Quiz3 参考答案 by morimoto

$\boxed{1}$

(1) $\beta(\mu) = \Pr(\bar{X} > 83 \mid \mu)$

注 $X_1, X_2, \ldots X_n \sim N(\mu, 10^2)$ $(n=25,\ \sigma=10)$

$$\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0,1)$$

$$\| $$

$$\frac{5(\bar{X}-\mu)}{10} = \frac{1}{2}(\bar{X}-\mu)$$

$$\therefore \beta(\mu) = \Pr(\bar{X} > 83 \mid \mu) = \Pr\Big(\underbrace{\tfrac{1}{2}(\bar{X}-\mu)}_{\sim N(0,1)} > \tfrac{1}{2}(83-\mu) \,\Big|\, \mu\Big)$$

$$= 1 - \Phi\left(\tfrac{1}{2}(83-\mu)\right)$$

(2) $\beta(80) = 1 - \Phi\left(\frac{1}{2}(83-80)\right) = 1 - \Phi\left(\frac{3}{2}\right)$

(3) $1 - \underbrace{\beta(86)}_{棄卻的機率} = 1 - \underbrace{\left(1 - \Phi\left(\tfrac{1}{2}(-3)\right)\right)}_{= \beta(86)} = \Phi\left(-\frac{3}{2}\right) = 1 - \Phi\left(\frac{3}{2}\right)$

$\boxed{2}$

棄卻的機率

(1) $\sup\limits_{\theta=\theta_0} E[\varphi_c(X)|\theta] = E[\varphi_c(X)|\theta_0] = \alpha$

under H_0

$\sup\limits_{\theta=\theta_0} E[\varphi_c^*(X)|\theta] = E[\varphi_c^*(X)|\theta_0] = \alpha$

under H_0

註 顯著水準… H_0 為真時 棄卻 H_0 的機率(的上限)

(2) $\varphi_c = 1 \Rightarrow \varphi_c - \varphi_c^* \geqq 0 \Rightarrow (X_1, X_2, \cdots X_n) \in C$

$(\because \varphi_c^* = 0 \text{ or } 1)$

$\Rightarrow \dfrac{\prod_{j=1}^{n} f(x_j|\theta_1)}{\prod_{j=1}^{n} f(x_j|\theta_0)} > h \Rightarrow \prod_{j=1}^{n} f(x_j|\theta_1) - h\prod_{j=1}^{n} f(x_j|\theta_0) > 0$

$\therefore \underbrace{(\varphi_c - \varphi_c^*)}_{\text{非負}} \underbrace{\left(\prod_{j=1}^{n} f(x_j|\theta_1) - h\prod_{j=1}^{n} f(x_j|\theta_0)\right)}_{\text{正}} \geqq 0$

$\varphi_c = 0 \Rightarrow \varphi_c - \varphi_c^* \leqq 0 \Rightarrow (X_1, X_2, \cdots X_n) \notin C$

$(\because \varphi_c^* = 0 \text{ or } 1)$

$\Rightarrow \dfrac{\prod_{j=1}^{n} f(x_j|\theta_1)}{\prod_{j=1}^{n} f(x_j|\theta_0)} \leqq h \Rightarrow \prod_{j=1}^{n} f(x_j|\theta_1) - h\prod_{j=1}^{n} f(x_j|\theta_0) \leqq 0$

$\therefore \underbrace{(\varphi_c - \varphi_c^*)}_{\text{負(零)}} \underbrace{\left(\prod_{j=1}^{n} f(x_j|\theta_1) - h\prod_{j=1}^{n} f(x_j|\theta_0)\right)}_{\text{負(零)}} \geqq 0$

(3) $\int_{(x_1,x_2,\dots,x_n)} g \, dx_1 dx_2 \cdots dx_n \geq 0$

$$\Leftrightarrow \int \phi_C \cdot \prod_{j=1}^{n} f(x_j|\theta_1)\, dx_1 \cdots dx_n - \int \phi_C^* \prod_{j=1}^{n} f(x_j|\theta_1)\, dx_1 \cdots dx_n$$

$$- k\int \phi_C \prod_{j=1}^{n} f(x_j|\theta_0)\, dx_1 \cdots dx_n + k\int \phi_C^* \prod_{j=1}^{n} f(x_j|\theta_0)\, dx_1 \cdots dx_n$$

$$= \underbrace{E[\phi_C|\theta_1]}_{\parallel \atop \beta_C(\theta_1)} - \underbrace{E[\phi_C^*|\theta_1]}_{\beta_{C^*}(\theta_1)} - k\underbrace{E[\phi_C|\theta_0]}_{\parallel \atop \alpha} + k\underbrace{E[\phi_C^*|\theta_0]}_{\parallel \atop \alpha}$$

$$= \beta_C(\theta_1) - \beta_{C^*}(\theta_1) \geq 0$$

(4) 對於 $H_0: \theta_0$ vs $H_1: \theta_1$ 的最強力檢定為：

(顯著水準 $=\alpha$)

$$\varphi(x) = \begin{cases} 1 & (x_1,\dots,x_n) \in C \\ 0 & (x_1,\dots,x_n) \notin C \end{cases}$$

$$C = \left\{ (x_1,x_2,\dots,x_n) \,\middle|\, \frac{\prod_{j=1}^{n} f(x_j|\theta_1)}{\prod_{j=1}^{n} f(x_j|\theta_0)} > k \right\}$$

k使得 $E[\varphi(x)|\theta_0] = \alpha$

$\boxed{3}$

(1) $\sup_{\theta \in \{\theta_3\}} \Pr(X|\theta) = \Pr(X|\theta_3) = \begin{cases} 0.9 & (X=0) \\ 0.08 & (X=1) \\ 0.02 & (X=2) \\ 0.00 & (X=3,4) \end{cases}$

$\sup_{\theta \in \{\theta_1, \theta_2, \theta_3\}} \Pr(X|\theta) = \begin{cases} 0.9 & (X=0) \\ 0.08 & (X=1) \\ 0.80 & (X=2) \\ 0.8 & (X=3) \\ 0.1 & (X=4) \end{cases}$

$\therefore \Lambda(X=0) = 1 \quad (0.9/0.9)$

$\Lambda(X=1) = 1 \quad (0.08/0.08)$

$\Lambda(X=2) = 0.025 \; (0.02/0.80)$

$\Lambda(X=3) = 0$

$\Lambda(X=4) = 0$

(2) $k = 0 \Rightarrow \{X \mid \Lambda(X) \leq k\} = \overset{C \atop \shortparallel}{\{3,4\}}$

$k = 0.025 \Rightarrow \{X \mid \Lambda(X) \leq k\} = \{2,3,4\}$

$k = 1 \quad \Rightarrow \{X \mid \Lambda(X) \leq k\} = \{0,1,2,3,4\}$

$\Pr(X \in \{2,3,4\} \mid \underset{H_0}{\underset{\shortparallel}{\theta_3}}) = 0.02 + 0.00 + 0.00 = 0.02$

$\therefore C = \{2,3,4\}$

(3) $\beta(\theta_1) = Pr(X \in \{2,3,4\} \mid \theta_1) = 0.5 + 0.30 + 0.1 = 0.95$

$\beta(\theta_2) = Pr(X \in \{2,3,4\} \mid \theta_2) = 0.80 + 0.10 + 0.00 = 0.9$

$\boxed{4}$

(1) $$\Lambda = \frac{\prod_{j=1}^{n} f(x_j \mid \theta_1)}{\prod_{j=1}^{n} f(x_j \mid \theta_0)} = \frac{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(x_j-\theta_1)^2\right)}{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(x_j-\theta_0)^2\right)}$$

$$= \exp\Big(\underbrace{\frac{(\theta_1-\theta_0)}{\sigma^2}}_{>0}\underbrace{(x_1+x_2+\dots+x_n)}_{\theta\text{的充份統計量}} - \frac{n\theta_1^2}{2\sigma^2} + \frac{n\theta_0^2}{2\sigma^2}\Big)$$

$\therefore x_1+x_2+\dots+x_n$ 增加 $\Leftrightarrow$ Λ 增加 ⊛

棄卻域 $C = \{(x_1, x_2, \dots x_n) \mid \Lambda \geq k\}$

$= \{(x_1, x_2, \dots x_n) \mid x_1+x_2+\dots x_n \geq c\}$ $\because$ ⊛

H_0 為真時，棄卻的機率 $= \alpha$ （$\Rightarrow$ 可以決定 c）

H_0 為真 $\Leftrightarrow$ $x_1, x_2, \dots x_n \sim N(\theta_0, \sigma^2)$

$$Pr(x_1+x_2+\dots x_n \geq c \mid \underbrace{\theta_0}_{H_0}) = Pr\Big(\underbrace{\frac{\sqrt{n}(\bar{X}-\theta_0)}{\sigma}}_{\sim N(0,1)} \geq \frac{\sqrt{n}(\frac{c}{n}-\theta_0)}{\sigma} \,\Big|\, \theta_0\Big)$$

$$= 1 - \Phi\left(\frac{\sqrt{n}}{\sigma}\left(\frac{c}{n}-\theta_0\right)\right) = \alpha$$

$\therefore \Phi^{-1}(1-\alpha) = \frac{\sqrt{n}}{\sigma}\left(\frac{c}{n} - \theta_0\right)$

$\therefore \frac{\sigma}{\sqrt{n}}\Phi^{-1}(1-\alpha) = \frac{c}{n} - \theta_0$

$\therefore c = n\left(\theta_0 + \frac{\sigma}{\sqrt{n}}\Phi^{-1}(1-\alpha)\right)$

$\therefore \varphi(x) = \begin{cases} 1 & \text{if } X_1+X_2+\cdots X_n \geq n\theta_0 + \sigma\sqrt{n}\,\Phi^{-1}(1-\alpha) \\ 0 & \text{otherwise} \end{cases}$

(2) 是. $\beta_\varphi(\theta_1) \geq \beta_{\varphi^*}(\theta_1)$ ($\forall \varphi^*$ with $E[\varphi^*|\theta_0]=\alpha$)

($\because$ φ為最強檢定 ⇒ 檢定力不會低於任意φ^*)

同理, 考慮 $H_0: \theta=\theta_0$ vs $H_1: \theta=\underline{\underline{\theta_2}}$ ($\theta_2 > \theta_0$) 時

φ對其依然擁有最強力,

$\therefore \beta_\varphi(\theta_2) \geq \beta_{\varphi^*}(\theta_2)$ ($\forall \varphi^*$ with $E[\varphi^*|\theta_0]=\alpha$)

⇒ 由此可知, $\beta_\varphi(\theta) \geq \beta_{\varphi^*}(\theta)$ ($\forall \theta > \theta_0$, $\forall \varphi^*$ $E[\varphi^*|\theta_0]=\alpha$)

⇒ φ為對 $H_0: \theta=\theta_0$ vs $H_1: \theta>\theta_0$

$\therefore$ 均勻最強力檢定 (顯著水準α下)

θ只要大於θ_0都可

(3) 證明 $\forall \theta \in \Theta_1$ $\beta_\varphi(\theta) \geq \alpha$

hint中的φ^* $\therefore$ 檢定力 $= E[\varphi(X)|\theta] = \alpha = \beta_{\varphi^*}(\theta)$ ($\forall\theta \in \Theta_1$)

β_φ為最強力 for all $\theta \in \Theta_1$ ⇒ $\beta_\varphi(\theta) \geq \beta_{\varphi^*}(\theta) = \alpha$ ($\forall \theta \in \Theta_1$)

$\therefore$ 證明完成