Introduction To Linear Regression Analysis 6.11 答案

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$$COVRATIO_i = \frac{\det\left|(X_{(i)}^t X_{(i)})^{-1} \cdot S_{(i)}^2\right|}{\det\left|(X^t X)^{-1} MS_{Res}\right|}$$

$(X^tX)^{-1}$, $X_{(i)}^t X_{(i)}$ 為 $p \times p$ 矩陣, 因此

$$= \frac{(S_{(i)}^2)^p}{(MS_{Res})^p} \cdot \frac{\det(X_{(i)}^t X_{(i)})^{-1}}{\det(X^tX)^{-1}}$$

$$(X_{(i)}^t X_{(i)})^{-1} = (X^tX)^{-1} + \frac{(X^tX)^{-1} x_i x_i^t (X^tX)^{-1}}{1-h_{ii}}$$

$$\therefore \det(X_{(i)}^t X_{(i)})^{-1} = \det(X^tX)^{-1} \det\left(I + \frac{x_i x_i^t (X^tX)^{-1}}{1-h_{ii}}\right)$$

$$\therefore COVRATIO_i = \frac{(S_{(i)}^2)^p}{(MS_{Res})^p} \cdot \det\left(I + \frac{x_i x_i^t (X^tX)^{-1}}{1-h_{ii}}\right)$$

因此剩下證明 $\det\left(I + \frac{x_i x_i^t (X^tX)^{-1}}{1-h_{ii}}\right) = \frac{1}{1-h_{ii}}$

考慮 $I + \frac{x_i x_i^t}{1-h_{ii}}(X^tX)^{-1}$ 之所有特徵值及特徵向量

① x_i ∵ $\left(I + \frac{x_i x_i^t}{1-h_{ii}}(X^tX)^{-1}\right) x_i = x_i + \frac{h_{ii}}{1-h_{ii}} x_i = \frac{x_i}{1-h_{ii}}$

$$\therefore \left(I + \frac{x_i x_i^t}{1-h_{ii}}(X^tX)^{-1}\right) x_i = \left(\frac{1}{1-h_{ii}}\right) x_i$$

因此可知 x_i 為 $I + \frac{x_i x_i^t (X^tX)^{-1}}{1-h_{ii}}$ 之特徵向量,

其特徵值 $= \frac{1}{1-h_{ii}}$

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② 接下來考慮 向量空間

$$W = \text{span}\{x_i\}$$

$$W^{\perp} = \{w \mid x_i^t w = 0\}$$

$$(\underbrace{W + W^{\perp}}_{= R^p},\ W \cap W^{\perp} = \vec{0} \quad \dim W^{\perp} = p-1)$$

$$V := \{v \mid x_i^t (X^t X)^{-1} v = 0\}$$

$$\forall v \in V \quad \left(I + \frac{x_i x_i^t (X^t X)^{-1}}{1-h_{ii}}\right) v = v + \vec{0} = v$$

∴ v 為 $I + \frac{x_i x_i^t (X^t X)^{-1}}{1-h_{ii}}$ 之特徵向量，其特徵值為 1 #

$W^{\perp}$ 與 V 為同構. ∴ $\dim W^{\perp} = \dim V = p-1$

考慮 $f: W \to V$ 之線性函數

$$f(w) = (X^t X)^{-1} w$$

$(X^t X)^{-1}$ 可逆矩陣 $\quad ((X^t X)^{-1})^{-1} = (X^t X)$

f: onto & one-to-one

∴ $I + \frac{x_i x_i^t (X^t X)^{-1}}{1-h_{ii}}$ 之 $\det = \left(\frac{1}{1-h_{ii}}\right)^{1} \cdot (1)^{(p-1)} = \frac{1}{1-h_{ii}}$

(所有特徵值之相乘)

∴ QED