

# The Maximum Intensity of Hurricanes

Speaker: Shih-Chieh Tai

## REFERENCES

- Emanuel, K. A., 1988: The maximum intensity of hurricanes. *J. Atmos. Sci.*, **45**, 1143–1155.
- Emanuel, K. A., 1994: *Atmospheric Convection*. Oxford University Press, New York, 580 pp.

# I. Introduction

1. Knowledge of the factors that set an **upper bound** on the intensity of hurricanes
2. Emanuel, K. A., 1986: An air-sea interaction theory for tropical cyclones. Part I. *J. Atmos. Sci.*, **43**, 585–604.

- Such an upper bound is determined by the **product** of
  - a. The **maximum possible latent heat** input from ocean to atmosphere
  - b. **Thermodynamic efficiency** proportional to the temperature difference between the sea surface and lower stratosphere
- Directly related to **the maximum possible pressure deficit** in the eye

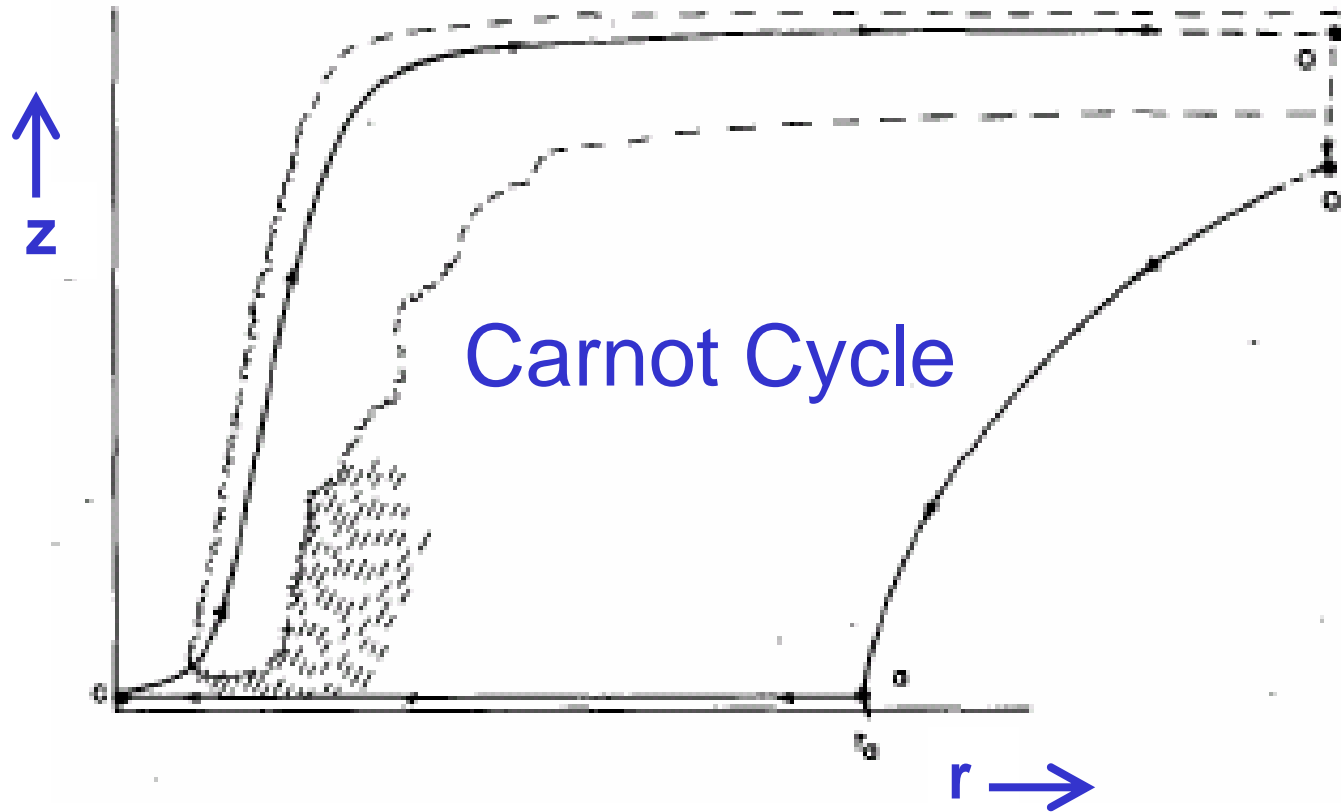
# Purpose?

- To derive an exact **equation for the maximum pressure drop**
  - a. Accounts for **fully reversible thermodynamics** and the **effects of water substance on density**
  - b. Point out that the equation has **NO** solution under certain conditions

## **II. The relation for minimum central pressure**

1. Constraints on the distributions of **entropy**, **total water** and **angular momentum**
2. Constraint on the **variation of entropy at the surface**

Consider a steady-state axisymmetric hurricane over an ocean with uniform temperature.



- Assume that outside a frictional boundary layer, and except at large radii in the outflow, three properties of the flow are **conserved**
  - a. Angular momentum per unit mass ( $M$ )
  - b. Total entropy ( $s$ )
  - c. Total water (liquid plus vapor,  $Q$ )



- $s$  and  $Q$  are functions of  $M$  alone, i.e., they are constant along the same  $M$  surface in the hurricane.

# The central pressure equation

Thermodynamic efficiency

$$\varepsilon \equiv \frac{T_s - \bar{T}_0}{T_s}$$

$$A = \frac{\varepsilon}{1 - \varepsilon} \frac{L_v}{R_v T_s} \frac{e_s}{p_{da}} \left\{ 1 - \underbrace{\frac{g \bar{z}_0^*}{\varepsilon L_v}}_{(a)} - \underbrace{\frac{1}{8} \frac{f^2 \bar{r}_1^{2*}}{\varepsilon L_v}}_{(b)} \right.$$

$$\left. + \underbrace{\frac{C_l T_s}{\varepsilon L_v} [\varepsilon + (1 - \varepsilon) \ln(1 - \varepsilon)]}_{(c)} \right\}$$

(19)

$$\ln x = -A \left[ \frac{1}{x} - B \right] \quad (18)$$

$$x = \frac{p_{dc}}{p_{da}}$$

$$B = \text{RH} \left[ 1 + \frac{e_s}{p_{da}} \frac{\ln(\text{RH})}{A} \right] + \frac{1}{4} \frac{f^2 r_a^2 \left( 1 + \text{RH} \frac{e_s}{p_{da}} \right)}{[R_d T_s (1 - \varepsilon) A]}$$

## Solutions?

$$\ln x = -A \left[ \frac{1}{x} - B \right] \quad (18)$$

Define  $y \equiv \frac{1}{x} = \frac{P_{da}}{P_{dc}}$

The maximum pressure drop that can be sustained by a given inward increase in vapor mixing ratio


$$y = \exp[A(y - B)] \quad (22)$$

Includes the pressure dependence of the core mixing ratio itself

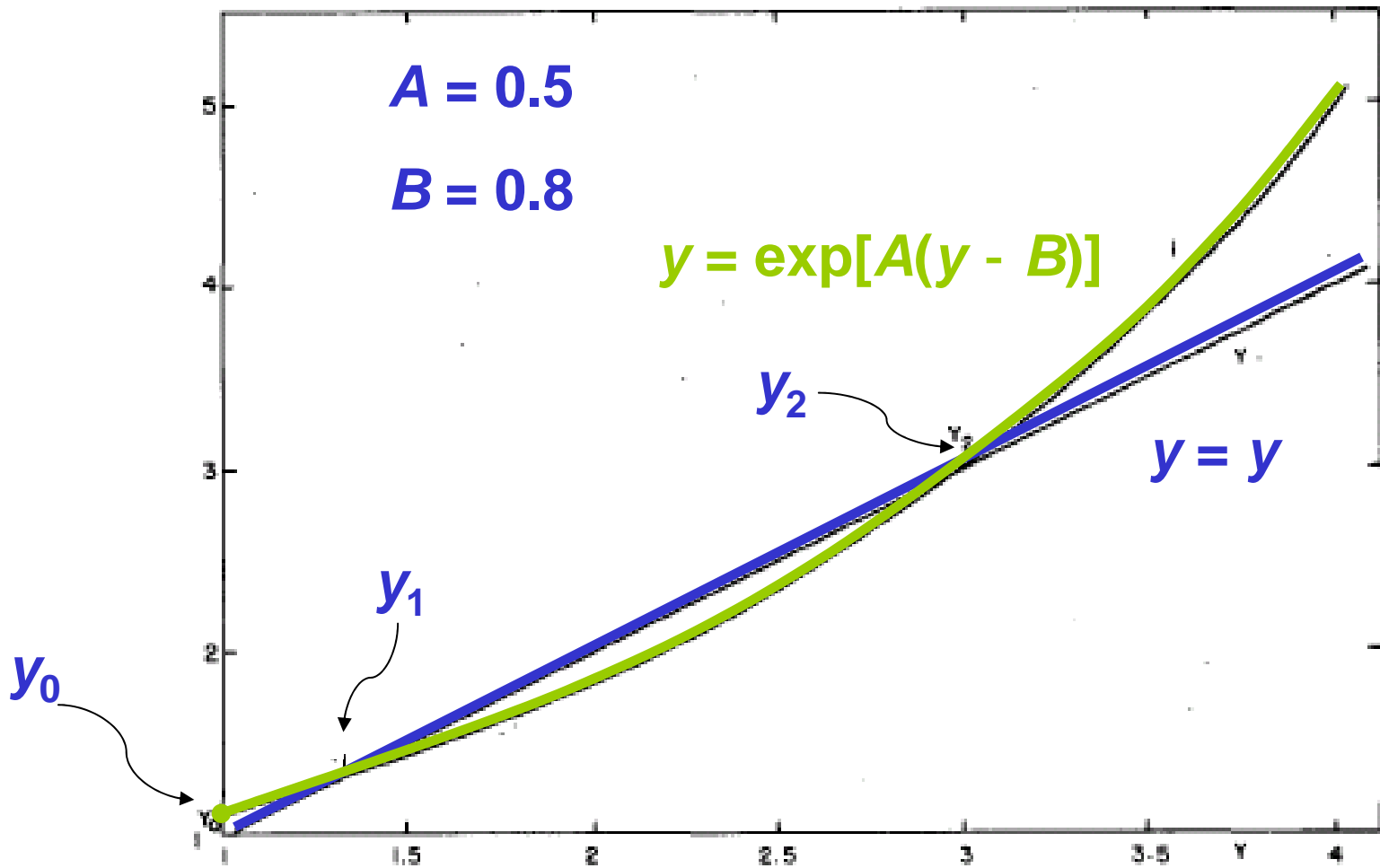


FIG. 1. The left- and right-hand sides of (22) as a function of  $y$ , for  $A = 0.5$  and  $B = 0.8$ . The central pressure deficit would be given by  $Y_0$  if there were no isothermal expansion effects.  $Y_1$  and  $Y_2$  denote the equilibrium solutions of (22).

Which one is the solution of our  
**INTEREST?**

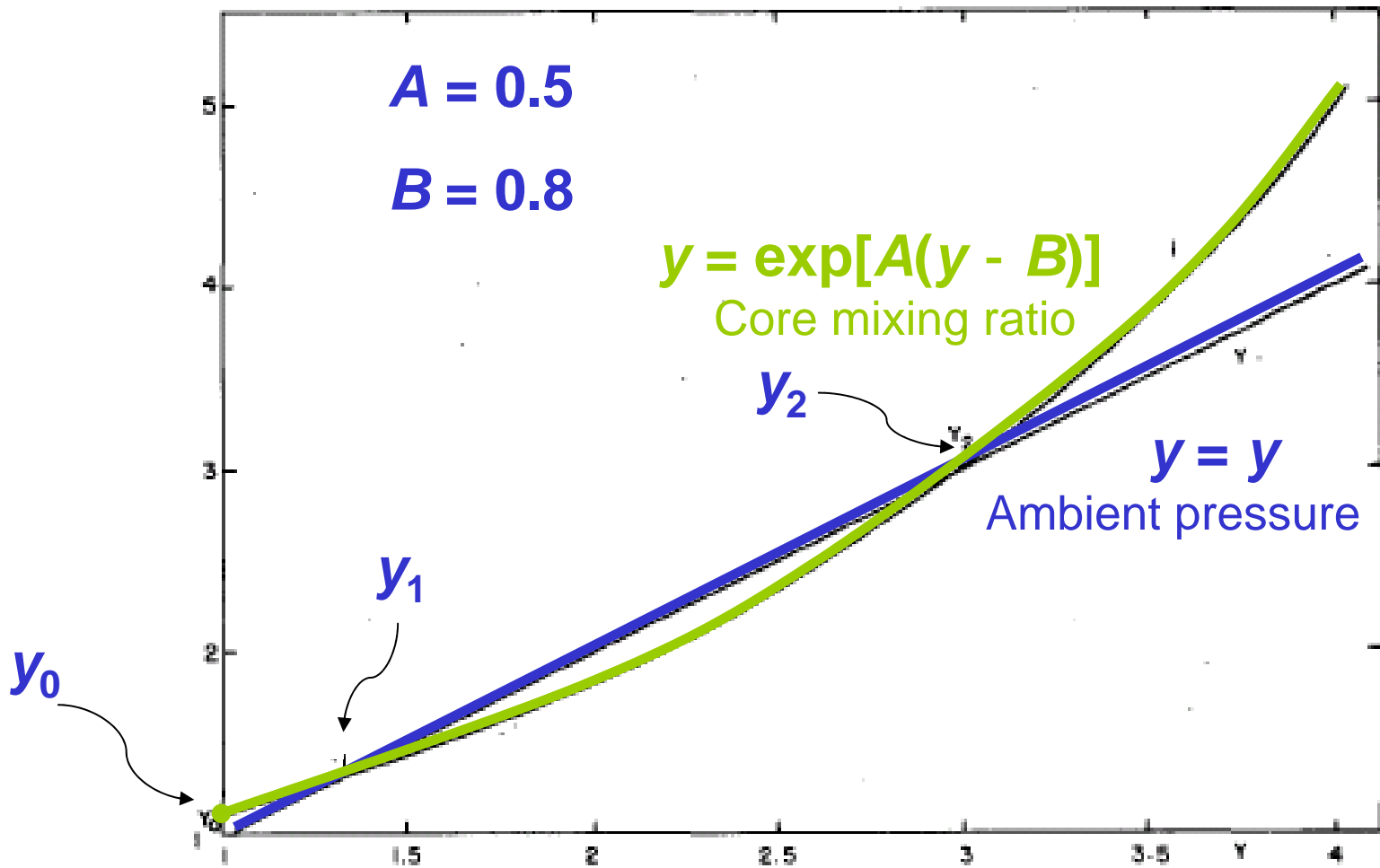


FIG. 1. The left- and right-hand sides of (22) as a function of  $y$ , for  $A = 0.5$  and  $B = 0.8$ . The central pressure deficit would be given by  $Y_0$  if there were no isothermal expansion effects.  $Y_1$  and  $Y_2$  denote the equilibrium solutions of (22).

★ Mean August conditions in the near-equatorial Western North Pacific

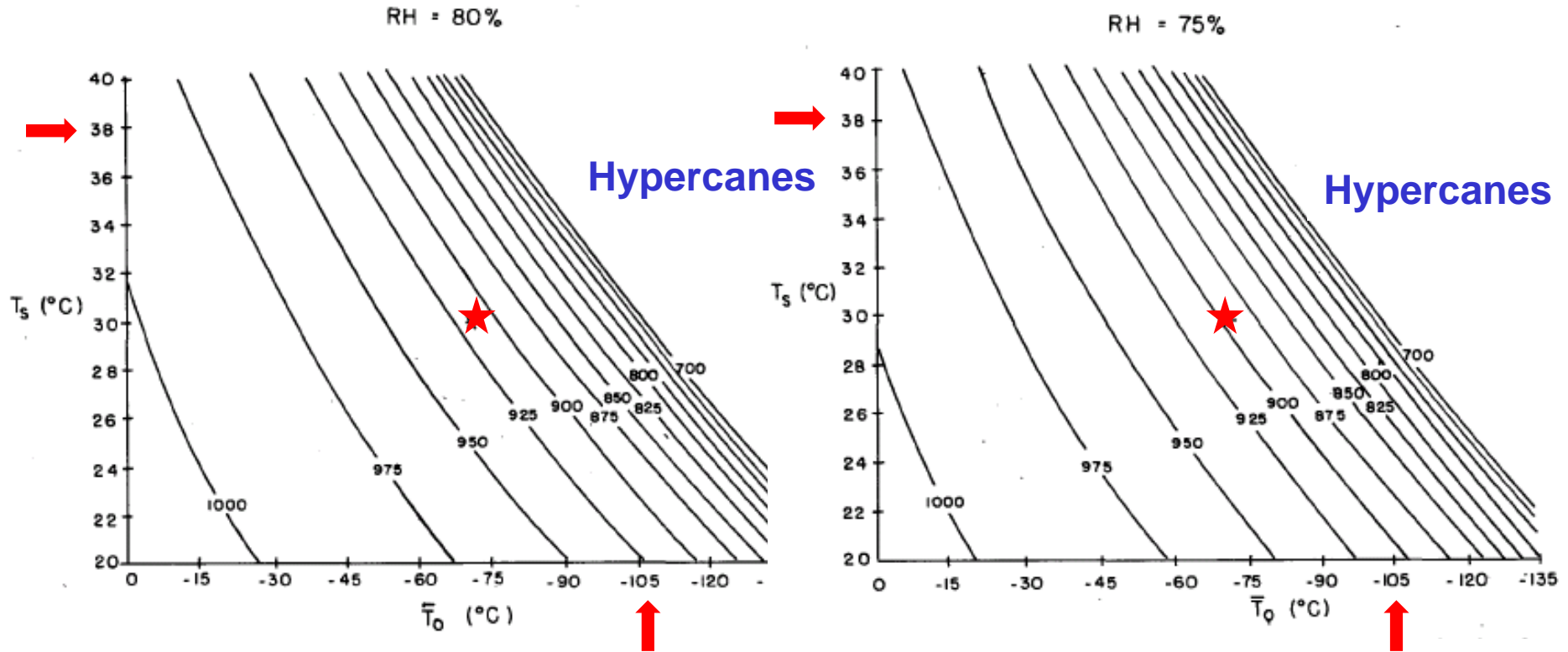


FIG. 3. (a) Minimum sustainable central pressure (mb) of tropical cyclones as a function of sea surface temperature ( $T_s$ ) and entropy-weighted mean outflow temperature ( $\bar{T}_o$ ), assuming reversible thermodynamics and an ambient surface relative humidity of 80%. The asterisk denotes mean August conditions in the near-equatorial western North Pacific. (b) As in (a) but for a surface relative humidity of 75%.



# **III. Hurricanes in the supercritical regime**

- From (18), the nature of processes that might actually **limit** hurricanes in the **supercritical regime** may be speculated:
  - a. Inflow in supercritical hurricanes is not approximately isothermal.
  - b. The outflow may not match up with the ambient environment.
  - c. The assumption that all of the dissipation occurs in the inflow layer and at large radii in the outflow **breaks down** and internal dissipation becomes important.

- Hurricanes occurring over oceanic mixed layers of finite depth are known to cool the sea surface temperatures by upwelling and mixing; this may limit hurricane intensity under some conditions.

Suppose for the present that whatever process limits the intensity of hypercanes nevertheless results in storms of extraordinary intensity.

What do they look like?

1. Hypercanes would penetrate large distances into the stratosphere due to the very high core values of the moist entropy.
2. The ratio of the radius of maximum winds to the outer radius would be relatively small in hypercanes.

For  $T_s = 40\text{C}$ ,  $T_0 = -73\text{C}$ ,  $\Rightarrow 13 \text{ km}$   
 $RH = 0.8$ , and  $x = 0.5$

For  $T_s = 30\text{C}$ ,  $T_0 = -73\text{C}$ ,  $\Rightarrow 2 \text{ km}$   
 $RH = 0.8$ , and  $x = 0.9$

## IV. Conclusions

1. Emanuel had derived an exact equation (18) governing the minimum sustainable central pressure of hurricanes in this paper.
2. The Carnot cycle derivation shows that the **only** approximations necessary in deriving (18) are
  - a. No radial temperature gradient in the mixed layer
  - b. No dissipation except within the inflow and at large radii in the outflow

## IV. Conclusions

3. Mature storms that might occur in the supercritical regime, called **Hypercanes**, would extend very high into the stratosphere and have either very large outer radii or very small eyes.

## IV. Conclusions

4. Holding the temperature of the lower stratosphere constant, sea surface temperatures would have to be **6C to 10C warmer** than present values to sustain hypercanes.
  - a. Some estimates indicate **middle Cretaceous** tropical sea surface temperatures as much as **7C warmer** than at present.
  - b. Hypercanes might have been possible at that time, unless the lower stratosphere was also substantially warmer.



## IV. Conclusions

5. The solutions to (18) under present conditions give quite reasonable estimates of the central pressures of the most intense storms on record (Emanuel, 1987), indicating that the upper bound provided by (18) is actually achieved in a small number of storms.
6. Is a tropical cyclone of **extraordinary intensity** and **large internal dissipation** actually **possible** in the supercritical regime?

**THE END**

## APPENDIX A

# **Expressions for Moist Entropy and Moist Static Energy Valid in Saturated and Unsaturated Air**

The following quantity can be shown to be conserved during reversible moist or dry adiabatic expansion:

$$s = (C_{pd} + QC_l) \ln T + \frac{L_v w}{T} - R_d \ln p_d - wR_v \ln(\text{RH}), \quad (\text{A1})$$

where  $C_{pd}$  and  $C_l$  are the heat capacities of dry air and liquid water, respectively,  $Q$  is the total water content,  $L_v$  is the heat of vaporization (a function of temperature),  $w$  the vapor mixing ratio,  $R_d$  and  $R_v$  are the gas constants for dry air and water vapor, respectively, and RH is the relative humidity.

Differentiation of (A1) yields

$$Tds = (C_{pd} + QC_l)dT + L_v dw - \frac{R_d T}{p_d} dp_d + w dL_v - \frac{L_v w}{T} dT - wR_v T d \ln(\text{RH}) - R_v T \ln(\text{RH}) dw. \quad (\text{A2})$$

The last term of (A2) vanishes since reversible changes in  $w$  can only occur at  $\text{RH} = 1$ . The temperature dependence of  $L_v$  is given by

$$dL_v = (C_{pv} - C_l)dT,$$

where  $C_{pv}$  is the heat capacity of water vapor at constant pressure. Also, the Clausius-Clapeyron equation may be written:

$$L_v \frac{dT}{T} = R_v T \frac{de_s}{e_s} = R_v T \left( \frac{de}{e} - \frac{d\text{RH}}{\text{RH}} \right),$$

since  $e_s = e/\text{RH}$ . Using these two expressions in (A2) gives

$$Tds = (C_{pd} + wC_{pv} + lC_l)dT + L_v dw - \frac{R_d T}{p_d} dp_d - wR_v T \frac{de}{e}. \quad (\text{A3})$$

Finally, we note that since  $w = \alpha_d/\alpha_v$ , where  $\alpha_d$  and  $\alpha_v$  are the specific volumes of dry air and water vapor, respectively, (A3) can be written

$$Tds = (C_{pd} + wC_{pv} + lC_i)dT + L_vdw - \alpha_d dp, \quad (\text{A4})$$

which is a direct statement of the first law of thermodynamics written as entropy changes per unit mass of dry air. This proves that (A1) is a uniformly valid expression for entropy of moist air.

A uniformly conserved moist static energy can also be derived from (15). Using the hydrostatic equation, this is

$$h = (C_{pd} + QC_i)T + L_v w + (1 + Q)gz. \quad (\text{A5})$$

It follows from (A5) that for reversible processes

$$\begin{aligned} dh &= (C_{pd} + QC_i)dT + L_v dw + w dL_v + (1 + Q)gdz \\ &= (C_{pd} + QC_i)dT + L_v dw + w(C_{pv} - C_i)dT \\ &\quad + (1 + Q)gdz \\ &= (C_{pd} + wC_{pv} + lC_i)dT + L_v dw - \alpha_d dp = 0. \end{aligned} \quad (\text{A6})$$

The last line of the above is simply a statement of the first law of thermodynamics. This shows that  $h$  is conserved for hydrostatic reversible displacements.

## APPENDIX B

# Maxwell's Relations for Reversible Moist Processes

We begin with the first law of thermodynamics, which can be obtained by differentiating (2) and making use of the Clausius-Clapeyron equation. The result is

$$Tds = -\alpha_d dp + (C_{pd} + QC_i)dT + d[L_v w] + C_i T \ln T dQ. \quad (\text{B1})$$

This is valid in both saturated and unsaturated air. The last term on the right arises because the system is not closed to exchange of water mass. Before proceeding further, we rewrite the first term on the right as follows:

$$\begin{aligned} -\alpha_d dp &= -d[\alpha_d p] + pd(\alpha_d) \\ &= -d\left[\alpha_d p_d + \frac{\alpha_d}{\alpha_v} \alpha_v e\right] + pd(\alpha_d) \\ &= -d[R_d T + wR_v T] + pd(\alpha_d). \end{aligned} \quad (\text{B2})$$

Combining (B2) with (B1) yields

$$Tds = pd(\alpha_d) + (C_{vd} + QC_i)dT + d[(L_v - R_v T)w] + C_i T \ln T dQ, \quad (\text{B3})$$

where  $C_{vd}$  is the heat capacity of dry air at constant volume.

We next define a moist enthalpy  $h$ :

$$h = p\alpha_d + (C_{vd} + C_l Q)T + (L_v - R_v T)w. \quad (\text{B4})$$

It then follows from (B3) and (B4) that

$$dh = Tds + \alpha_d dp + C_l T(1 - \ln T)dQ. \quad (\text{B5})$$

Using (B5) we then obtain the relations

$$\begin{aligned} \left(\frac{\partial h}{\partial s}\right)_{p,Q} &= T, \\ \left(\frac{\partial h}{\partial p}\right)_{s,Q} &= \alpha_d, \\ \left(\frac{\partial h}{\partial Q}\right)_{s,p} &= C_l T[1 - \ln T], \end{aligned} \quad (\text{B6})$$

from which it follows, by cross-differentiation, that

$$\begin{aligned} \left(\frac{\partial \alpha_d}{\partial s}\right)_{p,Q} &= \left(\frac{\partial T}{\partial p}\right)_{s,Q}, \\ \left(\frac{\partial \alpha_d}{\partial Q}\right)_{s,p} &= \left(\frac{\partial}{\partial p} [C_l T(1 - \ln T)]\right)_{s,Q}. \end{aligned} \quad (\text{B7})$$

Furthermore, since  $\alpha = \alpha_d/(1 + Q)$ , (B7) can be written

$$\left(\frac{\partial \alpha}{\partial s}\right)_{p,Q} = \frac{1}{(1 + Q)} \left(\frac{\partial T}{\partial p}\right)_{s,Q}, \quad (\text{B8})$$

$$\left(\frac{\partial \alpha}{\partial Q}\right)_{s,p} = \frac{1}{1 + Q} \frac{\partial}{\partial p} [C_l T(1 - \ln T)]_{s,Q} - \frac{\alpha}{1 + Q}. \quad (\text{B9})$$

Finally, the last term on the right of (B9) can be reexpressed using the hydrostatic equation

$$-\alpha = \frac{\partial \phi}{\partial p}, \quad (\text{B10})$$

where  $\phi = gz$  is the geopotential. Using the chain rule and gradient wind balance it is possible to show that

$$\left(\frac{\partial \phi}{\partial p}\right)_r = \left(\frac{\partial \Phi}{\partial p}\right)_M, \quad (\text{B11})$$

where  $\Phi = \phi + \frac{1}{2}V^2$  and  $V$  is the gradient wind. Using (B11) and (B10), (B9) becomes

$$\begin{aligned} \left(\frac{\partial \alpha}{\partial Q}\right)_{s,p} &= \frac{1}{1 + Q} \frac{\partial}{\partial p} [C_l T(1 - \ln T)]_{s,Q} \\ &\quad + \frac{1}{1 + Q} \left(\frac{\partial \Phi}{\partial p}\right)_M. \end{aligned} \quad (\text{B12})$$

The relations (B8) and (B12) are the desired Maxwell's relations.



## APPENDIX C

### **Carnot Cycle Derivation of (18)**

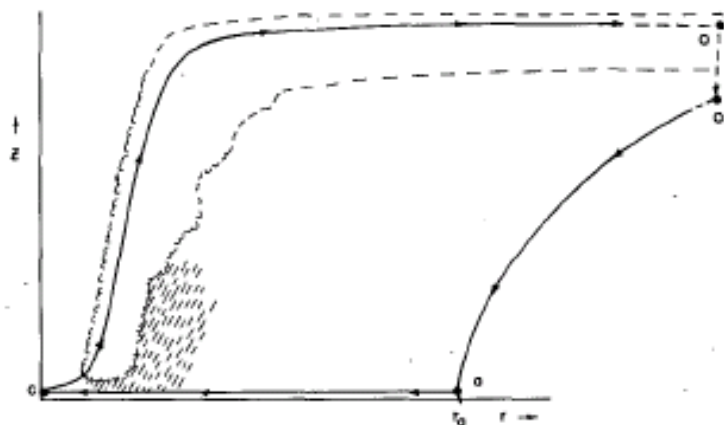


FIG. C1. Illustrating the path integral for the Carnot cycle. Points  $o$  and  $o'$  are taken to lie at very large radius.

We begin by writing a Bernoulli Equation for steady flow:

$$d\left(\frac{1}{2} V^2\right) + \alpha dp + g dz - \mathbf{F} \cdot d\mathbf{l} = 0, \quad (C1)$$

where  $V$  is the magnitude of the total velocity vector,  $\mathbf{F}$  is the vector friction force and  $\mathbf{l}$  is a unit vector parallel to streamlines. It is understood that the derivatives of (C1) are everywhere along streamlines. We next integrate (C1) around a closed streamline as indicated in Fig. C1. The result is

$$\oint \alpha dp = \oint \mathbf{F} \cdot d\mathbf{l}, \quad (C2)$$

which simply expresses a balance between pressure work and dissipation in steady flow. We also integrate (C1) between points  $a$  and  $c$  in Fig. C1. Since  $z = 0$  and  $V$  vanishes at both points the result is

$$\int_a^c \alpha dp = \int_a^c \mathbf{F} \cdot d\mathbf{l}. \quad (C3)$$

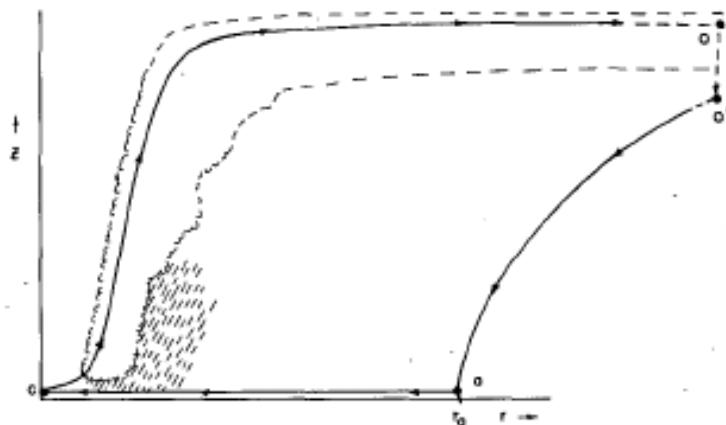


FIG. C1. Illustrating the path integral for the Carnot cycle. Points  $o$  and  $o'$  are taken to lie at very large radius.

We next assume that all frictional dissipation occurs only between points  $a$  and  $c$  and between points  $o$  and  $o'$  in Fig. C1. Thus

$$\oint \mathbf{F} \cdot d\mathbf{l} = \int_a^c \mathbf{F} \cdot d\mathbf{l} + \int_o^{o'} \mathbf{F} \cdot d\mathbf{l}. \quad (\text{C4})$$

Combining (C2), (C3) and (C4) results in

$$\int_a^c \alpha dp = \oint \alpha dp - \int_o^{o'} \mathbf{F} \cdot d\mathbf{l}. \quad (\text{C5})$$

Since  $\alpha = \alpha_d / (1 + Q)$ , it follows that

$$\alpha = \alpha_d - \alpha Q. \quad (\text{C6})$$

Using (C6) in (C5) we obtain

$$\begin{aligned} \int_a^c \alpha_d dp &= \oint \alpha_d dp - \int_o^{o'} \mathbf{F} \cdot d\mathbf{l} \\ &\quad + \int_a^c \alpha Q dp - \oint \alpha Q dp. \end{aligned} \quad (\text{C7})$$

The last two terms of (C7) can be written as

$$-\int_c^a Q \alpha dp,$$

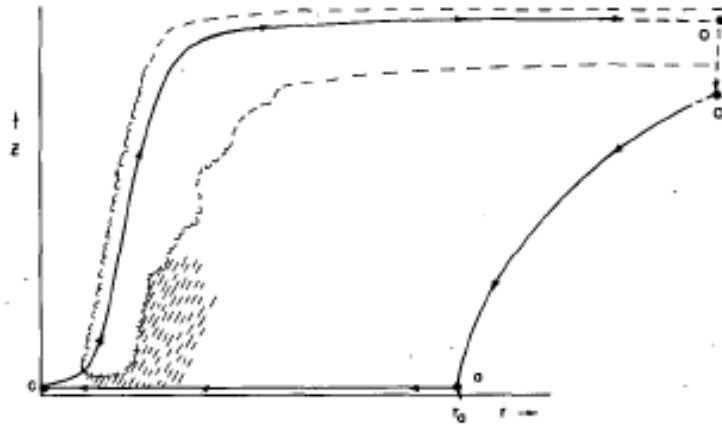


FIG. C1. Illustrating the path integral for the Carnot cycle. Points  $o$  and  $o'$  are taken to lie at very large radius.

where it is understood that the path of integration is  $c-o-o'-a$ . Once again using (C1) to replace  $\alpha dp$  in the above, the last two terms in (C7) can be written

$$\begin{aligned}
 -\int_c^a Q\alpha dp &= \int_c^a Q \left[ d\left(\frac{1}{2} V^2 + gz\right) - \mathbf{F} \cdot d\mathbf{l} \right] \\
 &= -\int_o^{o'} \left[ \left(\frac{1}{2} V^2 + gz\right) dQ + Q\mathbf{F} \cdot d\mathbf{l} \right], \quad (C8)
 \end{aligned}$$

where we have made use of the vanishing of  $z$  and  $V$  at the end points and the assumption that irreversible processes act only between  $o$  and  $o'$  along the integration path  $c-o-o'-a$ .

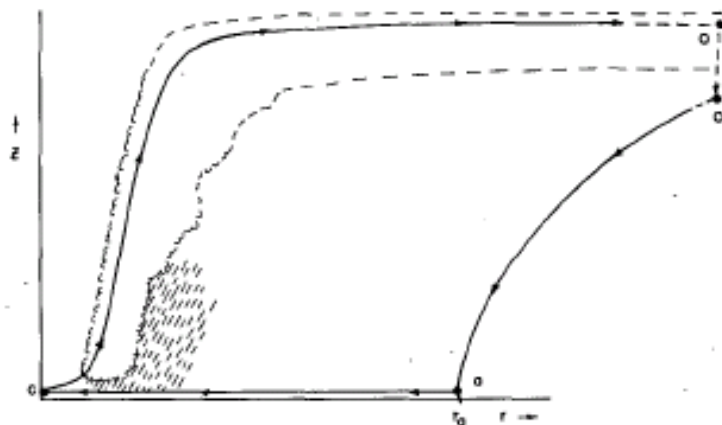


FIG. C1. Illustrating the path integral for the Carnot cycle. Points  $o$  and  $o'$  are taken to lie at very large radius.

We once again use (C1) to estimate the frictional dissipation necessary to close the Carnot cycle. Under the plausible assumption that the flow is hydrostatic between  $o$  and  $o'$ , (C1) shows that  $\mathbf{F} \cdot d\mathbf{l} = d(\frac{1}{2}V^2)$  in this region. Using this estimate in (C8) and (C7) the two relations may be combined to give

$$\begin{aligned} & \int_a^c \alpha_d dp \\ &= \oint \alpha_d dp - \int_o^{o'} \left[ (gz)dQ + d\left(1 + Q\right) \frac{1}{2} V^2 \right] \\ &= \oint \alpha_d dp - g\bar{z}_0^* Q \Big|_o^{o'} - \left[ \frac{1}{2} (1 + Q) V^2 \right] \Big|_o^{o'}, \quad (C9) \end{aligned}$$

where  $\bar{z}_0^*$  is defined as before.

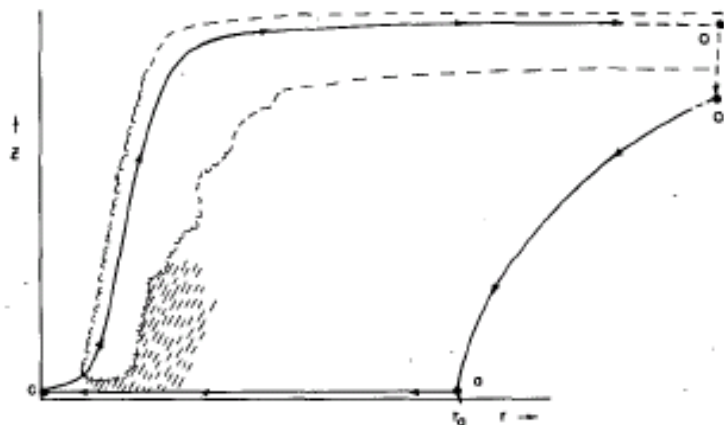


FIG. C1. Illustrating the path integral for the Carnot cycle. Points  $o$  and  $o'$  are taken to lie at very large radius.

Finally, we use (15) to eliminate  $\alpha_d dp$  from (C9). After some integrations by parts, and noting that entropy only changes at the surface and between  $o$  and  $o'$ , we obtain

$$\begin{aligned} \bar{T}_0(s_c - s_a) = & (w_c - w_a)[L_v + C_l T_s \\ & + C_l \bar{T}_0^* (\ln \bar{T}_0^* - 1) - g \bar{z}_0^*] + \left[ \frac{1}{2} (1 + Q) V^2 \right] \Big|_o^{o'}. \end{aligned} \quad (\text{C10})$$

Using (1) to relate  $V$  to  $r$  and  $M$  and noting that  $M_a = \frac{1}{2} f r_a^2$ , it can easily be seen that (C10) is equivalent to (17).

In the pseudo-adiabatic case we allow all condensate to fall out of the system immediately upon forming. To derive a pressure equation in this case, we start with (C7) and this time insist that  $Q = w$  everywhere. Then (C8) is instead written

$$\begin{aligned} - \int_c^a Q \alpha dp = & \int_c^a w \left[ d \left( \frac{1}{2} V^2 + gz \right) - \mathbf{F} \cdot d\mathbf{l} \right] \\ = & \left( \frac{1}{2} V_o^2 + gz_o \right) (\bar{w}_u - \bar{w}_d) - \int_o^{o'} w \mathbf{F} \cdot d\mathbf{l}, \end{aligned} \quad (\text{C11})$$

where  $\frac{1}{2} V_o^2 + gz_o$  is evaluated at point  $o$  and  $\bar{w}_u$  and  $\bar{w}_d$  are defined

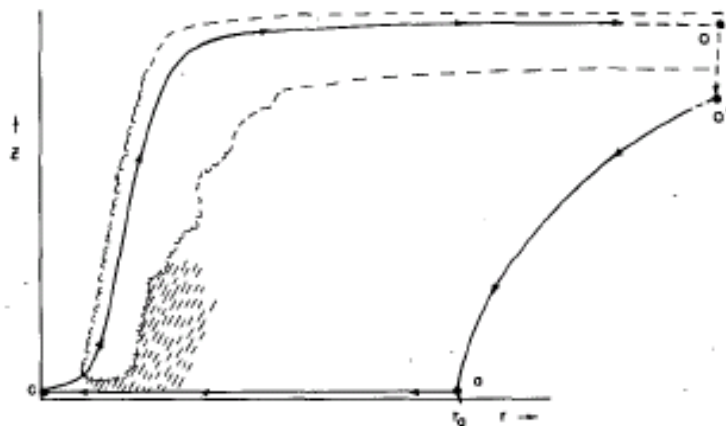


FIG. C1. Illustrating the path integral for the Carnot cycle.  
Points  $o$  and  $o'$  are taken to lie at very large radius.

$$\begin{aligned}\bar{w}_u &\equiv \frac{1}{\frac{1}{2}V_o^2 + gz_o} \int_c^o wd\left(\frac{1}{2}V^2 + gz\right), \\ \bar{w}_d &\equiv \frac{1}{\frac{1}{2}V_o^2 + gz_o} \int_a^o wd\left(\frac{1}{2}V^2 + gz\right).\end{aligned}\quad (C12)$$

Combining (C12) with (C7) gives

$$\begin{aligned}\int_a^c \alpha_d dp &= \oint \alpha_d dp - \int_o^{o'} (1+w)F \cdot dl \\ &\quad + \left(\frac{1}{2}V_o^2 + gz_o\right)(\bar{w}_u - \bar{w}_d).\end{aligned}\quad (C13)$$

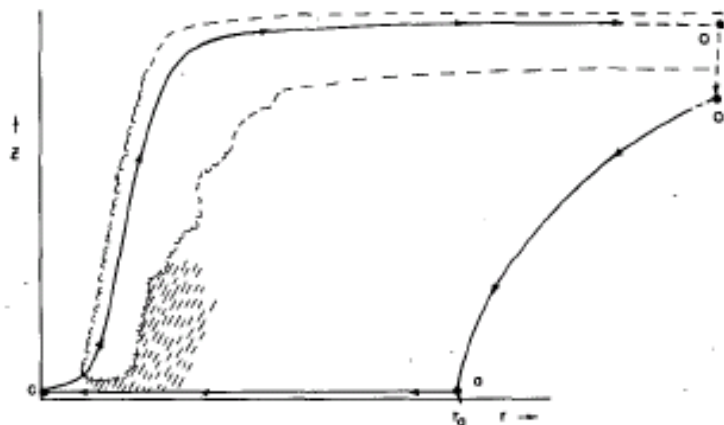


FIG. C1. Illustrating the path integral for the Carnot cycle. Points  $o$  and  $o'$  are taken to lie at very large radius.

As before, we use the first law to eliminate  $\alpha_d dp$  from (C13). In the pseudoadiabatic case, the first law can be written (see Iribarne and Godson, 1973)

$$-\alpha_d dp = T ds' - (C_{pd} + wC_l) dT - d[L_v w]. \quad (C14)$$

Unlike the reversible case,  $s'$  is a state variable as well as a conserved quantity so that its material derivative is the same as its general derivative. Substituting (C14) into (C13) yields

$$\begin{aligned} \bar{T}_0(s'_c - s'_a) = & L_v(w_c - w_a) + C_l(T_s - T_0)(\tilde{w}_u - \tilde{w}_d) \\ & - \left( \frac{1}{2} V_{o'}^2 + g z_{o'} \right) (\bar{w}_u - \bar{w}_d) + \int_o^{o'} (1 + w) d\left( \frac{1}{2} V^2 \right), \end{aligned} \quad (C15)$$

where

$$\tilde{w}_u = \frac{1}{T_s - T_o} \int_c^a w dT,$$

$$\tilde{w}_d = \frac{1}{T_s - T_o} \int_a^o w dT.$$

This can be directly compared to (C10) which applies to the reversible case. The main differences are the different definitions of  $s$  and  $s'$  and the replacement of  $w_c - w_a$  by weighted vertical averages of  $w_u - w_d$  in the heat capacity and gravitational terms. This means that the effects of heat capacity and weight of water substance are much less in the pseudo-adiabatic than in the reversible case.



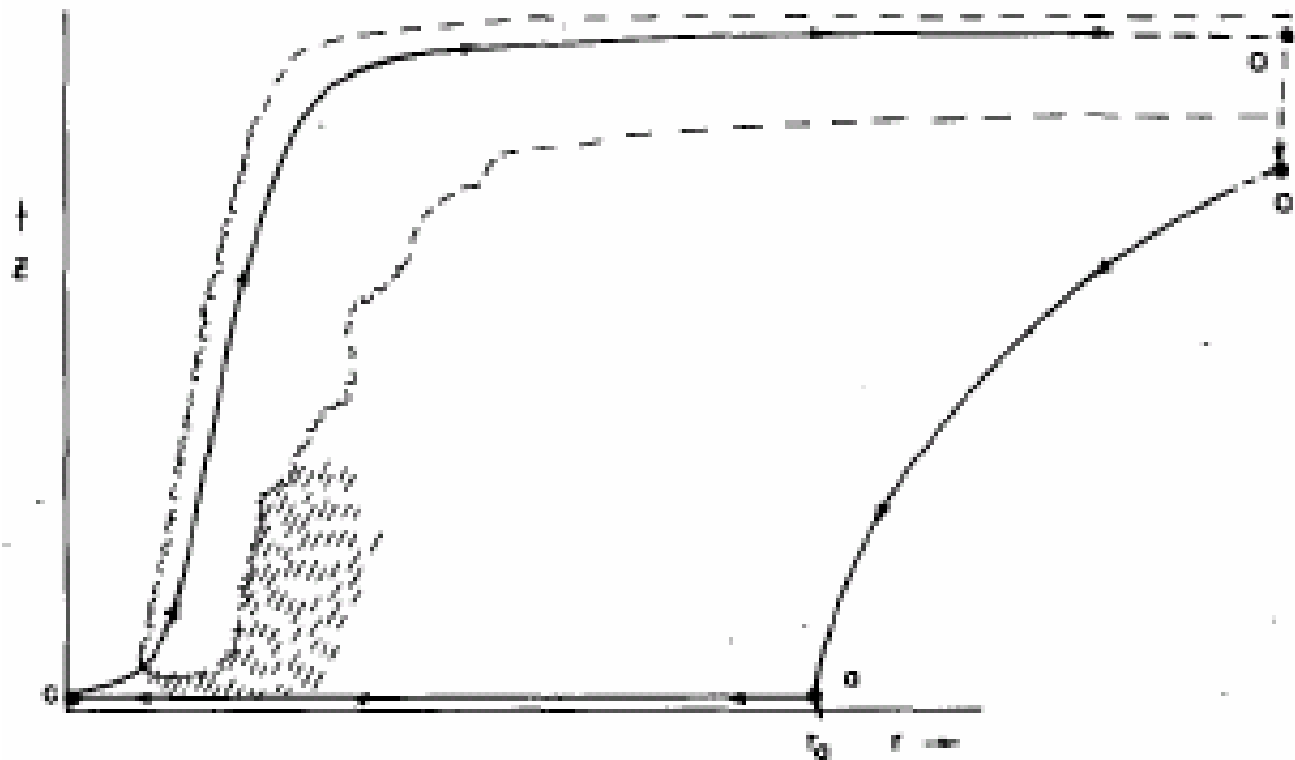


FIG. C1. Illustrating the path integral for the Carnot cycle.  
 Points  $o$  and  $o'$  are taken to lie at very large radius.

## Clausius-Clapeyron equation

$$\left( \frac{dp}{dT} \right)_{i,ii} = \frac{L_{i,ii}}{T(\alpha_{ii} - \alpha_i)}$$

1. We can speculate on several aspects of the structure of such storms based simply on their intensity and on some suppositions about their boundary layer structure.

In the first place, hypercanes would penetrate large distances into the stratosphere due to the very high core values of the moist entropy. The distance above the tropopause of the highest outflow from the eyewall is approximately

$$\delta z \approx \left( \frac{dz}{ds} \right)_s \delta s, \quad (28)$$

where  $(dz/ds)_s$  is the rate of increase of moist entropy with height in the stratosphere. We have neglected water loading of the updraft. If we approximate moist entropy by dry entropy in the ambient stratosphere and assume that the latter is isothermal with a temperature  $\bar{T}_0$ , we have

$$\delta z \approx \frac{\bar{T}_0}{g} \delta s. \quad (29)$$

The entropy increase  $\delta s$  is the increase between the ambient environment and the core along the surface. Neglecting effects due to the heat capacity of water, this is [from (2)]

$$\delta s = \frac{L_v}{T_s} \delta w - R_d \delta \ln p_d.$$

Using the definition of mixing ratio in the above, (29) becomes

$$\delta z = \frac{R_d \bar{T}_0}{g} \left[ \frac{L_v}{R_v T_s} \frac{e_s}{p_{da}} \left( \frac{1}{x} - \text{RH} \right) - \ln x \right], \quad (30)$$

where  $x = p_{dc}/p_{da}$ .

## Several further simplifying assumptions:

1. Neglect water loading and the effect of the heat capacity of water substance in (31) and in the definition of  $s$
2. Assume that  $\bar{T}_0$  does not depend on the averaging interval
3. Assume that the radial pressure gradient is equal to the radial gradient of the partial pressure of dry air
4. Neglect  $w$  where it multiplies  $f^2$  in (31)

## Angular momentum per unit mass ( $M$ )

$$M \equiv rV + \frac{1}{2} fr^2 \quad (1)$$

---

$r$  Radius

$V$  Azimuthal velocity

$f$  Coriolis parameter

## Total entropy ( $s$ )

$$s \equiv (C_{pd} + QC_l) \ln T + \frac{L_v w}{T} - R_d \ln p_d - wR_v \ln(\text{RH}) \quad (2)$$

---

$C_{pd}$  Heat capacity of dry air at constant pressure

$C_l$  Heat capacity of liquid water

$L_v$  Latent heat of vaporization

$R_d$  Gas constant of dry air

$w$  Mixing ratio

RH Relative humidity

## Total water ( $Q$ )

$$Q \equiv w + l \quad (3)$$

---

$l$  Mass of liquid per unit mass of dry air



# Thermal wind equation for axisymmetric flow

$$\frac{1}{r^3} \left( \frac{\partial M^2}{\partial p} \right)_r = - \left( \frac{\partial \alpha}{\partial r} \right)_p \quad (4)$$

Using the chain rule

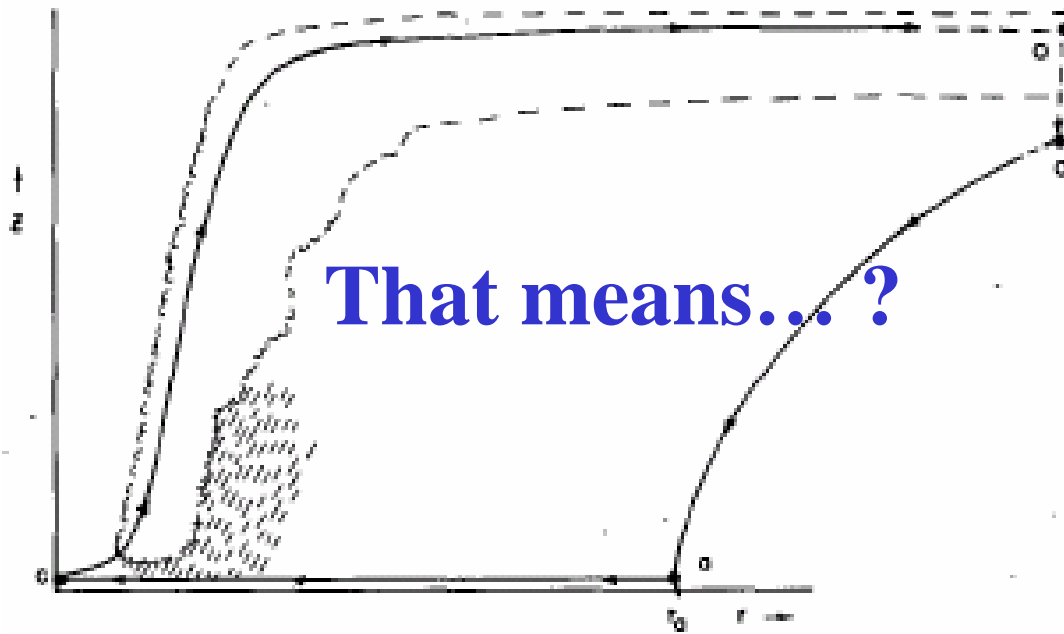
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$\alpha$  Specific volume (a function of the three variables  $p$ ,  $s$  and  $Q$ )

$$\alpha(p, s, Q)$$

$$\frac{1}{r^3} \left( \frac{\partial M^2}{\partial p} \right)_r = - \left( \frac{\partial \alpha}{\partial s} \right)_{p,Q} \frac{\partial s}{\partial r} \left( \frac{\partial \alpha}{\partial Q} \right)_{p,s} \frac{\partial Q}{\partial r} \quad (5)$$

Functions of  $M$  alone



$$\frac{1}{r^3} \left( \frac{\partial M^2}{\partial p} \right)_r = - \frac{\partial M}{\partial r} \left[ \left( \frac{\partial \alpha}{\partial s} \right)_{p,Q} \frac{ds}{dM} + \left( \frac{\partial \alpha}{\partial Q} \right)_{p,s} \frac{dQ}{dM} \right] \quad (6)$$

$$\frac{2M}{r^3} \left( \frac{\partial r}{\partial p} \right)_M = \left( \frac{\partial \alpha}{\partial s} \right)_{p,Q} \frac{ds}{dM} + \left( \frac{\partial \alpha}{\partial Q} \right)_{p,s} \frac{dQ}{dM} \quad (7)$$

Maxwell relations

$$\frac{1}{1+Q} \left( \frac{\partial T}{\partial p} \right)_{s,Q}$$

$$\frac{1}{1+Q} \left[ \left( \frac{\partial \Phi}{\partial p} \right)_M + C_l \frac{\partial}{\partial p} [T(1 - \ln T)]_{s,Q} \right]$$

$$\Phi \equiv gz + \frac{1}{2} V^2$$

$$-\frac{M}{r^2} = \frac{1}{1+Q} \left\langle \begin{aligned} &(T - T_0) \frac{ds}{dM} \\ &+ \frac{dQ}{dM} \{ \Phi - \Phi_0 + C_l [T(1 - \ln T) - T_0(1 - \ln T_0)] \} \end{aligned} \right\rangle \quad (10)$$

This constraint on the distributions of  $s$  and  $Q$  with respect to  $M$  is a direct consequence of the **thermal wind relation** and the specification that  $s$  and  $Q$  are functions of  $M$  alone.

---

subscript 0: as  $r \rightarrow \infty$

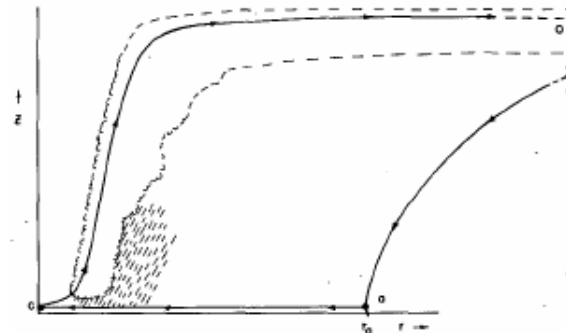


FIG. C1. Illustrating the path integral for the Carnot cycle. Points  $o$  and  $o'$  are taken to lie at very large radius.

# **Constraints on the variation of entropy at the surface**

$$-\frac{M}{r^2} = \frac{1}{1+Q} \left\langle \begin{aligned} &(T - T_0) \frac{ds}{dM} \\ &+ \frac{dQ}{dM} \{ \Phi - \Phi_0 + C_l [T(1 - \ln T) - T_0(1 - \ln T_0)] \} \end{aligned} \right\rangle \quad (10)$$

$$(10) \times \frac{\partial M}{\partial r}$$

$$\Phi \equiv gz + \frac{1}{2} V^2$$

$$M \equiv rV + \frac{1}{2} fr^2 \quad (1)$$

$$\begin{aligned}
& \boxed{-\frac{1+Q}{2r^2} \left( \frac{\partial M^2}{\partial r} \right)_z} = (T_s - T_0) \left( \frac{\partial s}{\partial r} \right)_z \\
& \boxed{+ \left( \frac{\partial Q}{\partial r} \right)_z} \left\{ -gz_0 \boxed{+ \frac{1}{2} \left| \frac{M^2}{r^2} \right.} + f^2 (r^2 - r_1^2) \right\} \\
& \quad + C_l [T_s (1 - \ln T_s) - T_0 (1 - \ln T_0)] \quad (11)
\end{aligned}$$

---

subscript  $s$ : at the surface ( $z = 0$ )

$r_1$  a very large radius

Energetically interpretable as the kinetic energy that must be used to spin up water mass in the anticyclonic outflow

On the left hand side, we have

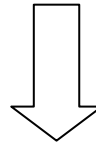
$$-\frac{1}{2r^2} \left( \frac{\partial}{\partial r} \right)_z \boxed{[(1+Q)M^2]}$$

From the gradient wind equation  
(Emanuel, 1986)

$$r^3 \alpha_d \left( \frac{\partial p}{\partial r} \right)_z + \frac{1}{4} f^2 r^4 (1+Q)$$



$$\begin{aligned}
-\frac{1+Q}{2r^2} \left( \frac{\partial M^2}{\partial r} \right)_z &= (T_s - T_0) \left( \frac{\partial s}{\partial r} \right)_z \\
&+ \left( \frac{\partial Q}{\partial r} \right)_z \left\{ -gz_0 + \frac{1}{2} \left[ \frac{M^2}{r^2} + f^2(r^2 - r_1^2) \right] \right. \\
&\quad \left. + C_A [T_s(1 - \ln T_s) - T_0(1 - \ln T_0)] \right\}, \quad (11)
\end{aligned}$$



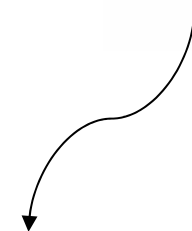
$$\begin{aligned}
\boxed{-\alpha_d \left( \frac{\partial p}{\partial r} \right)_z} &= \left( \frac{\partial}{\partial r} \right)_z \left[ \frac{1}{2} r \alpha_d \frac{\partial p}{\partial r} + \frac{1}{4} f^2 r^2 (1 + Q) \right] \\
&\quad - \frac{1}{8} f^2 r_1^2 \left( \frac{\partial Q}{\partial r} \right)_z + (T_s - T_0) \left( \frac{\partial s}{\partial r} \right)_z + \left( \frac{\partial Q}{\partial r} \right)_z \\
&\quad \times \{ -gz_0 + C_A [T_s(1 - \ln T_s) - T_0(1 - \ln T_0)] \}. \quad (14)
\end{aligned}$$

# The first law of thermodynamics

By differentiating (2) and making use of the Clausius-Clapeyron equation, the first law of thermodynamics can be written (for saturated *and* unsaturated processes)

$$-\alpha_d dp = T ds - (C_{pd} + C_l Q) dT - d[L_v w] - C_l T \ln T dQ. \quad (15)$$

$$\begin{aligned}
 -\alpha_d dp &= T ds - (C_{pd} + C_l Q) dT \\
 &\quad - d[L_v w] - C_l T \ln T dQ. \quad (15)
 \end{aligned}$$



$$\begin{aligned}
 -\alpha_d \left( \frac{\partial p}{\partial r} \right)_z &= \left( \frac{\partial}{\partial r} \right)_z \left[ \frac{1}{2} r \alpha_d \frac{\partial p}{\partial r} + \frac{1}{4} f^2 r^2 (1 + Q) \right] \\
 &\quad - \frac{1}{8} f^2 r_1^2 \left( \frac{\partial Q}{\partial r} \right)_z + (T_s - T_0) \left( \frac{\partial s}{\partial r} \right)_z + \left( \frac{\partial Q}{\partial r} \right)_z \\
 &\quad \times \{ -gz_0 + C_l [T_s (1 - \ln T_s) - T_0 (1 - \ln T_0)] \}. \quad (14)
 \end{aligned}$$

$$\begin{aligned}
T_0 \frac{\partial s}{\partial r} = & \frac{\partial}{\partial r} \left[ \frac{1}{2} r \alpha_d \frac{\partial p}{\partial r} + \frac{1}{4} f^2 r^2 (1 + Q) \right. \\
& \left. + L_v w + (C_{pd} + C_l Q) T_s \right] \\
& + \frac{\partial Q}{\partial r} \left[ -gz_0 + C_l (T_0 \ln T_0 - T_0) - \frac{1}{8} f^2 r_1^2 \right]. \quad (16)
\end{aligned}$$

Define an **outer radius**,  $r_a$  where  $\frac{\partial p}{\partial r} = 0$

$$\begin{aligned} \bar{T}_0(s_c - s_a) = & (w_c - w_a)[C_l T_s + L_v + C_l \bar{T}_0^* (\ln \bar{T}_0^* - 1) \\ & - \frac{1}{8} f^2 \bar{r}_1^{2*} - g \bar{z}_0^*] - \frac{1}{4} f^2 r_a^2 (1 + w_a) \end{aligned}$$

(17)

$$\bar{T}_0 \equiv \frac{1}{s_c - s_a} \int_a^c T_0 ds$$

$$\overline{(\quad)}^* \equiv \frac{1}{Q_a - Q_c} \int_a^c (\quad) dQ$$

# **The central pressure equation and its interpretation**

$$\bar{T}_0 (s_c - s_a) = (w_c - w_a) [C_l T_s + L_v + C_l \bar{T}_0^* (\ln \bar{T}_0^* - 1) - \frac{1}{8} f^2 \bar{r}_1^{2*} - g \bar{z}_0^*] - \frac{1}{4} f^2 r_a^2 (1 + w_a)$$

(17)

Evaluated at  $z = 0$  and assumed that air is saturated at  $r = 0$

**Eventually !!!!!**

$$\ln(x) - \frac{e_a}{p_{da}} \ln(\text{RH})$$

$$= -A \left[ \frac{1}{x} - \text{RH} \right] + \frac{1}{4} \frac{\left[ f^2 r_a^2 \left( 1 + \frac{e_a}{p_{da}} \right) \right]}{R_d T_s (1 - \varepsilon)} \quad (21)$$

1. The **1/x** term represents the **pressure** dependence of the saturation mixing ratio
2. **RH** represents the **thermodynamic disequilibrium** of the **air-sea system** which is the energy source of the hurricane.
3. **The smaller the relative humidity, the greater the air-sea entropy difference!!!**



★ Mean August conditions in the near-equatorial Western North Pacific

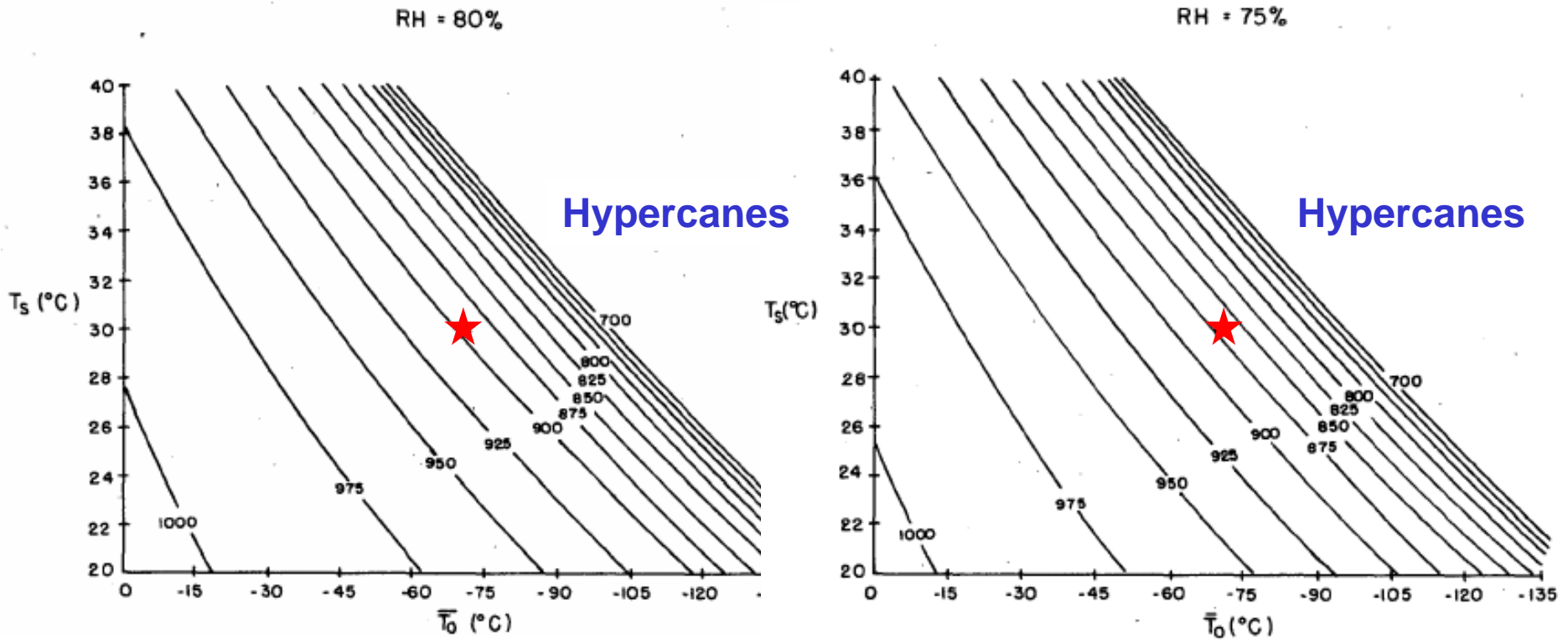


FIG. 4. As in Fig. 3 but for solutions of (18) ignoring condensate loading and heat capacity effects.

**Constraints on the distributions of  
entropy, total water and angular  
momentum**

- Assume that conserved variables ( $s$  and  $Q$ ) do not vary along angular momentum ( $M$ ) surfaces, when coupled with the thermal wind balance approximation

||

- The density of air above the boundary layer has the same value as the density of parcels lifted reversibly from the boundary layer

Betts (1982)

The threshold values of  $A$  or  $B$  beyond which no solutions exist can be found by requiring that the two curves in Fig.1 have the same slope at their intersection point

$$y = \exp[A(y - B)] \quad (22)$$

$$1 = A \exp[A(y - B)] = Ay$$

$$A = \exp(BA - 1), \quad B_c = \frac{1 + \ln A}{A}$$

**No solutions exist for  $B < B_c$ !!!**

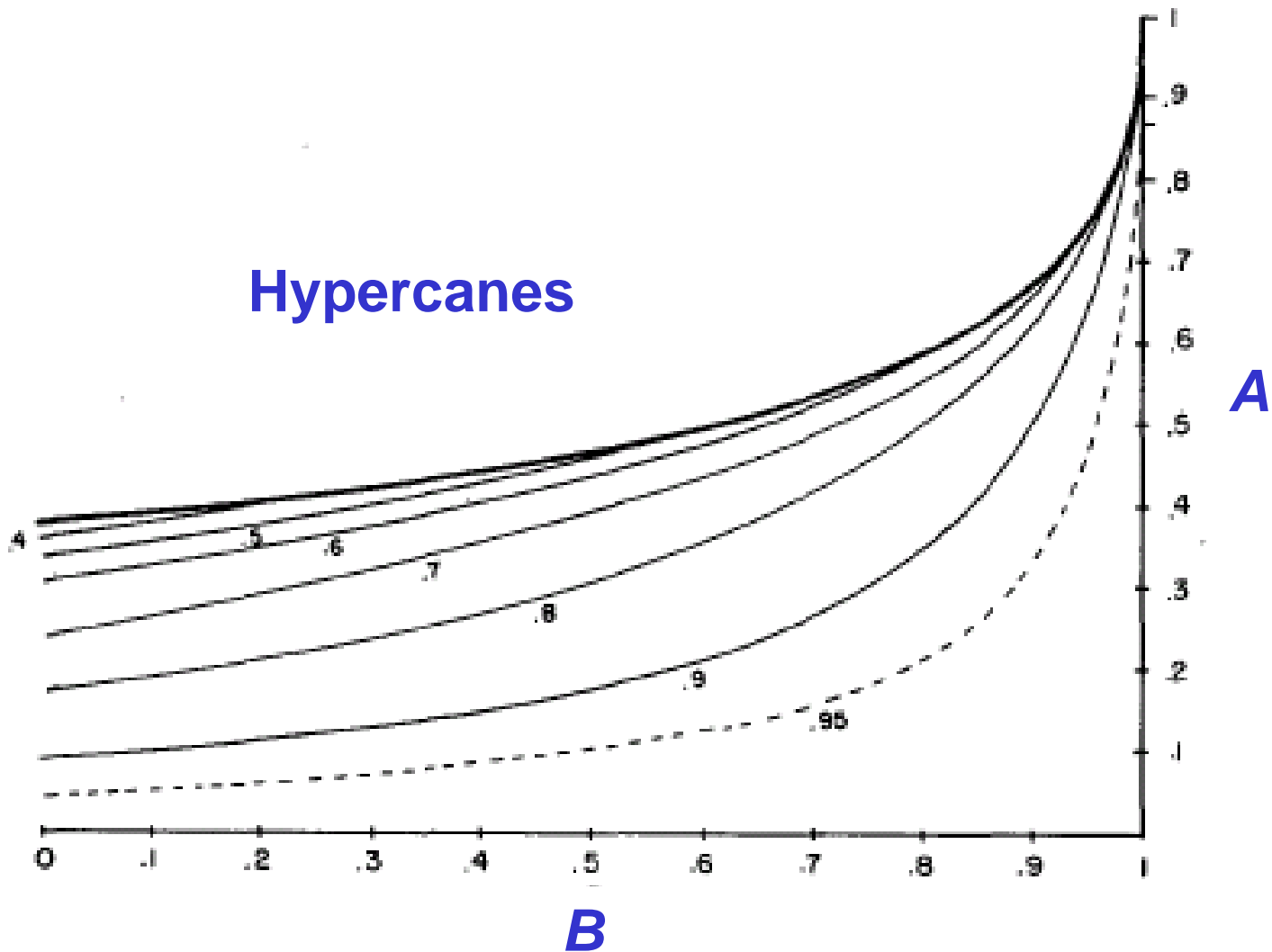


FIG. 2. General solutions of (18) for  $x$  ( $=p_{40}/p_{4a}$ ), as a function of  $A$  and  $B$ . The heavy solid curve denotes the critical condition beyond which no solutions exist.

# **Interpretation of the central pressure equation**