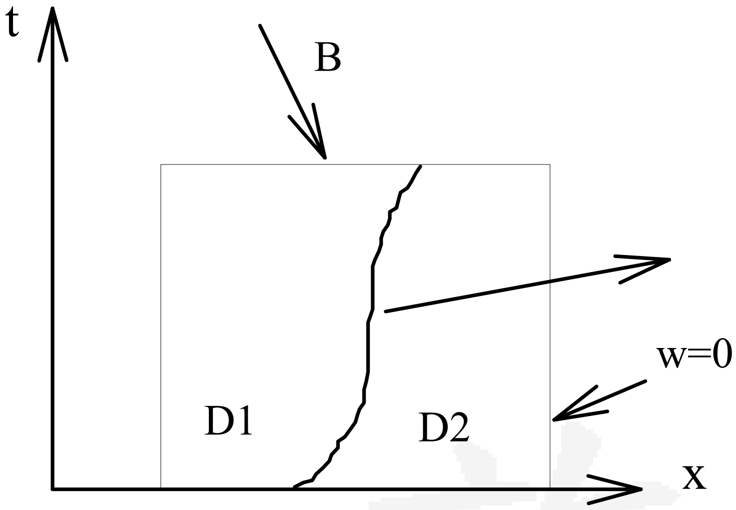
  It has the property $\int_j u_t dx = \frac{\partial}{\partial t}\int_j u dx = -\left(F^*_{j+1/2} - F^*_{j-1/2}\right)$

  Ensure the integral depends on the fluxes within the domain and not depends on the fluxes within the domain-numerical fluxes are functions of u at the mesh points.
- All the second-order, three-point central schemes of the Lax-Wendroff family have rather poor dissipative properties and generate oscillations around sharp discontinuities.
- In order to remove high frequency oscillations around discontinuities in second-order central schemes Von Neuman and Richtmeyer (1950) introduced the concept of artificial viscosity. This introduction of artificial viscosity should obtain the porperty.

  (i)locally around the discontinuity- can simulate the physical viscosity on the scale of mesh.

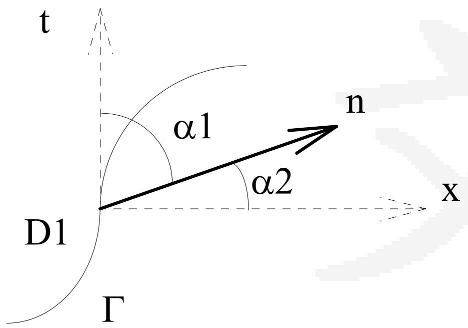
  (ii)In smooth region- can be negelected (i.e., of the order equal or higher than the truncation error)

  (iii)Requiring additional dissipation to avoid the appearance of expansion shocks where the sonic transitions occur.
- Any upwind scheme can be written as a central scheme plus dissipation terms(Can be verified)
- The added dissipation terms introduce an upwind correction to the central schemes, removing non-physical effects arising from the central discretization of wave propagation phenomena which arises mianly around discontinuities (A sudden change in the propagation direction of certain waves)
- The upwind scheme are defined in function of the signs of the propagation velocities.
- The introduction of second-order non-linear upwind algorithm can control and prevent the appearance of unwanted oscillations (TDV)



u is continuous in D_1 and D_2
Let w(x,t) be a test function which is continuous and has 1‘st continuous derivative.
It vanishs on boundary B.
Then

$$\iint_D (u_t + F_x)\, w(x,t)\, dx dt = 0$$



Integrating by parts

$$\iint_{D_1} (u_t + F_x)\, w dx dt + \iint_{D_2} (u_t + F_x)\, w dx dt + \int_\tau w\left([u]\frac{dt}{ds} + [F]\frac{dx}{ds} \right) ds = 0$$

since $u_t + F_x = 0$ over D1, D2, and w is arbitrary.
Then along the discontinuity surface (x,t)

$$[u]\frac{dt}{ds} + [F]\frac{dx}{ds} = 0$$

or $[u]\cos\alpha_1 + [F]\cos\alpha_2 = 0$ (4-136)

where ${}_1$:along between normal of and t axis
${}_2$:along between normal of and x axis

(4-136) is the condition that u is a weak solution for Burgers‘ equation

(A)Explicit method

(一)Lax Method (1954)

$u_t + F_x = 0$ is the model equation

since $u(x,t+\Delta t) = u(x,t) + \Delta t\left(\dfrac{\partial u}{\partial t}\right)_{x,t} + ...$

Then

$$u(x,t+\Delta t) = u(x,t) + \Delta t(-F_x)_{x,t} + ...$$

ie,

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{\Delta t}{\Delta x}\frac{1}{2}\left(F_{j+1}^n - F_{j-1}^n\right)$$

where $F = u^2/2$ in inviscid Burgers‘ equation

- The amplification factor is G

  $G = \cos\beta - i\dfrac{\Delta t}{\Delta x}u\sin\beta$
- First order accurate
- Stability limit $\left|\dfrac{\Delta t}{\Delta x}u_{max}\right| \le 1$

(二) Lax-Wendroff method (1960)

since $u_t = -F_x$

Then

$$u_{tt} = -\frac{\partial}{\partial t}(F_x) = -\frac{\partial}{\partial x}(F_t)$$
$$=-\frac{\partial}{\partial x}(F_u u_t) = -\frac{\partial}{\partial x}(Au_t)$$
$$=\frac{\partial}{\partial x}(AF_x)$$

Since

$$u(x,t+\Delta t) = u(x,t) + \Delta t\left(\frac{\partial u}{\partial t}\right)_{x,t} + \frac{\Delta t^2}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_{x,t} + ...$$

Then

$$u(x,t+\Delta t) = u(x,t) - \Delta t\frac{\partial F}{\partial x} + \frac{\Delta t^2}{2}\frac{\partial}{\partial x}\left(A\frac{\partial F}{\partial x}\right) + ...$$

or

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left(F_{j+1}^n - F_{j-1}^n\right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left[A_{j+\frac{1}{2}}^n\left(F_{j+1}^n - F_j^n\right) - A_{j-\frac{1}{2}}^n\left(F_j^n - F_{j-1}^n\right)\right]$$

or

$$u_j^{n+1} - u_j^n = -\frac{\Delta t}{\Delta x}\left(F_{j+1/2}^* - F_{j-1/2}^*\right)$$

where $A_{j+1/2}^n = A\left(\dfrac{u_j^n + u_{j+1}^n}{2}\right)$

since in inviscid Burgers‘ equation, $F = u^2/2, A = u$

then

$$A_{j+\frac{1}{2}} = \frac{1}{2}\left(u_j + u_{j+1}\right)$$

$$A_{j-\frac{1}{2}} = \frac{1}{2}\left(u_j + u_{j-1}\right)$$

- First second-order method for hyperbolic equation
- Amplification factor

$$G = 1 - 2\left(\frac{\Delta t}{\Delta x}u\right)^2 \left(1-\cos\beta\right) - 2i\frac{\Delta t}{\Delta x}u\sin\beta$$

- Stability limit $\left|\frac{\Delta \mathrm{t}}{\Delta \mathrm{x}} \mathrm{u}_{\max}\right| \le 1$
- Disperasive nature is evidenced through the presence of oscillations near the discontinuity(Homework)

(三) MacCormack Method

predictor: $\overline{u_j^{n+1}} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1}^n - F_j^n\right)$

corrector: $u_j^{n+1} = \frac{1}{2}\left[u_j^n + \overline{u_j^{n+1}} - \frac{\Delta t}{\Delta x}\left(\overline{F_j^{n+1}} - \overline{F_{j-1}^{n+1}}\right)\right]$

or $\overset{=}{u}_j = u_j^n - \frac{\Delta t}{\Delta x}\left(\overline{F}_j^n - \overline{F}_{j-1}^n\right)$

- Easier to apply then Lax-Wendroff scheme because the Jacobian doesn‘t appear
- Amplification and stability limit are the same as Lax-Wendroff scheme.
- non-linear Lax-Wendroff scheme

  For the nonlinear Burgers‘ equation

$$u_j^{n+1} - u_j^n = -\frac{\Delta t}{\Delta x}\left(F_{j+1/2}^* - F_{j-1/2}^*\right)$$

$$= -\frac{\Delta t}{\Delta x}\left\{\left[F_{i+1/2} - \frac{\Delta t}{2\Delta x}A_{i+1/2}\left(F_{i+1} - F_i\right)\right] - \left[F_{i-1/2} - \frac{\Delta t}{2\Delta x}A_{i-1/2}\left(F_i - F_{i-1}\right)\right]\right\}$$

requires the evaluation of Jacobian $\mathrm{A}_{i\pm1/2}$

by (Harten, 1983)

$$A_{i+1/2} = \frac{F_{i+1} - F_i}{u_{i+1} - u_i} \qquad \text{if } u_{i+1} - u_i \ne 0$$

$$= A\left(u_i\right) \qquad \text{if } u_{i+1} = u_i$$

$$\rightarrow u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left(F_{j+1}^n - F_{j-1}^n\right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left[A_{j+\frac{1}{2}}^2\left(u_{j+1}^n - u_j^n\right) - A_{j-\frac{1}{2}}^2\left(u_j^n - u_{j-1}^n\right)\right]$$

or $u_j^{n+1} - u_j^n = -\frac{\Delta t}{\Delta x}\left(F_{j+1/2}^* - F_{j-1/2}^*\right)$

where the numerical flux $F_{j+1/2}^*$ is

$$F_{j+1/2}^* = F_{j+1/2} - \frac{\Delta t}{2\Delta x} A_{j+1/2}^2 \left(u_{j+1} - u_j\right)$$

- Richtymer two step method and Morton (1967)

  $u_{i+1/2}^{n+1/2} = \frac{1}{2}\left(u_i^n + u_{i+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(F_{i+1}^n - F_i^n\right)$ (First order accuracy)

  (LF scheme)

  $u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^{n+1/2} - F_{i-1/2}^{n+1/2}\right)$ (Second order accuracy)

  (Leap-Forg scheme)

  $O\left(\Delta x^2, \Delta t^2\right)$ at $(i,n+1)$

- MacCormack scheme with artificial dissipation

  Predictor step:

  $$u_j^{\overline{n+1}} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1}^n - F_j^n\right) + \left(\frac{\Delta t}{\Delta x}\right)\left[D_{i+1/2}^n \left(u_{i+1} - u_i\right)^n - D_{i-1/2}^n \left(u_i - u_{i-1}\right)\right]$$

  Corrector step:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_j^{\overline{n+1}} - F_{j-1}^{\overline{n+1}}\right) + \left(\frac{\Delta t}{\Delta x}\right)\left[D_{i+1/2}^{\overline{n+1}} \left(u_{i+1} - u_i\right)^{\overline{n+1}} - D_{i-1/2}^{\overline{n+1}} \left(u_i - u_{i-1}\right)^{\overline{n+1}}\right]$$

  The associated numerical flux becomes

  $$F_{j+1/2}^{*(AV)} = \frac{1}{2}\left(F_{i+1}^n + F_i^{\overline{n+1}}\right) - \frac{1}{2}\left[D_{i+1/2}\left(u_{i+1} - u_i\right) + D_{i+1/2}^{\overline{n+1}}\left(u_{i+1}^{\overline{n+1}} - u_i^{\overline{n+1}}\right)\right]$$

Where Von Neumann-Richtymer artificial viscosity model for D is employed with =1.96.

→Oscillations at the shock are damped out.

- $u_i^{n+1} - u_i^n = -\tau\left(F_{j+1/2}^* - F_{j-1/2}^*\right)$

  where numerical flux $F_{j+1/2}^* = \frac{1}{2}\left(\overline{F_j} + F_{j+1}^n\right)$

- switched differencing in predictor and corrector steps.
- providing good resolution at discontinuities. The best resolution of discontinuities occures when the difference in the predictor is in the direction of propagation of discontinuity.
- High frequency errors generated at discontinuity, indicated by the mass flux error, is typical of all the central second order alogrithm.

  →Requiring the introduction of mechanism to damp out the high frequency error.

(四) Rusanov (Burstein-Mirin) Method

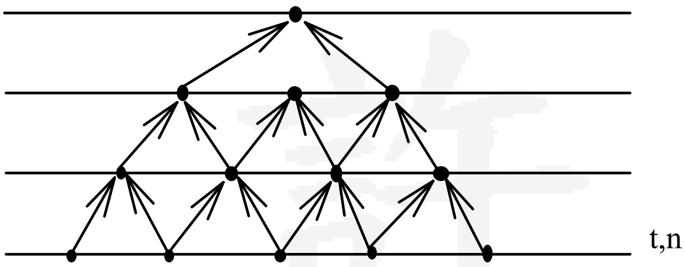
step 1: $u_{j+1/2}^{(1)} = \frac{1}{2}\left(u_{j+1}^n + u_j^n\right) - \frac{1}{3}\frac{\Delta t}{\Delta x}\left(F_{j+1}^n - F_j^n\right)$

step 2: $u_j^{(2)} = u_j^n - \frac{2}{3}\frac{\Delta t}{\Delta x}\left(F_{j+1/2}^{(1)} - F_{j-1/2}^{(1)}\right)$

$$u_j^{n+1} = u_j^n - \frac{1}{24}\frac{\Delta t}{\Delta x}\left(-2F_{j+2}^n + 7F_{j+1}^n - 7F_{j-1}^n + 2F_{j-2}^n\right)$$

step 3: $-\frac{3}{8}\frac{\Delta t}{\Delta x}\left(F_{j+1}^{(2)} - F_{j-1}^{(2)}\right)$

$$\underbrace{-\frac{\omega}{24}\left(u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n\right)}_{\text{explicity added fourth derivative terms for stability}}$$



- Third-order accurate
- Amplification factor

$$G = 1 - \left(\frac{\Delta t}{\Delta x}u\right)^2 \frac{\sin^2\beta}{2} - \frac{\omega}{6}\left(1-\cos\beta\right) + \frac{i\Delta t}{\Delta x}u\sin\beta$$

$$\times\left\{1 + \frac{1}{3}\left(1-\cos\beta\right)\left[1 - \left(\frac{\Delta t}{\Delta x}u\right)^2\right]\right\}$$

- Stability limit $|\nu| \le 1$ or $\left|\frac{\Delta t}{\Delta x}u_{max}\right| \le 1$

$$4\nu^2 - \nu^4 \le \omega \le 3$$

- Overshot exists on both sides of discontinuity.

(五) WKL Method (1973)

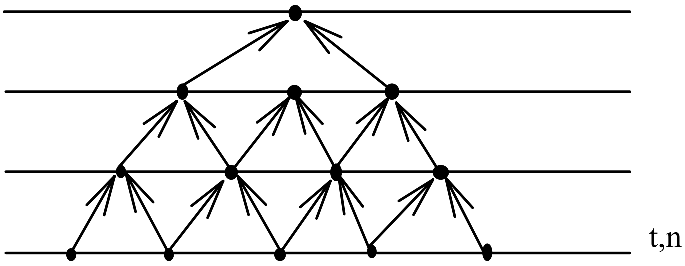
step 1: $u_j^{(1)} = u_j^n - \frac{2}{3}\frac{\Delta t}{\Delta x}\left(F_{j+1}^n - F_j^n\right)$

step 2: $u_j^{(2)} = \frac{1}{2}\left[u_j^n + u_j^{(n)} - \frac{2}{3}\frac{\Delta t}{\Delta x}\left(F_j^{(1)} - F_{j-1}^{(1)}\right)\right]$

$$u_j^{n+1} = u_j^n - \frac{1}{24}\frac{\Delta t}{\Delta x}\left(-2F_{j+2}^n + 7F_{j+1}^n - 7F_{j-1}^n + 2F_{j-2}^n\right)$$

step 3: $-\frac{3}{8}\frac{\Delta t}{\Delta x}\left(F_{j+1}^{(2)} - F_{j-1}^{(2)}\right)$

$$\underbrace{-\frac{\omega}{24}\left(u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n\right)}_{\text{explicity added to control stability}}$$



- Third-order accurate
- First two levels are employed by MacCormack method
- Advantage over Rusanov technique is that only values at integral mesh points are evaluated.
- Same stability limit as Rusanov method

(六) Tuned Third-order method

The parameter in (四), (五) and explicit artificial terms are replaced by

$$-\frac{\omega_{j+1/2}^{n}}{24}\left(u_{j+2}^{n}-3u_{j+1}^{n}+3u_{j}^{n}-u_{j-1}^{n}\right)+\frac{\omega_{j-1/2}^{n}}{24}\left(u_{j+1}^{n}-3u_{j}^{n}+3u_{j-1}^{n}-u_{j-2}^{n}\right)$$

where $\omega_{j\pm1/2}^{n}$ are chosen to min the dissipative or dispersive errors

$$\omega_{j\pm1/2}^{n}=\left(4\nu_{j\pm1/2}^{2}+1\right)\left(4-\nu_{j\pm1/2}^{2}\right)\Big/5$$

The effective Courant numbers are

$$\nu_{j+1/2}=\frac{1}{4}\left(\lambda_{j+2}+\lambda_{j+1}+\lambda_{j}+\lambda_{j-1}\right)\Delta t\Big/\Delta x$$

$$\nu_{j-1/2}=\frac{1}{4}\left(\lambda_{j+1}+\lambda_{j}+\lambda_{j-1}+\lambda_{j-2}\right)\Delta t\Big/\Delta x$$

and is the local eigenvalue

(B) Implicit method

(一) Time-centered implicit method (trapezoidal method) (Beam and Warming 1976)

$$u_t+F_x=0$$

since from (4-58)

$$u_j^{n+1}=u_j^{n}+\frac{\Delta t}{2}\left[\left(u_t\right)^{n}+\left(u_t\right)^{n+1}\right]_j+o\left[\left(\Delta t\right)^{3}\right]$$

then

$$u_j^{n+1}=u_j^{n}-\frac{\Delta t}{2}\left[\left(\frac{\partial F}{\partial x}\right)^{n}+\left(\frac{\partial F}{\partial x}\right)^{n+1}\right]$$

since F=F(u)

Beam and Warming (1976) suggested

$$F^{n+1}\approx F^{n}+\left(\frac{\partial F}{\partial u}\right)^{n}\left(u^{n+1}-u^{n}\right)=F^{n}+A^{n}\left(u^{n+1}-u^{n}\right)$$

Thus

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2}\left\{2\left(\frac{\partial F}{\partial x}\right)^n + \frac{\partial}{\partial x}\left[A\left(u_j^{n+1} - u_j^n\right)\right]\right\}$$

If x derivatives are replaced by second-order central differences,

$$\rightarrow \left(-\frac{\Delta t A_{j-1}^n}{4\Delta x}\right)u_{j-1}^{n+1} + u_j^{n+1} + \left(\frac{\Delta t A_{j+1}^n}{4\Delta x}\right)u_{j+1}^{n+1}$$

$$= -\frac{\Delta t}{2\Delta x}\left(F_{j+1}^n - F_{j-1}^n\right) - \frac{\Delta t A_{j-1}^n}{4\Delta x}u_{j-1}^n + u_j^n + \frac{\Delta t A_{j+1}^n}{4\Delta x}u_{j+1}^n$$

The tridiagonal system is solved by Thomas algorithm

- Explicit Damping

$$-\frac{\omega}{8}\left(u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n\right) \;,\quad 0 < \omega \le 1$$

is added since there is no even derivative term in the modified function.

The algorithm can be written in the delta form as：

Let $\Delta u_j = u_j^{n+1} - u_j^n$

Then $\Delta u_j = -\frac{\Delta t}{2}\left[\left(\frac{\partial F}{\partial x}\right)^n + \left(\frac{\partial F}{\partial x}\right)^{n+1}\right]$

Local linearization for F is

$$F_j^{n+1} = F_j^n + A_j^n \Delta u_j$$

$$\rightarrow \left(-\frac{\Delta t A_{j-1}^n}{4\Delta x}\right)\Delta u_{j-1} + \Delta u_j + \left(\frac{\Delta t A_{j+1}^n}{4\Delta x}\right)\Delta u_{j+1} = -\frac{\Delta t}{2\Delta x}\left(F_{j+1}^n - F_{j-1}^n\right)$$

- Simpler
- Tridiagonal coefficient matrix, the R.H.S. doesn‘t require the multiplication of the original algorithm

(二)Euler Implicit method (Beam and Warming, 1976)

$$u^{n+1} = u^n + \Delta t\left(\frac{\partial u}{\partial t}\right)^{n+1}$$

$$\rightarrow u^{n+1} = u^n - \Delta t\left(\frac{\partial F}{\partial x}\right)^{n+1}$$

If the same linearization is applied, then

$$\left(-\frac{\Delta t A_{j-1}^n}{2\Delta x}\right)u_{j-1}^{n+1} + u_j^{n+1} + \left(\frac{\Delta t A_{j+1}^n}{2\Delta x}\right)u_{j+1}^{n+1}$$

$$= -\frac{\Delta t}{2\Delta x}\left(F_{j+1}^n - F_{j-1}^n\right) - \left(\frac{\Delta t A_{j-1}^n}{2\Delta x}\right)u_{j-1}^n + u_j^n + \left(\frac{\Delta t A_{j+1}^n}{2\Delta x}\right)u_{j+1}^n$$

- Tridiagonal system of coefficient matrix

- Unconditional stable
- Explicit damping is added to insure the usable result.

Conclusion

For inviscid Burgers equation

- Implicit method is inferior to explicit method
  (1)more computation required per time step in implicit method
  (2)transient is usually desired
  (3)when discontinuities are present, explicit methods are superior to those of implicit methods using central differences
- Explicit MacCormack‘s scheme is recommended.