Quiz4 参考答案 by morimoto

$\boxed{1}$

(1) empirical cdf ; 経験累積分布函数

(2) $I_{[-\infty,x]}(X_1)=\begin{cases}1 & \text{if } X_1\le x\\ 0 & \text{else}\end{cases}$

$Pr(X_1\le x)=F(x)$

由此可知 $I_{[-\infty,x]}(X_1)\sim \text{bernoulli}(F(x))$

$\therefore \sum_{j=1}^{n} I_{[-\infty,x]}(X_j)\sim Bin(n, F(x))$

$\therefore n\hat{F}(x)\sim Bin(n, F(x))$

(3) $\mathrm{cov}\left[\frac{1}{n}\sum_{j=1}^{n} I_{(-\infty,u]}(X_j), \frac{1}{n}\sum_{j=1}^{n} I_{(-\infty,v]}(X_j)\right]$

$=\frac{1}{n^2}\mathrm{cov}\left[\sum_{j=1}^{n} I_{(-\infty,u]}(X_j), \sum_{j=1}^{n} I_{(-\infty,v]}(X_j)\right]$

$=\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{cov}\left[I_{(-\infty,u]}(X_i), I_{(-\infty,v]}(X_j)\right]$

$=\frac{1}{n^2}\Big(n\,\mathrm{cov}\left[I_{(-\infty,u]}(X_1), I_{(-\infty,v)}(X_1)\right)$

$+\,n(n-1)\underbrace{\mathrm{cov}\left[I_{(-\infty,u)}(X_1), I_{(-\infty,v)}(X_2)\right]}_{=0\ (\because \text{独立})}\Big)$ X_1 & X_2

$=\frac{1}{n}\mathrm{cov}\left[I_{(-\infty,u)}(X_1), I_{(-\infty,v)}(X_2)\right]$

$$\mathrm{Cov}\left[I_{(-\infty,u)}(X_1),\ I_{(-\infty,v)}(X_2) \right]$$

$$= E\left[I_{(-\infty,u)}(X_1) \cdot I_{(-\infty,v)}(X_1) \right] - E\left[I_{(-\infty,u)}(X_1) \right] \cdot E\left[I_{(-\infty,v)}(X_1) \right]$$

$$= \Pr(X_1 \leq u \cap X_1 \leq v) - \underbrace{\Pr(X_1 \leq u)}_{= F(u)} \cdot \underbrace{\Pr(X_1 \leq v)}_{= F(v)}$$

$$\Pr(X_1 \leq u \cap X_1 \leq v) = \Pr(X_1 \leq \min\{u, v\}) = F(\min\{u, v\})$$

$$= F(\min\{u,v\}) - F(u) F(v)$$

$$\fallingdotseq \frac{1}{n} \left(F(\min\{u,v\}) - F(u) F(v) \right)$$

$\boxed{2}$

(1) $Y_1 - X_1 \sim N(\mu_y - \mu_x,\ \sigma_x^2 + \sigma_y^2)$

$$Z = \frac{(Y_1 - X_1) - (\mu_y - \mu_x)}{\sqrt{\sigma_x^2 + \sigma_y^2}} \sim N(0,1)$$

$$\therefore Pr(Y_1 - X_1 > 0) = Pr\left(Z > \frac{-\mu_y + \mu_x}{\sqrt{\sigma_x^2 + \sigma_y^2}}\right)$$

$$= 1 - \Phi\left(\frac{\mu_x - \mu_y}{\sqrt{\sigma_x^2 + \sigma_y^2}}\right) = \Phi\left(\frac{\mu_y - \mu_x}{\sqrt{\sigma_x^2 + \sigma_y^2}}\right)$$

(2)

$$F = \frac{\frac{1}{m-1} \sum_{i=1}^{m} (X_i - \bar{X})^2 / \sigma_x^2}{\frac{1}{n-1} \sum_{j=1}^{n} (Y_i - \bar{Y})^2 / \sigma_y^2} \sim F_{m-1,n-1}$$

令 ψ 為 $F_{m-1,n-1}$ 之 cdf.

$$Pr\left(\psi^{-1}\left(\frac{\alpha}{2}\right) \leq F \leq \psi^{-1}\left(1-\frac{\alpha}{2}\right)\right) = 1-\alpha$$

$$\therefore \psi^{-1}\left(\frac{\alpha}{2}\right) \leq \frac{n-1}{m-1} \cdot \frac{\sigma_y^2}{\sigma_x^2} \cdot \frac{\sum_{i=1}^{m} (X_i - \bar{X})^2}{\sum_{j=1}^{n} (Y_i - \bar{Y})^2} \leq \psi^{-1}\left(1-\frac{\alpha}{2}\right)$$

$$\Leftrightarrow \frac{1}{\psi^{-1}\left(1-\frac{\alpha}{2}\right)} \cdot \frac{n-1}{m-1} \cdot \frac{\sum_{i=1}^{m} (X_i - \bar{X})^2}{\sum_{i=1}^{n} (Y_i - \bar{Y})^2} \leq \frac{\sigma_x^2}{\sigma_y^2} \leq$$

$$\frac{1}{\psi^{-1}\left(\frac{\alpha}{2}\right)} \cdot \frac{n-1}{m-1} \cdot \frac{\sum_{i=1}^{m} (X_i - \bar{X})^2}{\sum_{i=1}^{n} (Y_i - \bar{Y})^2}$$

$$\therefore \left[\frac{1}{\psi^{+}(1-\frac{\alpha}{2})} \frac{n-1}{m-1} \frac{\sum_{i=1}^{m}(X_i-\bar{X})^2}{\sum_{i=1}^{n}(Y_i-\bar{Y})^2},\ \frac{1}{\psi^{+}(\frac{\alpha}{2})} \frac{n-1}{m-1} \frac{\sum_{i=1}^{m}(X_i-\bar{X})^2}{\sum_{i=1}^{n}(Y_i-\bar{Y})^2} \right]$$

$\boxed{3}$

(1) Z_i…連續型，對稱於0.

$Pr(Z_i>0) = Pr(Z_i \leq 0) = \frac{1}{2}$

$\therefore I_i \sim \text{bernoulli}(\frac{1}{2}) \quad (i=1\sim n)$

(2) $W^{+} = I_1R_1 + I_2R_2 + \dots + R_nI_n$

$(\{R_1, R_2, \dots, R_n\} = \{1, 2, 3 \dots n\})$

由於 $I_1, I_2, \dots I_n$：iid，I_j, R_j：獨立，故

W^{+} 與 $(R_1, R_2, \dots R_n)$ 為獨立.

$$\left(\text{例如} \begin{array}{l} \underline{I_1} + \underline{2I_2} + 3I_3 + 4I_4 + \dots nI_n \\ \underline{2I_1} + \underline{I_2} + 3I_3 + 4I_4 + \dots nI_n \end{array} \right\} \text{同一分佈} \right)$$

$\therefore E[W^{+} \mid R_1=r_1, \dots R_n=r_n] = E[W^{+}]$ $\quad (r_1=1, r_2=2 \dots r_n=n)$

$V[W^{+} \mid R_1=r_1, \dots R_n=r_n] = V[W^{+}]$

我們考慮 $W^{+} \mid R_1=1, R_2=2, \dots R_n=n$

$= I_1 + 2I_2 + \dots nI_n$ 即可

$$E[I_1 + 2I_2 + \cdots nI_n] = \frac{1}{2}(1+2+\cdots+n) = \frac{n(n+1)}{4}$$

(3) $V[I_1 + 2I_2 + \cdots nI_n]$

$$= V[I_1] + 4V[I_2] + \cdots n^2 V[I_n] \quad (\because I_1 \sim I_n : \text{iid})$$

$$= \frac{1}{4}(1+4+9+\cdots n^2) = \frac{1}{24}n(n+1)(2n+1)$$

$\boxed{4}$

(1) $$\int_{-\infty}^{\infty} x \hat{f}(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{nh} \sum_{j=1}^{n} x K\left(\frac{x-x_j}{h}\right) dx$$

$$= \sum_{j=1}^{n} \frac{1}{nh} \underbrace{\int_{-\infty}^{\infty} x K\left(\frac{x-x_j}{h}\right) dx}$$

$$\downarrow \qquad y \overset{def}{=} \frac{x-x_j}{h} \qquad \frac{dy}{dx} = \frac{1}{h}$$

$$\int_{-\infty}^{\infty} (hy + x_j) K(y) \cdot h\,dy$$

$$= \sum_{j=1}^{n} \frac{1}{nh} \int_{-\infty}^{\infty} (\underbrace{h^2 y K(y)}_{\to 0} + h x_j K(y))\,dy \quad (\because \int_{-\infty}^{\infty} x K(x)\,dx = 0)$$

$$= \sum_{j=1}^{n} \frac{x_j}{n} \underbrace{\int_{-\infty}^{\infty} K(y)\,dy}_{=1}$$

$$\therefore \sum_{j=1}^{n} \frac{x_j}{n} = \underline{\bar{X}}$$

(2) $E[X^2 | X_1, X_2, \dots X_n]$

$$= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{nh} \sum_{j=1}^{n} K\left(\frac{x-x_j}{h}\right) dx$$

$$= \frac{1}{nh} \sum_{j=1}^{n} \int_{-\infty}^{\infty} x^2 K\left(\frac{x-x_j}{h}\right) dx \qquad y = \frac{x-x_j}{h}$$

$$= \frac{1}{nh} \sum_{j=1}^{n} \int_{-\infty}^{\infty} (hy+x_j)^2 K(y) \cdot h\,dy$$

$$= \frac{1}{n} \sum_{j=1}^{n} \int_{-\infty}^{\infty} (h^2y^2 + \underbrace{2hx_jy}_{\to 0} + x_j^2) K(y)\,dy$$

$$= \frac{1}{n} \sum_{j=1}^{n} (h^2 \sigma_K^2 + x_j^2)$$

$$= h^2 \sigma_K^2 + \frac{1}{n} \sum_{j=1}^{n} X_j^2$$

$$\therefore V[X] = E[X^2 | X_1 \dots X_n) - E[X | X_1, \dots, X_n]^2$$

$$= (h^2 \sigma_K^2 + \frac{1}{n} \sum_{j=1}^{n} X_j^2) - (\bar{X})^2$$

$$= h \sigma_K^2 + \frac{1}{n} \sum_{j=1}^{n} (X_j - \bar{X})^2$$