#### **Chapter Three** Cloud Microphysics

"... fleecy plies dissolved in dew drops..."42

As noted in the Introduction, cloud physics consist of two branches: cloud microphysics and cloud dynamic. While the topic of this book is the latter, it is impossible to divorce a discussion of dynamics from a-knowledge of the microphysics. Just as the discussions in Chapters 5-12 assume a certain level of background knowledge of atmospheric dynamics, so do they assume some background in cloud microphysics. To provide this background, the present chapter summarizes the aspects of cloud microphysics that are crucial to the discussions of later chapters. First, we describe some of the basic microphysical process that are involved in the formation, growth, shrinkage, breakup, and fallout of cloud and precipitation particles.<sup>43</sup> In sec. 3.1, we describe the microphysics of warm cloud, where the temperature is everywhere above 0°C. Section 3.2 ex-tends the review of microphysical processes to cold clouds, in which the temperature drops below 0°C and both ice and liquid particles may exist. After this review of the individual microphysical processes that may occur in clouds, we consider in Secs. 3.3-3.6 how these microphysical processes occur simultaneously in a real cloud and how they may be linked to the cloud dynamics through a set of water-continuity equations.

### **3.1** Microphysics of Warm Clouds

#### **3.1.1** Nucleation of Drops

The particles in a cloud form by a process referred to as *nucleation*, in which water molecules change from a less ordered to a more ordered state. For example, vapor molecules in the air may come together by chance collisions to form a liquid-phase drop. To see how this process takes place, consider the conditions required for the formation of a drop of pure water from vapor. This case is called homogeneous nucleation to distinguish it from the case of heterogeneous nucleation, which refers to the collection of molecules onto a

<sup>&</sup>lt;sup>42</sup> Goethe realizes that clouds are composed of microscopic particles.

<sup>&</sup>lt;sup>43</sup> these microphysical processes are described in more detail in basic texts on cloud microphysics, such as Fletcher(1966), Mason(1971), Pruppacher and Klett (1978), and Rogers and Yau (1989). The physics of ice is presented comprehensively by Hobbs(1974).

foreign substance. If the embryonic drop of pure water has radius R, then the net energy required to accomplish its nucleation is

$$\Delta E = 4\pi R^2 \sigma_{\nu l} - \frac{4}{3}\pi R^3 n_l (\mu_{\nu} - \mu_l)$$
(3.1)

The first term on the right is the work required to create a surface of vapor-liquid interface around the drop. The factor  $\sigma_{\nu l}$  is the work required to create a unit area of the interface. It is called the surface energy or surface tension. The second term on the right of (3.1) is the energy change associated with the vapor molecules going into the liquid phase. It is expressed as the change in the Gibbs free energy of the system. The Gibbs free energy of a single vapor molecule is  $\mu_{\nu}$ , while that of a liquid molecule is  $\mu_{l}$ , and the factor  $n_l$  is the number of water molecules per unit volume in the drop. If the work required to create the surface exceeds the change in Gibbs free energy ( $\Delta E > 0$ ), the embryonic drop formed by chance aggregation of molecules has no chance of surviving and immediately evaporates. If, on the other hand, the work required to create the surface is less than the change in Gibbs free energy ( $\Delta E < 0$ ), then the drop survives and is said to have been nucleated.

It can be shown<sup>44</sup> that

$$\mu_{v} - \mu_{l} = k_{B}T \ln \frac{e}{e_{s}}$$
(3.2)

where  $k_B$  is Boltzmann's constant, *e* is the vapor pressure, and  $e_s$  is the saturation vapor pressure over a plane surface of water. Substituting this expression into (3.1), seeking the condition for which the work required to change the drop's surface is exactly matched by the change in Gibbs free energy ( $\Delta E=0$ ), and rearranging terms, we obtain an expression for the critical radius  $R_c$  at which this equilibrium condition holds. This expression is

$$R_c = \frac{2\sigma_{vl}}{n_l k_B T \ln(e/e_s)}$$
(3.3)

and is referred to as *Kelvin's formula*.<sup>45</sup> This radius is evidently crucially dependent on the *relative humidity*( defined as  $e/e_s \times 100\%$ ). Air is said to be *saturated* whenever the relative humidity is 100%(  $e/e_s=1$ ). However, it is clear from (3.3) that it is impossible for a cloud droplet to form under saturated <sup>44</sup> See problem 2.19 of Wallace and Hobbs (1977).

<sup>&</sup>lt;sup>45</sup> Named after Lord Kelvin, who first derived it.

conditions since  $R_c$  as  $e/e_s$  1. Rather, the air must be supersaturated  $(e/e_s>1)$  for  $R_c$  to be positive. The greater the *supersaturation* [defined, in percent, as  $[(e/e_s - 1)\times100\%]$ , the smaller the size of the drop that must be exceeded by the initial chance collection of molecules.

It should be noted that  $R_c$  is also a function of temperature. Not only does T appear in the denominator of (3.3) explicitly, but  $\sigma_{vl}$  and  $e_s$  are functions of T. However, at atmospheric temperatures, the dependence of  $R_c$  on temperature is comparatively weak. In view of the primary dependence of  $R_c$  on ambient humidity, it is not surprising that the rate of nucleation of drops exceeding the critical size  $R_c$  is a strong function of the degree of supersaturation.



Figure 3.1 A spherical-cap embryo of liquid (L) in contact with its vapor (V) and a nucleating surface (C). (From Fletcher, 1966. Reprinted with permission from Cambridge University Press.)

The rate at which the vapor molecules collide to form aggregates of various sizes can be computed using principles of statistical quantum mechanics applied to an ideal gas whose molecules are in a state of random motion.<sup>46</sup> This rate of formation of drops exceeding the critical size is the *nucleation rate*. It is found to increase from undetectably small values to extremely large values over a very narrow range of  $e/e_s$ . The value of  $e/e_s$  at which this rise occurs is in the range of 4-5. Thus, the air must be supersaturated by 300-400% for a drop of pure water to be nucleated homogeneously. Since supersaturation in the atmosphere seldom exceeds 1%, one concludes that homogeneous nucleation of water drops plays no role in natural clouds. However, the physics of the process are nonetheless relevant, as will become evident below. Heterogeneous nucleation is the process whereby cloud drops actually form. The atmosphere is filled with small *aerosol particles*, and molecules of vapor may collect onto the surface of aerosol particles as illustrated ideally in Fig. 3.1. If the surface

<sup>&</sup>lt;sup>46</sup> See Chanter 2 of Fletcher.(1996)

tension between the water and the nucleating surface is sufficiently low, the nucleus is said to be *wettable*, and the water may form a spherical cap on the surface of the particle. A particle onto which the molecules collect in this manner is referred to as a cloud condensation nucleus (CCN).

If a CCN is insoluble in water, the physics governing the survival of an embryonic cloud droplet are the same as in the case of homogeneous nucleation. It can be shown that Eq. (3.3) still applies, but  $R_c$  has the more general interpretation in that it refers to the critical radius of *curvature* of the embryonic drop. Since the radius of curvature of the droplet forming on a particle is greater than what it would be if the same number of molecules were to aggregate in the absence of the particle (Fig. 3.1), the aggregation of the vapor molecules has a greater chance of producing a drop exceeding the critical radius. If the aggregated water molecules form a film of liquid completely surrounding a particle, then a complete droplet is formed whose radius is larger than it would be in the absence of the nucleus. Clearly, the larger such a nucleus is, the more likely is the survival of a drop formed by a film around it. For this reason, the larger the aerosol particle, the more likely it is to be a site for drop formation in a natural cloud.

If the cloud condensation nucleus happens to be composed of a material that is soluble in water, the efficacy of the nucleation process is further enhanced. Since the saturation vapor pressure over the liquid solution is generally lower than that over a surface of pure water,  $e/e_s$  is increased. According to (3.3), the critical radius is then reduced, and nucleation is easier to achieve at the ambient vapor pressure.

There are generally more than enough wettable aerosol particles in the air to accommodate the formation of all cloud droplets. However, the physics of the nucleation process just described indicate that the first droplets in a cloud will tend to form around the largest and most soluble CCN. The sizes and compositions of the aerosol particles in a sample of air thus have a profound effect on the size distribution of particles nucleated in a cloud.

#### **3.1.2** Condensation and Evaporation

Once formed, water drops may continue to grow as vapor diffuses toward them. This process is called *condensation*. The reverse process, drops decreasing in size as vapor diffuses away from them, is called *evaporation*. Particle growth by condensation and evaporation may be represented quantitatively by assuming that the flux of water vapor molecules through air is proportional to the gradient of the concentration of vapor molecules.<sup>47</sup> In this case, the vapor density  $\rho_v$  (defined as the mass of vapor per unit volume of air) is governed by the diffusion equation

$$\frac{\partial \rho_{\nu}}{\partial t} = \nabla \cdot \left( D_{\nu} \nabla \rho_{\nu} \right) = D_{\nu} \nabla^{2} \rho_{\nu}$$
(3.4)

where  $D_{\nu}\Delta\rho_{\nu}$  is the flux of water vapor by molecular diffusion and  $D_{\nu}$  is the diffusion coefficient (assumed constant) for water vapor in air. The concentration of vapor around a spherical pure-water drop of radius *R* is assumed to be symmetric about a point located at the center of the drop, and the diffusion is assumed to be in a steady state. Under these assumptions,  $\rho_{\nu}$  depends only on radial distance *r* from the center of the drop, and (3.4) reduces to

$$\nabla^2 \rho_v(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \rho_v}{\partial r} \right) = 0$$
(3.5)

The vapor density at the surface is  $\rho_{\nu}(R)$ . As r, the vapor density approaches the ambient or free-air value  $\rho_{\nu}($ ). The solution to (3.5) satisfying these boundary conditions is

$$\rho_{\nu}(r) = \rho_{\nu}(\infty) - \frac{R}{r} [\rho_{\nu}(\infty) - \rho_{\nu}(R)]$$
(3.6)

If the drop has mass m, the flux of molecules causes its mass to increase or decrease at a rate given by

$$\mathfrak{m}_{\mathrm{dif}} = 4\pi R^2 D_{\nu} \frac{d\rho_{\nu}}{dr} |_{R}$$
(3.7)

<sup>&</sup>lt;sup>47</sup> This assumption is called Fick's first law of diffusion.

where  $D_v d\rho_v / dr_R$  is the flux of vapor in the radial direction across a spherical surface of radius *R*. Substitution of (3.6) into (3.7) yields

$$\dot{\mathfrak{m}}_{\rm dif} = 4\pi R D_{\nu} [\rho_{\nu}(\infty) - \rho_{\nu}(R)]$$
(3.8)

Since  $m \propto R^3$ , there are two unknowns in (3.8),  $\rho_v(R)$  and either *m* or *R*. Conditions in the environment (r = -) are assumed to be known. To obtain a solution for m or *R*, other relationships are needed. First, a heat-balance equation is introduced. In the condensation of water vapor on a drop, latent heat is released at a rate  $L\dot{m}_{dif}$ , where *L* is the latent heat of vaporization. Assuming that heat is conducted away from the drop as rapidly as it is being released, we have by analogy to (3.8)

$$L\dot{\mathbf{m}}_{\rm cif} = 4\pi\kappa_a R[T(R) - T(\infty)]$$
(3.9)

where  $\kappa_a$  is the thermal conductivity of air and *T* is temperature.

The equation of state for an ideal gas applied to water vapor under saturated conditions over a plane surface of pure water is

$$e_s = \rho_{vs} R_v T \tag{3.10}$$

where  $R_v$  is the gas constant for a unit mass of water vapor, and  $e_s$  and  $\rho_{vs}$  are the saturation vapor pressure and density over a planar surface of water. Since  $e_s$  depends only on temperature,<sup>48</sup> it is evident from (3.10) that  $\rho_{vs}$  is a known function of *T*. If it is then assumed that the vapor density at the drop's surface is given by the saturation vapor density, we may write

$$\rho_{\nu}(R) = \rho_{\nu s}[T(R)] \tag{3.11}$$

and (3.8), (3.9), and (3.11) can be solved numerically for  $m_{\text{dif}}$ , T(R), and  $\rho_v(R)$ . These equations can, moreover, be combined analytically for the special case of a drop growing or evaporating in a saturated environment (i.e., for the case in which  $e() = e_s[T()]$ ). In this special case, use is made of the Clausius-Clapeyron equation:<sup>49</sup>

$$\frac{1}{e_s}\frac{de_s}{dT} \cong \frac{L}{R_v T^2} \tag{3.12}$$

<sup>&</sup>lt;sup>48</sup> See pp.72-73 of Wallace and Hobbs (1977).

<sup>&</sup>lt;sup>49</sup> See p.95 of Wallace and Hobbs (1977).

Combination of (3.10) and (3.12) yields

$$\frac{d\rho_{vs}}{\rho_{vs}} = \frac{L}{R_v} \frac{dT}{T^2} - \frac{dT}{T}$$
(3.13)

Then (3.8), (3.9), (3.11), and (3.13) may be combined<sup>50</sup> under saturated environmental conditions to obtain

$$\dot{\mathfrak{m}}_{dif} = \frac{4\pi R \tilde{S}}{F_K + F_D} \tag{3.14}$$

where  $\tilde{S}$  depends on the humidity of the environment,  $F_K$  on the heat conductivity, and  $F_D$  on the vapor diffusivity. More specifically,  $\tilde{S}$  is the ambient supersaturation (expressed as a fraction):

$$\widetilde{S} \equiv \frac{e(\infty)}{e_s(\infty)} - 1 \tag{3.15}$$

The other factors are given by

$$F_{K} \equiv \frac{L^{2}}{\kappa_{a} R_{v} T^{2}(\infty)}$$
(3.16)

and

$$F_D = \frac{R_v T(\infty)}{D_v e_s(\infty)} \tag{3.17}$$

From (3.14)-(3.17), it is evident that the diffusional growth rate of a drop depends on the temperature and humidity of the environment and on the radius of the drop.

The relation (3.11) used in deriving (3.14) assumes that saturation at the drop's surface may be approximated as if it obtained over a plane surface of water (i.e., that the growing drop were large enough for the curvature of the drop's surface to have negligible influence upon the equilibrium vapor pressure). The drop has also been assumed to be sufficiently dilute with respect to dissolved nuclei or other impurities that the drop may be regarded as being composed of pure water. For very small drops, however, curvature and solution effects must be included. If a drop is growing on a water-soluble nucleus,  $\rho_v(R)$  becomes

$$\rho_{\nu}(R) = \rho_{\nu s}\left[T(R)\right] \left(1 + \frac{\hat{a}}{R} - \frac{\hat{b}}{R^3}\right)$$
(3.18)

<sup>50</sup> See pp.99-102 of Rogers and Yau (1989) for details of the derivation.

where the term  $\hat{a}/R$  represents the effect of drop curvature on the equilibrium vapor pressure above the drop. The factor  $\hat{a}$  is given by

$$\hat{a} = \frac{2\sigma_{vl}}{\rho_L R_v T} \tag{3.19}$$

where  $\sigma_{\nu l}$  is the surface tension of liquid-vapor interface and  $\rho_L$  is the density of liquid water. The term  $\hat{b}/R$  represents the effect of salt dissolved in the drop on the equilibrium vapor pressure above the drop. The factor  $\hat{b}$  is given by

$$\hat{b} = \frac{3i_{\rm vH}m_sM_w}{4\pi\rho_L M_s} \tag{3.20}$$

where  $i_{vH}$  is the van't Hoff factor,<sup>51</sup>  $m_s$  and  $M_s$  are the mass and molecular weight of the dissolved salt, respectively, and  $M_w$  is the molecular weight of water.

Replacing (3.11) with (3.18) leads, following steps similar to those leading to (3.14), to the equation

$$\dot{\mathsf{m}}_{\mathsf{dif}} = \frac{4\pi R}{F_{\kappa} + F_D} \left( \widetilde{S} - \frac{\hat{a}}{R} + \frac{\hat{b}}{R^3} \right)$$
(3.21)

which applies when the air is saturated. When the air is unsaturated, (3.8), (3.9), and (3.18) must be solved numerically to obtain  $\dot{m}_{dif}$  for the evaporation rate of the drop.

When drops are falling relative to the surrounding air, the diffusion of vapor and heat is altered. To account for this process, the right-hand sides of (3.8) and (3.9) may be multiplied by a ventilation factor  $V_F$ . In this case, (3.14) and (3.21), the growth/evaporation rate under saturated conditions become

$$\dot{\mathsf{m}}_{\mathsf{dif}} = \frac{4\pi R V_F \hat{S}}{F_\kappa + F_D} \tag{3.22}$$

and

$$\dot{\mathsf{m}}_{\mathsf{dif}} = \frac{4\pi R V_F \widetilde{S}}{F_{\kappa} + F_D} \left( \widetilde{S} - \frac{\hat{a}}{R} + \frac{\hat{b}}{R^3} \right)$$
(3.23)

respectively.<sup>52</sup>

<sup>51</sup> This factor is equal to the number of ions into which each molecule of salt dissociates. See p. 162 of Wallace and Hobbs (1977).

<sup>&</sup>lt;sup>52</sup> See pp. 440-463 of Pruppacher and Klett (1978).

#### **3.1.3** Fall Speeds of Drops

Growing cloud droplets are subject to downward gravitational force. This force can lead to their fallout as precipitation particles. The gravitational force on a drop is, however, largely offset by the frictional resistance of the air. As a particle is accelerated downward by gravity, its motion is increasingly retarded by the growing frictional force. Its final speed is called the terminal fall speed V. For drops of water in air, V is a function of the drop radius R. Generally V is negligible until the drops reach a radius of about 0.1 mm. This is usually considered to be the threshold size separating *cloud droplets*, which are suspended in the air indefinitely, from falling *precipitation* drops. The smallest precipitation drops (taken by con vention<sup>53</sup> to be those 0.1-0.25 mm in radius) are called drizzle. Drops >0.25 mm in radius are called rain. Drizz1e and raindrops have terminal fall speeds that increase with increasing drop radius. We will represent this function as V(R). For drops <500  $\mu$  m in radius, V increases approximately linearly with increasing drop radius (Fig. 3.2). For larger drops, V(R) increases at a lower rate (Fig. 3.3), becoming a constant at a radius of about 3 mm. This asymptotic behavior is associated with the fact that a drop becomes increasingly flattened, into the shape of a horizontally oriented disc, at larger sizes (see Fig. 4.2).



Figure 3.2 Fall velocity of water drops  $< 500 \ \mu m$  in radius for various atmospheric conditions. (From Beard and Pruppacher, 1969. Reprinted with permission from the American Meteorological Society.)

<sup>&</sup>lt;sup>53</sup> See the Glossary of Meteorology (Huschke, 1959).



Figure 3.3 Fall velocity of water drops  $> 500 \ \mu m$  in radius. (From Beard, 1976. Reprinted with permission from the American Meteorological Society.)

#### **3.1.4 Coalescence 3.1.4.1 Continuous Collection**

Cloud drop growth by coalescence with other drops can be envisioned in terms of a drop of mass m falling through a cloud of particles of mass m'. The water contained in the particles of mass m' is assumed to be distributed uniformly through the cloud with liquid water content  $\rho q_{m'}$  (g m<sup>-3</sup>), where  $q_{m'}$  is the cloud water mixing ratio (mass of cloud water per mass of air). As it falls, the particle of mass m is assumed to increase in mass continually at a rate given by the continuous collection equation,

$$\dot{\mathbf{m}}_{col} = A_{\mathbf{m}} | V(\mathbf{m}) - V(\mathbf{m}') | \rho q_{\mathbf{m}'} \sum_{c} (\mathbf{m}, \mathbf{m}')$$
(3.24)

where V represents the fall speed of the drops of masses m and m' (Figs. 3.2 and 3.3),  $\rho$  is the density of the air,  $\Sigma_c(m, m')$  is *the collection efficiency*, and  $A_m$  is the effective cross-sectional area swept out by a particle of mass m. The absolute value notation is used in (3.24) since it is only the relative motion of the particles that matters for collectional growth. For the case of a large drop collecting smaller drops, the absolute value symbol is redundant since the fall velocity of the larger drop always exceeds that of the smaller drops. However, (3.24) may also be used to calculate the increase of mass of a smaller drop coalescing with larger drops. If the absolute value were not used in that case, negative growth would be calculated. Moreover, as will be seen below, (3.24) is applied also to cold clouds where in some special cases (e.g., an ice particle collecting water drops) the fall velocity of the larger particle may not be the greater of the two.

For the purpose of calculating collectional growth, water drops are usually assumed to be spherical. In that case, the factor  $A_m$  in (3.24) is given by

$$A_{\rm m} = \pi \left( R + R' \right)^2 \tag{3.25}$$

where *R* and *R'*, are the radii of drops of mass m and m', respectively. This area is based on the sum of the drop radii since any drop centered within a distance R + R' of the center of the drop of radius *R* can be intercepted by that drop.



Figure 3.4 Collision efficiency for collector drops of radius R1 with droplets of radius R2. The dashed portions of the curve represent regions of doubtful accuracy. (From Wallace and Hobbs, 1977.)

The collection efficiency is the efficiency  $\Sigma_c(m, m')$  is the efficiency with which a drop intercepts and unites with the drops it overtakes. It is the product of a collision efficiency and a coalescence efficiency. The collision efficiency (Fig. 3.4) is determined primarily by the relative airflow around the falling drop. Smaller particles may be carried out of the path of a larger particle (efficiency < 1), or small particles not in the direct path of a large particle may collide with the large particle if they are pulled into its wake (efficiency > 1). The coalescence efficiency expresses the fact that a collision between two drops does not guarantee coalescence; the drops may bounce off each other or remain united only temporarily. Under most conditions, coalescence efficiency is high, especially if the droplets are electrically charged or if an electric field is present. The electrical conditions are often met in clouds, and little else is known about the coalescence efficiency. Hence, the most common practice in theoretical or modeling studies is to assume a coalescence efficiency of unity. The collection efficiency then reduces to the collision efficiency.

A more general version of (3.24) may be written for the case in which a particle of mass m is falling relative to a population of particles of varying size. For that case, the generalized continuous collection equation is

(3.26) 
$$\dot{m}_{col} = \int_0^\infty A_m |V(m) - Vm'| m' N(m') \sum_c (m, m') dm'$$

where N (m') dm' is the number of particles per unit volume of air in the size range m' to m' + dm'.

#### 3.1.4.2 Stochastic Collection

Cloud drop growth by coalescence is actually not a continuous process, as assumed in (3.24), but rather proceeds in a discrete, stepwise, probabilistic manner. In a time interval  $\Delta t$  drops of a given initial size do not grow uniformly. Some may undergo more than the average number of collisions and thus grow faster than others. Consequently, a drop size distribution develops.

The probabilistic nature of collection may be accounted for by considering the size distribution N(m,t), where N(m,t) dm is the number of particles per unit volume of air in mass range m to m + dm at time t. The change in N(m,t) with time is computed as follows. The rate at which the space within which a particle of mass m' is located is swept out by a particle of mass m is given by the *collection kernel*, defined as

$$\hat{K}(\mathsf{m},\mathsf{m}') \equiv A_{\mathsf{m}} | V(\mathsf{m}) - V(\mathsf{m}') | \sum_{c} (\mathsf{m},\mathsf{m}')$$
(3.27)

The probability that a particular drop of mass m will collect a drop of mass m', in time interval  $\Delta t$  is

$$\hat{P} \equiv N(\mathbf{m}', t) d\mathbf{m}' \hat{K} \Delta t \tag{3.28}$$

where it is assumed that  $\Delta t$  is small enough that the probability of more than one collection in this time is negligible. Making use of (3.27) and (3.28), we note that the mean number of drops of mass m that will collect drops of mass m' at time  $\Delta t$  is

$$\hat{P}N(\mathsf{m}',t)d\mathsf{m} = \hat{K}(\mathsf{m},\mathsf{m}')N(\mathsf{m}',t)N(\mathsf{m},t)d\mathsf{m}d\mathsf{m}'\Delta t \qquad (3.29)$$

Rearranging this expression we obtain

$$\frac{\hat{P}N(\mathsf{m},t)}{\Delta t} = \hat{K}(\mathsf{m},\mathsf{m}')N(\mathsf{m}',t)N(\mathsf{m},t)d\mathsf{m}'$$
(3.30)

which expresses the rate at which the number of drops of mass m is reduced as a result of coalescence with drops of mass m' per unit volume of air. It follows that the rate of decrease of the number concentration of drops of mass m as a result of their coalescence with drops of all other sizes is given by the integral

$$I_{1}(\mathbf{m}) = \frac{1}{2} \int_{0}^{\mathbf{m}} \hat{K}(\mathbf{m}, \mathbf{m}') N(\mathbf{m}', t) N(\mathbf{m}, t) d\mathbf{m}'$$
(3.31)

By reasoning similar to that given above we may express the rate of generation of drops of mass m by coalescence of smaller drops as

$$I_{2}(\mathbf{m}) = \frac{1}{2} \int_{0}^{\mathbf{m}} \hat{K}(\mathbf{m} - \mathbf{m}', \mathbf{m}') N(\mathbf{m} - \mathbf{m}', t) N(\mathbf{m}', t) d\mathbf{m}'$$
(3.32)

where the factor of 1/2 is included to avoid counting each collision twice. The net rate of change in the number density of drops of mass m is obtained by subtracting (3.32) from (3.31) and may be written as

$$\left(\frac{\partial N(\mathbf{m},t)}{\partial t}\right)_{col} = I_2(\mathbf{m}) - I_1(\mathbf{m})$$
(3.33)

This result is referred to as the stochastic collection equation.

Computations may be made with (3.33) starting with some arbitrary initial drop size distribution N(m,0). The result obtained by integrating (3.33) over time yields the drop size distribution altered by the stochastic collection process. In addition to the initial distribution, one must also assume reasonable values of the collection efficiencies and fall velocities appearing in (3.31) and (3.32). For realistic conditions, it is generally found that a large portion of the liquid water accumulates in the tail of the distribution. An example of such a calculation is shown in Fig. 3.5. The drop size distribution at successive times is plotted as mass distribution  $g_m = mN(m)$ , rather than number distribution N(m), so that the area under each curve is proportional to the total liquid water content in the distribution. The mass distribution is plotted versus the radius of a drop of mass m on a logarithmic scale. This plotting convention emphasizes the result that a large portion of the liquid water becomes concentrated in the large drops as time progresses. The two peaks in the mass distribution after 30 min correspond to the amount of water contained in cloud droplets (radii  $\sim 10^{-3}$  cm) and raindrops (radii  $\sim 10^{-1}$  cm). The two dashed lines following the centers of the two peaks correspond to the means of the number and mass concentrations. The mean of the number distribution follows the cloud droplet peak. This result illustrates that the cloud droplets are far more numerous than the raindrops but that the latter nonetheless contain a large part of the liquid water after half an hour of stochastic collection. Stochastic collection can thus quickly convert cloud water to rainwater.



Figure 3.5 Example of the evolution of a drop size distribution as a result of stochastic collection.  $g_m$  is the mass distribution function; R is the drop radius.

The two dashed lines show the radii  $(R_n \text{ and } R_g)$  corresponding to the means of the number and mass distributions, respectively. (From Berry and Reinhardt,

1973. Reprinted with permission from the American Meteorological Society.)



Figure 3.6 The probability  $P_B(m)$  that a drop of radius *R* breaks up per unit time. Based on empirical foumula of Srivastava (1971).

#### 3.1.5 Breakup of Drops

When raindrops achieve a certain size, they become unstable and break up into smaller drops. Breakup has been studied in the laboratory, and empirical functions based on the experimental data are used to describe breakup quantitatively.<sup>54</sup> One empirical function is the probability  $P_B(\mathbf{m})$  that a drop of mass  $\mathbf{m}$  breaks up per unit time. It is nearly zero for drops less than about 3.5 mm and increases exponentially with size for radii greater than this value (Fig. 3.6). The function shown in the plot is

$$P_B(\mathbf{m}) = 2.94 \times 10^{-7} \exp(3.4R) \tag{3.34}$$

where *R* is the radius in millimeters of a drop of mass m and  $P_B(m)$  is in s<sup>-1</sup>. A second empirical function is  $Q_B(m',m)$ , which is defined such that  $Q_B(m',m) dm$ 

is the number of drops of mass m to m + dm formed by the breakup of one drop of mass m'.  $Q_B(m',m)$  is approximately exponential. It is given by

$$Q_B(\mathsf{m}',\mathsf{m}) = 0.1R'^3 \exp(-15.6R)$$
 (3.35)

<sup>54</sup> The formulation of breakup presented in this subsection was developed by Srivastava (1971).

where the radii are in cm. The empirical functions  $P_B(m)$  and  $Q_B(m',m)$  can be used to determine the net effect of breakup on the drop size distribution N(m,t). The net rate of production of drops of mass m by breakup implied by these functions is

$$\left(\frac{\partial N(\mathbf{m},t)}{\partial t}\right) = -N(\mathbf{m},t)P_B(\mathbf{m}) + \int_m^\infty N(\mathbf{m}',t)Q_B(\mathbf{m}',\mathbf{m})P_B(\mathbf{m}')d\mathbf{m}'$$
(3.36)

## 3.2 Microphysics of Cold Clouds3.2.1 Homogeneous Nucleation of Ice Particles

Ice particles in clouds may be nucleated from either the liquid or vapor phase. Homogeneous nucleation of ice from the liquid phase is analogous to nucleation of drops from the vapor phase. An embryonic ice particle can be considered a polyhedron of volume  $\alpha_i 4\pi R^3/3$  and surface area  $\beta_i 4\pi R^2$ , where *R* is the radius of a sphere that can just be contained within the polyhedron, and  $\alpha_i$ and  $\beta_i$  are both greater than unity but approach unity as the polyhedron tends toward a spherical shape. By reasoning analogous to that leading to (3.3), the expression for the critical radius  $R_{ci}$  of the inscribed sphere is

$$R_{ci} = \frac{2\beta_i \sigma_{il}}{\alpha_i n_i k_B T \ln(e_s/e_{si})}$$
(3.37)

where  $\sigma_{il}$  is the free energy of an ice-liquid interface,  $n_i$  is the number of molecules per unit volume of ice, and  $e_{si}$  is the saturation vapor pressure with respect to a plane surface of ice. The saturation vapor pressures of liquid and

ice in the denominator and the free energy in the numerator are all functions of temperature. The critical radius is thus a function of temperature.

<sup>55</sup> Larger drops freeze homogeneously at slightly higher temperatures than smaller ones (Rogers and Yau, 1989, p.151).

Theoretical and empirical results indicate that homogeneous nucleation of liquid water occurs at temperatures lower than about -35 to -40 , depending somewhat on the size of the drops being subjected to the low temperature.<sup>55</sup> This threshold lies within the range of temperatures in natural clouds, which may have cloud-top temperatures below -80 . It is therefore possible, in a natural cloud, to have unfrozen liquid (i.e., supercooled) drops in the temperature range of 0 to about -40 . However, wherever the temperature in the cloud is below about -40 , any liquid drops that happen to be present freeze spontaneously by homogeneous nucleation. This conclusion is consistent with the fact that at temperatures below -40 atmospheric clouds are always composed entirely of ice, in which case they are said to be glaciated.

In principle, an ice particle may be nucleated directly from the vapor phase in the same manner as a drop. The critical size for homogeneous nucleation of an ice particle directly from the vapor phase is given by an expression similar in form to (3.3). In this case, the critical size depends strongly on both temperature and ambient humidity. Theoretical estimates of the rate at which molecules in the vapor phase aggregate to form ice particles of critical size indicate, however, that nucleation occurs only at temperatures below -65 and at supersaturations  $\sim 1000\%$ . Such high supersaturations do not occur in the atmosphere. Since liquid drops would nucleate from the vapor phase before these supersaturations were reached, and since the liquid drops thus formed would freeze homogeneously below -40 , it is concluded that homogeneous nucleation of ice directly from the vapor phase never occurs in natural clouds.

#### 3.2.2 Heterogeneous Nucleation of Ice Particles

From observations of the particles in clouds it is readily determined that ice and -40 . Since homogeneous crystals form at temperatures between 0 nucleation does not occur in this temperature range, the crystals must form by a heterogeneous process. As in the case of heterogeneous nucleation of liquid drops, the foreign surface on which an ice particle nucleates reduces the critical size that must be attained by chance aggregation of molecules. However, in the case of drops nucleating from the vapor phase, the atmosphere has no shortage of wettable nuclei. In contrast, ice crystals do not form readily on many of the particles found in air. The principal difficulty with the heterogeneous nucleation of the ice is that the molecules of the solid phase are arranged in a highly ordered crystal lattice. To allow the formation of an interfacial surface between the ice embryo and the foreign substance, the latter should have a lattice structure similar to that of ice. Figure 3.7 illustrates schematically an ice embryo which has formed on a crystalline substrate with a crystal lattice different from that of the ice. There are two ways in which the embryo could form. Either the ice could retain its normal lattice dimensions right to the interface, with dislocations in the sheets of molecules, or the ice lattice could deform elastically to join the lattice of the substrate. The effect of dislocations is to increase the surface tension of the ice-substrate interface. The effect of elastic strain is to raise the free energy of the ice molecules. Both of these effects lower the ice-nucleating efficiency of a substance. These effects, moreover, are temperature dependent in the sense that the higher the temperature, the more the surface tension and elastic strain are increased.



Figure 3.7 Schematic illustration of an ice embryo growing upon a crystalline substrate with a slight misfit. Dislocations of the interface are indicated by arrows. (From Fletcher, 1966. Reprinted with permission from Cambridge University Press.)

There are several modes of action by which an ice nucleus can trigger the formation of an ice crystal. An ice nucleus contained within a supercooled drop may initiate heterogeneous freezing when the temperature of the drop is lowered to the value at which the nucleus can be activated. There are two possibilities in this case. If the cloud condensation nucleus on which the drop forms is the ice nucleus, the process is called *condensation nucleation*. If the nucleation is caused by any other nucleus suspended in supercooled water, the process is referred to as *immersion freezing*. Drops may also be frozen if an ice nucleus in the air comes into contact with the drop; this process is called contact nucleation. Finally, the ice may be formed on a nucleus directly from the vapor phase, in which case the process is called deposition nucleation.

From the above considerations, it is evident that the probability of ice particle nucleation should increase with decreasing temperature and that substances possessing a crystal lattice structure similar to that of ice should be the most likely to serve as a nucleating surface. In this respect, ice itself provides the best nucleating surface; whenever a supercooled drop at any 0°C comes into contact with a surface of ice it immediately temperature freezes. Other than ice, the natural substances possessing a crystal lattice structure most similar to that of ice appear to be certain clay minerals found in many soil types and bacteria in decayed plant leaves. They may nucleate ice at but appear to occur in low concentrations in the temperatures as high as -4 atmosphere. Most ice particle nucleation in clouds occurs at temperatures lower than this. In general, particles in the air on which ice crystals are able to form are called *ice nuclei*. Measurements can be made to indicate how many ice nuclei can be activated by lowering the temperature of a sample of air in an expansion chamber.<sup>56</sup> Generally, these measurements do not distinguish among condensation, immersion, contact, or deposition nucleation, nor do they indicate the composition of the nuclei. They also do not indicate the effect of varying the humidity. However, extensive measurements of this type indicate that the average number of ice nuclei  $N_I$  per liter of air generally increases exponentially with decreasing temperature according to the empirical formula

$$\ln N_{I} = a_{I} (253 \,^{\circ}\text{K} - T) \tag{3.38}$$

where  $\alpha_I$  varies with location but has values in the range of 0.3-0.8. Note that according to this relationship, there is only about one ice nucleus per liter at -20°C. For a value of  $\alpha_I = 0.6$ , the concentration increases by approximately a factor of ten for every 4 of temperature decrease.

<sup>56</sup> See p.184 of Wallace and Hobbs (1977) for a description of the technique.

#### **3.2.3** Deposition and Sublimation

Growth of an ice particle by diffusion of ambient vapor toward the particle is called deposition. The loss of mass of an ice particle by diffusion of vapor from its surface into the environment is called sublimation. These processes are the ice-phase analogs of condensation and evaporation. However, since ice particles take on a variety of shapes, the spherical geometry assumed in evaluating the growth and evaporation of drops by vapor diffusion (Sec. 3.1.2) may not always be assumed in calculations of the change of mass of ice particles. Diffusion of vapor toward or away from nonspherical ice particles is accounted for by replacing *R* in (3.8), and thus in (3.14) and (3.22), by a shape factor  $\tilde{C}$ , which is analogous to electrical capacitance.<sup>57</sup> Thus, the analog to (3.8) is

$$\dot{\mathsf{m}}_{\mathsf{dif}} = 4\pi \widetilde{C} D_{v} \big[ \rho_{v} \big( \infty \big) - \rho_{vsfc} \big]$$
(3.39)

where  $\rho_{vsfc}$  is the vapor density at the particle's surface. It follows that the analogs to (3.14) and (3.22) are

-----

$$\dot{\mathsf{m}}_{\mathsf{dif}} = \frac{4\pi \widetilde{CS}_i}{F_{\kappa i} + F_{Di}} \tag{3.40}$$

and

$$\dot{\mathsf{m}}_{\mathsf{dif}} = \frac{4\pi \widetilde{C} V_F \widetilde{S}_i}{F_{\kappa i} + F_{Di}} \tag{3.41}$$

respectively.  $\tilde{S}_i$ ,  $F_{Ki}$  and  $F_{Di}$  are the same as  $\tilde{S}$ ,  $F_K$ , and  $F_D$  in (3.15)-(3.17) except that *L* is replaced by the latent heat of sublimation  $L_s$  in (3.16), and  $e_s()$  is replaced by the saturation vapor pressure over a plane surface of ice  $e_{si}()$  in (3.15) and (3.17). The relations (3.40) and (3.41), like (3.14) and (3.22), apply only when the air is saturated (in this case with respect to ice). As in the case of drops,  $\dot{m}_{dif}$  must be obtained numerically if the air is unsaturated.

<sup>57</sup> The analogy between the vapor field around an ice crystal and the field of electrostatic potential around a conductor of the same size and shape was first applied by Houghton (1950). See Hobbs (1974) for further notes on the origin of the analogy.

The shape, or habit, adopted by an ice crystal growing by vapor diffusion is a sensitive function of the temperature T and supersaturation  $\tilde{S}_i$  of the air.<sup>58</sup>

These growth modes are known from observations in the laboratory and in clouds themselves. The basic crystal habits exhibit a hexagonal face. Let a crystal be imagined to have an axis normal to its hexagonal face. If this axis is long compared to the width of the hexagonal face, it is said to be prismlike. If this axis is short compared to the width of the hexagonal face, the crystal is said to be platelike. The basic crystal habits are illustrated schematically in Fig. 3.8. The habits change back and forth between prismlike and platelike as the ambient temperature changes (Table 3.1). The effect of increasing the ambient supersaturation is to increase the surface-to-volume ratio of the crystal. The additional surface area gives the increased ambient vapor more space on which to deposit. The multiarmed, fernlike crystals that appear at temperatures of -12to -16°C have six main arms and several secondary branches (Fig. 3.8c). They may be thought of as hexagonal plates with sections deleted to increase the surface-to-volume ratio of the crystal. They occur in the temperature range where the difference between the saturation vapor pressure over water (an approximation to the actual vapor pressure in many cold clouds) and the saturation vapor pressure over ice (an approximation to the condition at the surface of the crystal) is greatest.



Figure 3.8 Schematic representation of the main shapes of ice crystals: (a) columnar, or prismlike; (b) plate; (c) dendrite. (Adapted from Rogers and Yau, 1989.)

<sup>58</sup> See Chapters 8 and 10 of Hobbs (1974).

#### 3.2.4 Aggregation and Riming

If ice particles collect other ice particles, the process is called aggregation. If ice particles collect liquid drops, which freeze on contact, the process is called *riming*. The continuous collection equation (3.24) may be used to describe the growth of ice particles by aggregation or riming.

Aggregation depends strongly on temperature. The probability of adhesion of colliding ice particles becomes much greater when the temperature increases to above -5 , at which the surfaces of ice crystals become sticky. Another factor affecting aggregation is crystal type. Intricate crystals, such as dendrites, become aggregated when their branches become entwined. These facts are known from laboratory experiments and observations of natural snow. The sizes of collected snow aggregates are shown as a function of the temperature at which they were observed in Fig. 3.9. The sizes increase sharply at temperatures above -5 , while aggregation does not appear to exist below -20 . A secondary maximum occurs between -10 and -16 , where the arms of

the dendritic crystals growing at these temperatures apparently become entangled. In correspondence to these observations, the collection efficiency for aggregation is often assumed to be an exponentially increasing function of temperature in calculations using (3.24) or (3.26).

Table	<b>3.1</b>

variations in the Basic Habits of ice Crystals with Temperature				
		Types of crystal at slight water		
Temperature (°C)	Basic habit	Supersaturation		
0 to – 4	Platelike	Thin hexagonal plates		
- 4 to - 10	Prismlike	Needles $(-4 \text{ to} - 6 ^{\circ}\text{C})$		
		Hollow columns ( $-5$ to $-10$ °C)		
		Sector plates $(-10 \text{ to} - 12 \text{ °C})$		
-10 to $-22$	Platelike	Dendrites $(-12 \text{ to} - 16 ^{\circ}\text{C})$		
		Sector plates $(-16 \text{ to} - 22 \text{ °C})$		
-22 to $-50$	Prismlike	Hollow columns		

Variations in the Basic Habits of Ice Crystals with Temperature

Source: Wallace and Hobbs (1997).



Maximum dimensions of natural aggregates of ice crystals as a Figure 3.9 function of the temperature of the air where they were collected. xindicates crystals collected from an aircraft. Circles represent crystals collected on the ground. (From Hobbs, 1973b. Reprinted with permission from Oxford University Press.)



Figure 3.10 (a) A lightly rimed needle; (b) densely rimed column; (c) densely rimed plate; (d) densely rimed stellar; (e) lump graupel; (f) cone graupel. (From Wallace and Hobbs, 1977.)

The collection efficiency for riming is not well known theoretically or empirically, but it is generally thought to be quite high and often assumed to be unity in calculations using (3.24) or (3.26). If the ice particle is viewed as the collector and the liquid drops as the collected particles in (3.24), the degree of riming that is achievable is determined primarily by the mixing ratio of the liquid water  $(q_{m'})$ . Lightly to moderately rimed crystals retain vestiges of the original crystal habit of the collector (Fig. 3.10 a-d). Under heavy riming the identity of the collector becomes lost, and the particle is referred to as graupel, which may be in the form of lumps or cones (Fig. 3.10e and f).

#### 3.2.5 Hail

Extreme riming produces *hailstones*. These particles are commonly 1 cm in diameter but have been observed to be as large as 10-15 cm. They are produced

as graupel or frozen raindrops collect supercooled cloud droplets. So much liquid water is accreted in this fashion that the latent heat of fusion released when the collected water freezes significantly affects the temperature of the hailstone. The hailstone may be several degrees warmer than its environment. This temperature difference has to be taken into account in calculating the growth of hail particles, which is determined by considering the heat balance of the hailstone.

The rate at which heat is gained as a result of the riming of a hailstone of mass **m** is

$$\dot{Q}_f = \dot{m}_{col} \left\{ L_f - c_w [T(R) - T_w] \right\}$$
 (3.42)

The factor  $\dot{m}_{col}$  is the rate of increase of the mass of the hailstone as a result of collecting liquid water. It is given by (3.26). The hailstone is assumed to be spherical with radius *R*.  $L_f$  is the latent heat of fusion released as the droplets freeze on contact with the hailstone. The second term in the curly brackets is the heat per unit mass gained as the collected water drops of temperature  $T_w$ come into temperature equilibrium with the hailstone. The factor  $c_w$  is the specific heat of water. If the air surrounding the particle is subsaturated, the temperature  $T_w$  is approximated by the wet-bulb temperature of the air, which is the equilibrium temperature above a surface of water undergoing evaporation at a given air pressure.<sup>59</sup> This temperature may be several degrees less than the actual air temperature when the humidity of the air is very low. If the air surrounding the particle is saturated  $T_w = T($  ).

The rate at which the hailstone gains heat by deposition (or loses heat

by sublimation) is obtained from a modified form of (3.8)

$$\dot{Q}_{c} = 4\pi R D_{\nu} \left[ \rho_{\nu}(\infty) - \rho_{\nu}(R) \right] V_{Fc} L_{s}$$
(3.43)

where  $V_{Fs}$  is a ventilation factor for sublimation, and  $L_s$  is the latent heat of sublimation.

The rate at which heat is lost to the air by conduction is obtained from a modified version of (3.9), which may be written as

$$\dot{Q}_c = 4\pi R \kappa_a [T(R) - T(\infty)] V_{Fc}$$
(3.44)

where  $V_{Fs}$  is a ventilation factor for conduction.

In equilibrium we have

$$\dot{Q}_f + \dot{Q}_s = \dot{Q}_c \tag{3.45}$$

which upon substitution from (3.42)-(3.44) may be solved for the hailstone equilibrium temperature as a function of size. As long as this temperature remains below 0, the surface of the hailstone remains dry, and its development is called dry growth. The diffusion of heat away from the hailstone, however, is generally too slow to keep up with the release of heat associated with the riming (depositional growth is much less than the riming). Therefore, if a hailstone remains in a supercooled cloud long enough, its equilibrium temperature can rise to 0 . At this temperature, the collected supercooled droplets no longer freeze spontaneously upon contact with the hailstone. Some of the collected water may then be lost to the warm hailstone by shedding. However, a considerable portion of the collected water becomes incorporated into a water-ice mesh forming what is called *spongy hail*. This process is called wet growth. During its lifetime, a hailstone may grow alternately by the dry and wet processes as it passes through air of varying temperature. When hailstones are sliced open, they often exhibit a layered structure, which is evidence of these alternating growth modes.

<sup>59</sup> See pp.75-76 of Wallace and Hobbs (1977)

#### **3.2.6** Ice Enhancement

When the concentrations of ice particles are measured in natural clouds, it is often found that there are far more ice particles present than can be accounted for by the typical concentrations of ice nuclei activated by lowering the temperature of air in expansion chambers.<sup>60</sup> Figure 3.11 compares some measurements of ice particle concentrations in cumuliform clouds with the concentration of ice nuclei expected from expansion-chamber measurements to be active at the cloud-top temperature. The latter concentration is calculated from (3.38). The cloud-top temperature is the lowest temperature anywhere in the cloud and hence provides an estimate of the maximum possible ice nucleus concentration in the cloud according to (3.38). The actual particle concentration is seen typically to exceed the maximum possible nucleus concentration by one or more orders of magnitude.

These high concentrations are found in many cold clouds. They do not, however, occur in uniform spatial and temporal patterns within a cloud. A common characteristic is that they occur in older rather than newly formed portions of clouds, and they are found in association with supercooled cloud droplets. They are most likely to be found when the size distribution of the droplets is broad, with largest drops exceeding about 20  $\mu$ m in diameter. The high concentrations develop initially near the tops of clouds, and the high concentrations may develop suddenly (e.g., the concentration may rise from 1 to 1000 1<sup>-1</sup> in less than 10 min).

The microphysical process, or processes, by which the concentrations of ice particles become so highly enhanced relative to the number of nuclei which would appear to be active according to (3.38) are not certain. Some hypotheses that have been suggested are<sup>61</sup>

(i) *Fragmentation of ice crystals*. Delicate crystals may break into pieces as a result of collisions and/or thermal shock.

<sup>60</sup> For a more complete account of the observations and hypotheses regarding the occurrence of high ice particle concentrations in clouds, see Hobbs and Rangno (1985,1990) and Rangno and Hobbs (1988,1991).

<sup>61</sup> For a more full discussion and many references to the literature, see the Hobbs and Rangno papers mentioned in the previous footnote.

(ii) *Ice splinter production in riming*. It has been found in laboratory experiments that when supercooled droplets >23  $\mu$ m in diameter collide with an ice surface at a speed of 1.4 m s<sup>-1</sup> at temperatures of -3 to -8 , small ice splinters are produced.<sup>62</sup>

(iii) *Contact nucleation*. It is thought that when certain aerosol particles come into contact with supercooled droplets they can cause nucleation at higher temperatures than they would through other forms of nucleation.

(iv) *Condensation or deposition nucleation*. There is evidence that the icenucleating activity of atmospheric aerosol particles by either condensation or deposition nucleation is greatly increased when the ambient supersaturation rises above 1% with respect to water. Ice nucleus counters whose data lead to the expression in (3.38) are usually operated near water saturation. A pocket of high supersaturation in a cloud might be favorable for the sudden appearance of a large number of ice particles.

The latter two mechanisms, (iii) and (iv), do not require the pre-existence of ice particles and may thus help to account for the sudden appearance of high concentrations in cloudy air at relatively high temperatures.



Figure 3.11 Maximum ice particle concentrations observed in mature and aging maritime (open humps), continental (closed humps), and transitional (half-open humps) cumuliform clouds. The line represents the concentrations expected at the cloud-top temperature. (From Hobbs and Rangno, 1985. Reprinted with permission from the American Meteorological Society.)

<sup>62</sup> The laboratory experiments were performed by Hallett and Mossop (1974). This ice-enhancement process is often called the Hallett-Mossop mechanism.

#### **3.2.7 Fall Speeds of Ice Particles**

The fall speeds of ice particles encompass a wide range. Observations show that these speeds depend on particle type, size, and degree of riming and that the more heavily rimed a particle, the more its fall speed depends on its size. Individual snow crystals (lower curves of Fig. 3.12) and unrimed to moderately rimed aggregates of crystals (Fig. 3.13) drift downward at speeds of 0.3-1.5 m s<sup>-1</sup>, with the aggregates showing a tendency to increase slightly in fall speed as they approach 12 mm in dimension. Graupel fall speeds increase sharply from 1 to3m s<sup>-1</sup> over a narrow size range of 1-3 mm (upper curve of Fig. 3.12 and Fig. 3.14). Empirical formulas for fall speeds of snow and graupel for the data set represented in Fig. 3.12 and Fig. 3.14 are listed in Table 3.2. These formulas apply near the earth's surface. They do not take into account the fact that the fall speed also depends on the density of the air through which the particles are

falling. The dependence on air density has been determined experimentally and theoretically.<sup>63</sup>



Figure 3.12 Terminal fail speeds of snow crystals as a function of their maximum dimensions. Open circle (uppermost curve) indicates graupel fall speeds. Other curves are for rimed crystals (dot in circle), needles (filled circle with slash), spatial dendrites (triangle), powder snow (x), and dendrites (filled circle). (From Nakaya and Terada, 1935.)

<sup>63</sup> See, for example, Bohm (1989).

Hailstone fall velocities are an order of magnitude larger than those for snow and graupel. At a pressure of 800 mb and a temperature of 0  $\,$ , they obey the empirical formula<sup>64</sup>

$$V(ms^{-1}) \approx 9D_h^{0.8}$$
 (3.46)

where  $D_h$  is the diameter of the hailstone in cm. This formula was obtained for hailstones in the size range 0.1-0.8 cm. Over this range, the fall speeds indicated by (3.46) are roughly 10-50 m s<sup>-1</sup>. These large values imply that updrafts of comparable magnitude must exist in the cloud to support the hailstones long enough for them to grow. Hence, hail is found only in very intense thunderstorms, of the types considered in Chapters 8 and 9.



Figure 3.13 Terminal velocity and maximum dimension measurements for unrimed to moderately rimed aggregates. Combinations of: sideplanes (a type of branched crystal), bullets and columns (circles, dotted curve); sideplanes (triangles, solid curve); radiating assemblages of dendrites (asterisks, dashed curve); dendrites (squares, dash-dot curve). (From Hobbs, 1974, based on data from Locatelli and Hobbs, 1974. Reprinted with permission from Oxford University Press.)

<sup>64</sup> From Pruppacher and Klett (1978, p.345), based on data of Auer (1972).

### Table 3.2

Empirical Relationships between the Terminal Velocities  $v(\text{in ms}^{-1})$  of Solid Precipitation Particles and Their Maximum Dimension  $D_m$  (in mm)

	Type of particle <sup><i>a</i></sup>	$v$ - $D_m$ relationship
Graupel	and Cone shaped groupel	$v = 1 - 2 D^{0.65}$
Graupe-like		$V$ 1 · $2D_m$

snow	Hexagonal graupel	$v = 1 \cdot 1 D_m^{0.57}$
	Graupel-like snow of lump type	$v = 1 \cdot 1 D_m^{0.28}$
	Graupel-like snow of hexagonal type	$v = 0 \cdot 86 D_m^{0.25}$
	Combination of sideplanes, plates, bullets, and columns	$v = 0 \cdot 69 D_m^{0.41}$
Unrimed aggregates	Sideplanes	$v = 0 \cdot 82 D_m^{0.12}$
	Radiating assemblages of dendrite or dendrites	$v=0 \cdot 8 D_m^{0.16}$
Densely rimed aggregates	Radiating assemblages of dendrites or dendrites	$v = 0 \cdot 79 D_m^{0.27}$
Densely rimed columns		$v = 1 \cdot 1 D_m^{0.56}$

Source: Locatelli and Hobbs (1974).

<sup>a</sup> Based on Magono and Lee's (1966) classification.



Figure 3.14 Terminal velocity and maximum dimension measurements for graupel and graupel-like snow. Cone-shaped graupel (circles, dash-dot curve); hexagonal graupel (triangles, dashed curve); graupel-like snow of hexagonal type (asterisks). (From Hobbs, 1974, based on data from Locatelli and Hobbs, 1974. Reprinted with permission from Oxford University Press.)

#### 3.2.8 Melting

Ice particles can change into liquid water when they come into contact with air or water that is above 0 . A quantitative expression for the rate of melting of an ice particle of mass m can be obtained by assuming heat balance for the particle during the melting. According to (3.45), this balance may then be written as

$$-L_{f}\dot{\mathfrak{m}}_{\text{mel}} = 4\pi R \kappa_{a} [T(\infty) - 273 \text{K}] V_{F_{c}} + \dot{\mathfrak{m}}_{\text{col}} c_{w} (T_{w} - 273 \text{K}) + \dot{Q}_{s} \qquad (3.47)$$

where  $\dot{m}'_{mel}$  is the rate of change of the mass of the ice particle as a result of melting. The first term on the right is the diffusion of heat toward the particle from the air in the environment. The second term is the rate at which heat is transferred to the ice particle from water drops of temperature  $T_w$  that are collected by the melting particle. The third term,  $\dot{Q}_s$ , is the gain or loss of heat

by vapor diffusion [given by (3.43) in the case of a spherical hailstone]. If both the air and drop temperatures exceed 273 K, then both the first and second term in (3.47) contribute to the melting.

#### **3.3** Types of Microphysical Processes and Categories of Water Substance in Clouds

From the foregoing review of cloud microphysics (Secs. 3.1and 3.2), it is evident that water substance can take on a wide variety of forms in a cloud and that these forms develop under the influence of *seven basic types of* microphysical processes:

- 1. Nucleation of particles
- 2. Vapor diffusion
- 3. Collection
- 4. Breakup of drops
- 5. Fallout
- 6. Ice enhancement
- 7. Melting

These individual processes may sometimes be isolated for study in numerical models or in the laboratory. However, in a natural cloud several or all of these processes occur simultaneously, as the entire ensemble of particles comprising the cloud forms, grows, and dies out. Thus, the various forms of water and ice particles coexist and interact within the overall cloud ensemble. It is the behavior of the overall ensemble that is of primary interest in cloud dynamics, and it is generally unnecessary to keep track of every particle in the cloud in order to describe the cloud's gross behavior. At the same time, it is also impossible to ignore the microphysical processes and accurately represent the cloud's overall behavior. To retain the essentials of the microphysical behavior, it is convenient to group the various forms of water substance in a cloud into several broad categories of water substance:

- *Water vapor* is in the gaseous phase.
- *Cloud liquid water* is in the form of small suspended liquid-phase droplets (i.e., drops that are too small to have any appreciable terminal fall speed constitute cloud liquid water and therefore are generally carried along by the air in which they are suspended).
- *Precipitation liquid water* is in the form of liquid-phase drops that are large enough to have an appreciable fall speed toward the earth. This water may be subdivided into drizzle (drops 0.1-0.25 mm in radius) and rain (drops >0.25 mm in radius), as defined in Sec. 3.1.3.

- *Cloud ice* is composed of particles that have little or no appreciable fall speed. These particles may be in the form of pristine crystals nucleated directly from the vapor or water phase, or they may be tiny particles of ice produced in some form of ice enhancement process.
- *Precipitation ice* refers to ice particles that have become large and heavy enough to have a terminal fall speed ~ $0.3 \text{ m s}^{-1}$  or more. These particles may be pristine crystals, larger fragments of particles, rimed particles, aggregates, graupel, or hail. To simplify the description of the gross behavior of a cloud these particle types are sometimes grouped into categories according to their density or fall speed. Such groupings are arbitrary; however, a commonly used scheme (employed in discussions below) is to divide these particles into *snow*, which has lower density and falls at speeds of ~ $0.3 1.5 \text{ m s}^{-1}$  (see Figs. 3.12-3.13); *graupel*, which falls at speeds of ~ $1 3 \text{ m s}^{-1}$  [see Eq. (3.46)].

According to these categories of particles, the water substance in a sample of air may be represented by eight mixing ratios:

- $q_v$  mass of *water vapor* / mass of air
- $q_c$  mass of *cloud liquid water* / mass of air
- $q_d$  mass of *drizzle* / mass of air
- $q_r$  mass of *rainwater* / mass of air
- $q_I$  mass of *cloud ice* / mass of air (3.48)
- $q_s$  mass of *snow* / mass of air
- $q_g$  mass of graupel / mass of air
- $q_h$  mass of *hail* / mass of air

The evolution of a cloud can be characterized in terms of fields of these mixing ratios of water substance. The various categories are interactive. For example, drops grow at the expense of vapor during nucleation and condensation, precipitation ice grows at the expense of cloud and precipitation liquid water during riming, rainwater is produced at the expense of precipitation ice during melting, and so on. The many conversions of water substance from one form to another in a given sample of air are illustrated in Fig. 3.15 for a cloud whose water substance is divided into water vapor, cloud

liquid water, rainwater, cloud ice, snow, and graupel. Also indicated are the possible gains or losses by precipitation fallout. The six categories in Fig. 3.15 are a subset of the eight listed above, drizzle and hail being omitted. This six-category scheme is frequently used in cloud dynamics. Sometimes hail rather than graupel is used as the sixth category. To be perfectly general, all the categories should be included. It is easy to see from Fig. 3.15 that the number of interactions to be considered would increase greatly by expanding to a seven- or eight- category scheme. It is for this reason that the number of categories is often limited to six or less.

#### 3.4 Water-Continuity Equations

In cloud dynamics one is concerned with the overall development of a cloud, in which any combination of the categories of water substance defined above and any combination of the microphysical processes linking the various categories (as shown in Fig. 3.15) may be present simultaneously in the context of a particular set of air motions. It is therefore necessary to have a way to keep track of all of the processes in some systematic way. For this purpose, the water-continuity equations (2.21) are used. They allow one to account numerically for the amount of water contained in the form of vapor and in the form of particles of different types and sizes throughout a cloud, as it evolves. This system of equations is referred to as a *water-continuity model*. In (2.21), each category of water is assigned a mixing ratio  $q_i$ , where the subscript *i* refers to a particular category of water and the mixing ratio is defined as the mass of water of category *i* per unit mass of air. The total water substance in a parcel of air is then given by the sum of the water contained in each of the categories:

$$q_T = \sum_{i=1}^{n} q_i$$
 (3.49)

where *n* is the total number of categories into which the total water  $q_T$  in the air parcel has been divided.



Figure 3.15 Conversions of water substance from one form to another in a bulk water-continuity model in which there are six categories of water substance. Dashed lines indicate various interactions leading to production of graupel: collection of cloud ice by rain freezes the ice by contact nucleation and produces either snow or graupel; or riming of snow by collection of either cloud water or raindrops can also lead to production of graupel. In the model, the mass of ice produced by these two processes passes temporarily through the snow category. (Adapted from Rutledge and Hobbs, 1984. Reproduced with permission from the American Meteorological Society.)

There is no limit to the number of categories into which the total water in a parcel of air can be divided. To begin with, we can divide the water into the eight categories listed in (3.48). Each of these categories, however, can be further subdivided. The cloud liquid water, drizzle, and rainwater categories can be subdivided according to drop size. Each drop size category can be further subdivided according to such factors as the type of nucleus on which the drops -formed, chemical composition of the drops, etc. The snow category can be subdivided by type of particle (columns, plates, aggregates, ice splinters, etc.), and these particle types can be further subdivided according to particle size, density, or other factors. The graupel may be subdivided by size and shape (some graupel particles are cone shaped while some are lumpy), and hail may be subdivided by particle size, whether it is spongy or hard, and other factors. Obviously, calculations can become highly complex if all of these subdivisions are employed. Generally the strategy in cloud dynamics is to identify and use only the categories and subcategories of water substance that are essential to keep account of in the particular problem being considered. Hence, practically every water-continuity model employed is tailored to the problem at hand.

Once the *n* categories to be used in a particular water-continuity model have been established, the source terms on the right-hand side of (2.21) must be formulated to allow for all the possible interactions among the different categories of water (e.g., the interactions shown for the six-category model in Fig. 3.15). The source terms are formulated in terms of the seven basic cloud microphysical mechanisms mentioned earlier (nucleation, vapor diffusion, collection, particle breakup, fallout, ice enhancement, and melting). Two general strategies have been employed to formulate the source terms. In *bulk* models, the liquid and ice water mixing ratios are grouped into categories according to particle type only. In *explicit* models, the hydrometeors are subdivided according to size within each particle-type grouping. The following sections summarize the salient features of these two types of models.

### 3.5 Explicit Water-Continuity Models3.5.1 Genera

Hydrometeors may be grouped into categories according to particle type as in (3.48). In an explicit water-continuity model, one or more of these categories are subdivided according to particle size. Since the mass of water contained in particles of different sizes is calculated, the size distributions of particles are able to evolve naturally in each air parcel associated with the cloud. The only disadvantage of the explicit model is computational. A large number of size categories (~10-100) and associated interactions have to be included to represent the size distribution of the particles of a given category of water substance accurately. From a physical standpoint, the explicit method is the more direct approach. The microphysicalprinciples reviewed in previous sections can be applied directly to the calculation of the size distributions within a given category of water substance.

# 3.5.2 Explicit Modeling of Warm Clouds3.5.2.1 Drops Subdivided by Size

We consider first clouds without ice. Therefore, none of the ice categories in (3.48) are relevant;  $q_I = q_s = q_g = q_h = 0$ . The cloud and precipitation liquid water categories are combined into a single total liquid water mixing ratio  $q_L$ 

 $q_c + q_d + q_r$ . The total liquid water content is thus viewed as a continuous

spectrum of drops, ranging from small, freshly nucleated cloud droplets to large raindrops. The size distribution is represented by N(m), where N(m) dm is the number of particles per unit volume of air in mass range m to m + dm, in which case

(3.50) 
$$q_L = \frac{1}{\rho} \int_0^\infty \mathsf{m} N(\mathsf{m}) d\mathsf{m}$$

For computational purposes the size distribution is approximated by k discrete mass categories so that  $q_L$  is approximated by

(3.51) 
$$q_L \approx \frac{1}{\rho} \sum_{i=1}^k \mathsf{m}_i N_i (\Delta \mathsf{m})_i$$

The water continuity in the cloud may then be expressed by k + 1 equations<sup>65</sup>

$$\frac{DN_i}{Dt} = -N_i \nabla \cdot \mathbf{v} + N_i + D_i + C_i + B_i + F_i$$

$$i = 1, \dots, k$$
(3.58)

and

(3.53) 
$$\frac{Dq_{\nu}}{Dt} = -\frac{1}{\rho} \sum_{i=1}^{k} \mathsf{m}_{i} (\mathsf{N}_{i} + \mathsf{D}_{i}) (\Delta \mathsf{m})_{i}$$

The first term on the right-hand side of (3.52) represents changes in  $N_i$  associated with air motions. It expresses the decrease (increase) in concentration associated with the expansion (contraction) of the volume of air within which the population of drops is located. The remainder of the terms in (3.52) represent the microphysical processes affecting the drop size distribution. They express the changes in the concentration of drops in size range *i* resulting from nucleation from the vapor phase  $N_i$ , vapor diffusion (condensation or evaporation)  $D_i$ , collection  $C_i$ , drop breakup  $B_i$ , and sedimentation (or fallout)  $F_i$ . These microphysical processes correspond to the first five of the seven basic microphysical mechanisms listed in Sec. 3.3. The remaining two on that list, ice enhancement and melting, do not apply in a warm cloud.

<sup>&</sup>lt;sup>65</sup> This type of water-continuity model was used in early studies by Takeda (1971), Ogura and Takahashi (1973), and Soong (1974).

The nucleation term  $N_i$  in (3.52) is calculated by first making assumptions about the characteristics of the condensation nuclei present in the air. Then the appropriate concentration of drops is nucleated according to the supersaturation of the air and the relations governing nucleation (Sec. 3.1.1). The diffusion term  $D_i$  represents the rate of change in the number concentration of drops resulting from differing diffusional growth or evaporation rates of drops in adjacent size categories. This rate of change is expressed by

$$(3.54) D_i = -\frac{\partial}{\partial m} [\dot{m}_{dif} N]_{m=m_i}$$

where  $\dot{m}_{dif}$  is the diffusional growth (or evaporation) rate of an individual drop of mass m. If the air is saturated with respect to liquid water,  $\dot{m}_{dif}$  is given by (3.22). If not, it is obtained by solving (3.8), (3.9), and (3.11) numerically. The term  $D_i$  maybe thought of as the convergence of number concentration of

drops as a result of the growth rate  $\dot{m}_{dif}$  varying with drop size. The collection term  $C_i$  in (3.52) is given by the stochastic collection equation (3.33), which may be written for the *i*-th drop size category as

(3.55) 
$$C_{i} = \left(\frac{\partial N(\mathbf{m}_{i}, t)}{\partial t}\right)_{col} = I_{2}(\mathbf{m}_{i}) - I_{1}(\mathbf{m}_{i})$$

The rate of production of drops of mass  $m_i$  by breakup  $B_i$  can be expressed according to (3.36) as

(3.56) 
$$B_{i} = \left(\frac{\partial N(\mathsf{m}_{i}, t)}{\partial t}\right)_{bre}$$
$$= -N(\mathsf{m}_{i}, t)P_{B}(\mathsf{m}_{i}) + \int_{\mathsf{m}_{i}}^{\infty} N(\mathsf{m}', t)Q_{B}(\mathsf{m}', \mathsf{m}_{i})P_{B}(\mathsf{m}')d\mathsf{m}'$$

Finally, the sedimentation term  $F_i$  in (3.52) represents the local accumulation of drops of mass  $m_i$  at a point as a result of their fall speeds. It is given by

$$(3.57) F_i = \frac{\partial}{\partial z} (N_i V_i)$$

where  $V_i$  is the terminal velocity (defined to be positive downward) of a drop of mass  $M_i$ .

#### **3.5.2.2** Variable Drop Composition

During the early stages of drop growth, the condensation nucleus around which the drop formed may become dissolved and affect the rate of growth of the drop by vapor diffusion. To account for this effect, we can further subdivide the drop size distribution according to nuclear mass n, with N(m, n) dm dn representing the number of particles of mass m to m + dm and nuclear mass n to n + dn per unit volume of air.<sup>66</sup> The distribution could be further subdivided if nuclei of different types of dissolved substances were considered. For the purpose of illustration, we will consider the nuclei all to consist of the same substance but to be of different sizes. Each of the *k* discrete categories of drop size can then be subdivided into *l* discrete categories of nucleus size. The subdivided water-continuity equations are then

$$\frac{DN_{ij}}{Dt} = -N_{ij}\nabla \cdot v + N_{ij} + D_{ij} + C_{ij} + B_{ij} + F_{ij}$$
(3.58)

where i = 1, ..., k, j = 1, ..., l, and

$$\frac{Dq_{\nu}}{Dt} = -\frac{1}{\rho} \sum_{j=1}^{1} (\Delta n)_{j} \sum_{i=1}^{k} m_{i} (N_{ij} + D_{ij}) (\Delta m)_{i}$$
(3.59)

The change in the concentration  $N_{ij}$  of drops in size category *i* and nucleus category *j* as a result of nucleation is represented by  $N_{ij}$  and calculated by postulating a spectrum of nuclei to be present in the air initially and activating them in accordance with the supersaturation of the air and the relations governing nucleation (Sec. 3.1.1). The change in the concentration  $N_{ij}$  of drops in size category *i* and nucleus category *j* as a result of vapor diffusion  $D_{ij}$  is given by

$$D_{ij} = -\frac{\partial}{\partial \mathbf{m}} \left[ \dot{\mathbf{m}}_{dif} N_{ij} \right]_{\mathbf{m} = \mathbf{m}_i, \mathbf{n} = \mathbf{n}_j}$$
(3.60)

<sup>&</sup>lt;sup>66</sup> Subdivision of the drop size spectrum according to the size of condensation nuclei has been used by Silverman (1970), Tag *et al.* (1970), Arnason and Greenfield (1972), Clark (1973), and Silverman and Glass (1973). Further details of the technique can be found in these articles.

which is similar in form to (3.54). However, the drop growth rate  $\dot{m}_{dif}$  in this case is given, under saturated conditions, by (3.23), which takes into account solution effects in depositional growth. If the air is unsaturated, it is obtained by numerical solution of (3.8), (3.9), and (3.18). The collection term  $C_i$  is conceptually more difficult when the nuclear composition of drops is accounted for. Appropriate assumptions must be made concerning the coalescence or noncoalescence of the nuclei of coalescing drops. If the nuclei are assumed to coalesce whenever two parent drops coalesce, the nuclear mass n of the newly formed drop is determined by adding the n's of the coalescing drops.<sup>67</sup> In this case, the stochastic collection equation is

$$C_{ij} = \frac{1}{2} \int_{0}^{m_{i}} \int_{0}^{n_{i}} \hat{K}(m_{i} - m', m') N(m_{i} - m', n_{j} - n', t) N(m', n', t) dn' dm'$$
  
(3.61) 
$$- \int_{0}^{\infty} \int_{0}^{\infty} \hat{K}(m_{i}, m') N(m_{i}, n_{j}, t) N(m', n', t) dn' dm'$$

where the first term on the right-hand side is the rate of formation of drops of mass  $m_i$  and nuclear mass  $n_j$  by coalescences of drops with masses smaller than  $m_i$  and  $n_j$ . The second term is the rate of removal by combinations of drops mass  $m_i$  and nuclear mass  $n_j$  with other drops. Drop breakup  $B_{ij}$  is also conceptually difficult. One must assume something about what happens to the nucleus during the breakup process in order to calculate this term, and there is no standard way to do this.<sup>68</sup> Finally, the sedimentation  $F_{ij}$  is expressed by

(3.62) 
$$\mathsf{F}_{ij} = \frac{\partial}{\partial z} \left( N_{ij} V_i \right)$$

which is similar to (3.57).

<sup>&</sup>lt;sup>67</sup> Suggested by Silverman (1970).

<sup>&</sup>lt;sup>68</sup> One attempt to include this effect was made by Silverman and Glass (1973).

#### 3.5.3 Explicit Modeling of Cold Clouds

In an explicit water-continuity model of cold clouds one must subdivide the ice categories listed in (3.48) as well as the liquid categories. This case is particularly complex since the liquid-phase drops in each size category potentially may interact with ice particles of every size and type. A staggering multiplicity of interactions is possible. Nonetheless, explicit water-continuity models have been developed that treat a substantial subset of the possible microphysical interactions that can occur in a mixed-phase cloud. Figure 3.16 indicates the categories of hydrometeors included in one example of such a model.<sup>69</sup> The categories of particle types in this model parallel those of the six-category model illustrated in Fig. 3.15, except that cloud ice has been subdivided into pristine "ice crystals" and "ice splinters," the latter being the product of a postulated ice-enhancement mechanism. The other categories, "cloud droplets," "drops." "snowflakes," and "graupel," correspond to the categories cloud liquid water, rainwater, snow, and graupel in Fig. 3.15. The ice crystals are subdivided into categories of crystal habit (plates, columns, and dendrites), where the crystal habit is determined by temperature. The graupel particles are subdivided according to particle density, ranging from 0.1 to 0.8  $\times$ 10<sup>3</sup> kg m<sup>-3</sup>. Size distributions are computed for each particle category or subcategory. Stochastic collection concepts are used in the generation of drops from cloud droplets, snowflakes (aggregates) from ice crystals, and graupel from the riming of ice crystals. Graupel is also allowed to form when drops freeze.

#### 3.6 Bulk Water-Continuity Models

As noted in Sec. 3.5.1, the disadvantage of the explicit water-continuity models is that one has to keep track of so many individual categories of water. The basic idea of *bulk water-continuity models* is to assume as few categories of water as possible in order to minimize the number of equations and calculations in the water-continuity model. To accomplish this simplification, the shapes and size distributions of particles must be assumed and the basic microphysical processes must be parameterized. This method is used extensively in cloud dynamics and the following subsections outline its essential features.

<sup>&</sup>lt;sup>69</sup> This example is from Scott and Hobbs (1977). Another cold-cloud explicit water-continuity model was developed by Hall (1980).



Figure 3.16 Microphysical interactions among different categories of water substance in an explicit model of a mixed-phase cloud. (From Scott and Hobbs, 1977. Reprinted with permission from the American Meteorological Society.)

#### 3.6.1 Bulk Modeling of Warm Clouds

The simplest type of cloud is a warm, nonprecipitating cloud. The minimum number of categories that describe it is two: vapor, represented by  $q_v$ , and cloud liquid water, represented by  $q_c$ . The total water-substance mixing ratio,  $q_T = q_v + q_c$ , is conserved, and the water-continuity model (2.21) consists simply of the two equations

$$\frac{Dq_v}{Dt} = -C \tag{3.63}$$

$$\frac{Dq_c}{Dt} = C \tag{3.64}$$

where *C* represents the condensation of vapor when C > 0 and evaporation when C < 0.

For a warm precipitating cloud, rain is included as an additional category of water substance (drizzle is ignored). The water-continuity model (2.21) then consists of three equations:

$$\frac{Dq_v}{Dt} = -C_c + K_c + E_r \tag{3.65}$$

$$\frac{Dq_{c}}{Dt} = C_{c} - E_{c} - A_{c} - K_{c}$$
(3.66)

$$\frac{Dq_r}{Dt} = A_c + K_c - E_r + F_r \tag{3.67}$$

where  $q_r$  is the mixing ratio of rainwater, as defined in (3.48),  $C_c$  is the condensation of cloud water, Ec is the evaporation of cloud water, and Er is the evaporation of rainwater.  $A_c$  is the autoconversion, which is the rate at which cloud water content decreases as particles grow to precipitation size by coalescence and/or vapor diffusion.  $E_c$  is the collection of cloud water, which is the rate at which the precipitation content increases as a result of the large falling drops intercepting and collecting small cloud droplets lying in their paths.  $F_r$  is the sedimentation of the raindrops in the air parcel; it is the net convergence of the vertical flux of rainwater relative to the air. All of the terms on the right of (3.65)-(3.67) are defined to be positive quantities, except for  $F_r$ , which is positive or negative depending on whether more rain is falling into or out of the air parcel. According to this model, cloud water  $q_c$  first appears by condensation  $C_c$ . Vapor is not condensed directly onto raindrops. Once sufficient cloud water has been produced, microphysical processes can then lead to the autoconversion  $(A_c)$  of some of the cloud water to rain. After autoconversion has begun to act, the amount of precipitation can then increase further through either  $A_c$  or  $K_c$ , or both. Once sufficient rainwater  $q_r$  has been produced, the additional microphysical processes  $E_r$  and  $F_r$  can become active.<sup>70</sup>

To calculate the microphysical sources and sinks of rainwater,  $A_c$ ,  $K_c$ ,  $E_r$ , and  $F_r$ , in terms of the mixing ratios  $q_v$ ,  $q_c$ , and  $q_r$ , several key assumptions are made about the raindrops. First, the terminal fall speeds (defined to be positive in the downward direction) of individual raindrops are related to drop diameter Dsuch that

$$V = V(D) > 0$$
 (3.68)

<sup>&</sup>lt;sup>70</sup> The bulk warm-cloud water-continuity model, consisting of the three categories vapor, cloud water, and rain interrelated by autoconversion and collection, was proposed by Kessler (1969). Virtually all

bulk water-continuity models used today are a direct outgrowth of the concepts introduced in Kessler's seminal monograph.

where V(D) is given by an empirical curve like that in Fig. 3.3. Second, it is assumed that all the rain in a parcel of air falls with the mass-weighted fall velocity

$$\hat{V} = \frac{\int_0^\infty V(D)\mathsf{m}(D)N(D)dD}{\int_0^\infty \mathsf{m}(D)N(D)dD}$$
(3.69)

where m(D) is the mass of a drop of diameter *D*, and N(D) dD is the number of drops per unit volume of air with diameter *D* to D + dD. Third, the precipitation particles are assumed to be exponentially distributed in size:

$$N = N_0 \exp(-\lambda_r D) \tag{3.70}$$

where  $N_o$  is an empirically determined constant. This distribution is called the Marshall-Palmer distribution.<sup>71</sup> A typical value of N<sub>o</sub> in rain is ~ 8 ×10<sup>6</sup> m<sup>-4</sup>. The quantity  $\lambda_r$  is a function of the total rainwater mixing ratio  $q_r$ . Its value is determined by inverting the integral:

$$q_r = \rho^{-1} \int_0^\infty \mathsf{m}(D) N_0 \exp(-\lambda_r D) dD \qquad (3.71)$$

The above assumptions allow the microphysical source/sink terms  $K_c$ ,  $E_r$ , and  $F_r$  to be calculated. According to the continuous collection equation (3.24), the rate of increase of the mass m of an individual raindrop of diameter D is given by

$$\dot{\mathfrak{m}}_{col} = \frac{\pi D^2}{4} V(D) \rho q_c \sum_{rc} \qquad (3.72)$$

where  $\Sigma_{rc}$  is the collection efficiency of raindrops collecting cloud drops. The variables  $\Sigma_{rc}$  and V(D) are assumed to be known empirically. The fall velocity of the cloud liquid water is assumed to be zero, in accordance with the definitions given in Sec. 3.3.  $\Sigma_{rc}$  is usually taken to be ~1, which is an approximation that is representative of the majority of rainfall situations (Sec. 2.5.1). The depletion of cloud water by collection  $K_c$  by all of the raindrops in a parcel of air is computed from the integral

<sup>71</sup> After Marshall and Palmer (1948), who analyzed images of a large sample of raindrops collected on treated filter paper.

$$K_c = \rho^{-1} \int_0^\infty \dot{\mathsf{m}}_{\rm col} N(D) dD \tag{3.73}$$

where N(D) is assumed to have the exponential form (3.70). The bulk rate of evaporation of rainwater mass from all raindrops  $E_r$  is computed in an analogous manner from the integral

$$E_r = \rho^{-1} \int_0^\infty \dot{\mathfrak{m}}_{\rm dif} N(D) dD \qquad (3.74)$$

where  $\mathfrak{m}_{dif}$  is the rate of evaporation by diffusion of vapor mass away from a single raindrop of diameter *D* falling through unsaturated air. A relationship for  $\mathfrak{m}_{dif}$  as a function of *D* must be obtained by numerical solution of (3.8), (3.9), and (3.11). It is not given in general by (3.22), which applies only when the air is saturated (i.e., in cloud). Often the rain is falling below cloud base, where the air is quite unsaturated.

The mass-weighted fall speed of the rain, given by (3.69), can be used to calculate the sedimentation of raindrops in the air parcel according to

$$F_r = \frac{\partial}{\partial z} \left( \hat{V} q_r \right) \tag{3.75}$$

The autoconversion rate  $A_c$  is usually assumed to be proportional to the amount by which the cloud liquid water mixing ratio exceeds a selected threshold; that is,

$$A_c = \widetilde{\alpha} \left( q_c - a_T \right) \tag{3.76}$$

where  $a_T$  is the autoconversion threshold (often assumed arbitrarily to be 1gkg<sup>-1</sup>) and  $\tilde{\alpha}$  is a positive constant when  $q_c > a_T$  and 0 otherwise.<sup>72</sup> Thus, whenever cloud water exceeds the threshold amount, it is converted to rainwater at an exponential rate.

<sup>&</sup>lt;sup>72</sup> This autoconversion formulation was postulated intuitively by Kesslre (1969) as part of his basic warm-cloud bulk parameterization scheme. Other autoconversion formulas have been developed from

the physics of droplet coalescence (Cotton, 1972; Berry and Reinhardt, 1973). Cotton (1972), however, showed that Kessler's simple formula works about as well as the other formulas.

#### **3.6.2** Bulk Modeling of Cold Clouds

The bulk water-continuity model described in the previous subsection can be extended to cold clouds by adding the categories of cloud ice, snow, and graupel, represented by the mixing ratios  $q_I$ ,  $q_s$ , and  $q_g$  defined in (3.48).<sup>73</sup> We thus obtain the six-category water-continuity scheme illustrated in Fig. 3.15. The water-continuity equations (2.21) may then be written as

$$\frac{Dq_{\nu}}{Dt} = \left(-C_c + E_c + E_r\right)\delta_4 + S_{\nu}$$
(3.77)

$$\frac{Dq_{c}}{Dt} = (C_{c} - E_{c} - A_{c} - K_{c})\delta_{4} + S_{c}$$
(3.78)

$$\frac{Dq_{r}}{Dt} = (A_{c} + K_{c} - E_{r} + F_{r})\delta_{4} + S_{r}$$
(3.79)

$$\frac{Dq_I}{Dt} = S_I \tag{3.80}$$

$$\frac{Dq_s}{Dt} = F_s + S_s \tag{3.81}$$

$$\frac{Dq_g}{Dt} = F_g + S_g \tag{3.82}$$

where the S terms represent all of the sources and sinks associated with ice-phase microphysical processes, except for the sedimentation of snow and graupel, which are represented by terms  $F_s$  and  $F_g$ , respectively. The term  $\delta_4$  is defined as

$$\delta_4 = \begin{cases} 0 & \text{if } T < -40 \,^{\circ}\text{C} \\ 1 & \text{otherwise} \end{cases}$$
(3.83)

 $^{73}$  This extension of the bulk water-continuity method to cold clouds was developed by Lin *et al.* (1983).

Thus, it is assumed that, if the air temperature drops below  $-40^{\circ}$ C, all supercooled water freezes by homogeneous nucleation (Sec. 3.2.1) and hence all the terms in the liquid-water part of the model are set to zero.

The terms on the right in (3.77)-(3.82) include all of the possible interactions among the six categories of water, as illustrated in Fig. 3.15. Among these interactions are several bulk collection terms of the form (3.73). These represent graupel collecting cloud water and rain water, snow collecting cloud ice, etc. There are also several evaporation terms of the form (3.74). These include the sublimation and depositional growth of snow, graupel, and cloud ice. In addition, there are melting terms representing the increase of rainwater mixing ratio as a result of the melting of snow and graupel. The process of shedding liquid water collected by but not frozen to the surface of graupel or hail particles is also included. There are also three-way interactions that can occur, such as rain collecting cloud ice to produce graupel or hail.

To obtain mathematical expressions for the  $F_g$ ,  $F_s$ , and S terms in (3.77)-(3.82), the same types of basic assumptions are made about the precipitating ice particles as were made for raindrops in the warm-cloud scheme. Crude assumptions are made regarding the collection efficiencies of ice particles, since very little is known about them. For riming (i.e., ice particles collecting liquid particles), the collection efficiency is usually assumed to be  $\sim 1$ . The collection efficiencies of ice particles collecting other ice particles is sometimes assumed to be a function of temperature that drops off exponentially from a value  $\sim 1$  at 0°C to zero at lower temperatures. This assumption mirrors the observation of more frequent aggregation of falling particles as they near the melting level (Fig. 3.9). The precipitation particles are assumed to be exponentially distributed, as in (3.70), but with different values of  $N_{o}$ . For example,  $N_o$  might be assumed to be ~ 8 ×10<sup>6</sup>- 2 ×10<sup>7</sup> m<sup>-4</sup> for snow, <sup>74</sup> ~ 4 ×10<sup>6</sup> m<sup>-4</sup> for graupel,<sup>75</sup> and  $\sim 3 \times 10^4$  m<sup>-4</sup> for hail.<sup>76</sup> The fall speeds of snow and high-density ice particles are assumed to be known empirically as functions of particle diameter, as in (3.68), and the precipitation in a parcel of air is assumed to fall with the mass-weighted fall velocity, similar to that expressed by (3.69).<sup>77</sup>

<sup>&</sup>lt;sup>74</sup> This value exhibits a temperature dependence. See Houze *et al.* (1979).

<sup>&</sup>lt;sup>75</sup> See Rutledge and Hobbs (1983, 1984).

<sup>&</sup>lt;sup>76</sup> See Lin *et al.* (1983).

<sup>&</sup>lt;sup>77</sup> See Lin *et al.* (1983) for further details of how the cold-cloud bulk parameterization terms may be formulated. Rutledge and Hobbs (1983) give a concise summary of the technique.