台大、數學系 105-2、統計學

Quiz 2 參考答案

$\boxed{1}$ $X_1, X_2 \cdots X_n \sim iid$

(1) $E[X_i] = \mu$. $V[X_i] = \sigma^2 < \infty$

$$\Rightarrow \lim_{n\to\infty} Pr\left(\left| \frac{X_1+X_2+\cdots+X_n}{n} - \mu \right| \geq \varepsilon \right) = 0 \quad (\text{for all } \varepsilon > 0)$$

$\boxed{\text{條件}}$ $V[X_i] < \infty$

(2) $\boxed{\text{證明}}$ … 利用 Chevyshev's 不等式

$$\overline{X} \overset{def}{=} \frac{X_1+X_2+\cdots+X_n}{n}$$

$$V[\overline{X}] = \frac{\sigma^2}{n} \quad E[\overline{X}] = \mu$$

$$Pr\left(|\overline{X} - \mu| \geq \underbrace{k \cdot \sqrt{\frac{\sigma^2}{n}}}_{= \varepsilon} \right) \leq \frac{1}{k^2}$$

$$k = \frac{\varepsilon}{\sqrt{\frac{\sigma^2}{n}}}$$

$$k^2 = \frac{n\varepsilon^2}{\sigma^2}$$

$\therefore$

$$Pr(|\overline{X} - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \to 0 \ \text{as} \ (n \to \infty)$$

$$\lim_{n\to\infty} Pr(|\overline{X} - \mu| \geq \varepsilon) = 0 \quad (\text{for all } \varepsilon > 0)$$

CHA SHIN

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(参考) Khintchine's Week Law of Large Number (別的版本)

$X_1 \sim X_n \sim^{iid}$ 且 $E[X_i] = \mu < \infty$

$\Rightarrow \dfrac{X_1+X_2+\cdots X_n}{n} \xrightarrow{P} \mu$

(証明) 令 ϕ 為 X_i 之特徵函數

$\bar{X}$ 之特徵函數 $= E\left[e^{it\bar{X}}\right]$

$= E\left[e^{\frac{it}{n}(X_1+X_2+\cdots+X_n)}\right] = \left(\phi\left(\frac{t}{n}\right)\right)^n$

$= \left(1 + i\frac{t}{n}\mu + o\left(\frac{1}{n^2}\right)\right)^n$ (Taylor展開)

$\downarrow$ 收斂速度比 $\frac{1}{n}$ 更快的項.

$\lim_{n\to\infty}\left(1 + i\frac{t}{n}\mu + o\left(\frac{1}{n^2}\right)\right)^n \to e^{it\mu}$ (退化)

由此可知 $\bar{X} \xrightarrow{P} \mu$

$\boxed{2}$

(1) $\sum_{j=1}^{n} Z_j^2 \sim \chi_n^2$

(2) $\dfrac{Z_{n+1}}{\sqrt{\frac{1}{n}\sum_{j=1}^{n} Z_j^2}} \sim t_n$

(3) $\dfrac{\frac{1}{m}\sum_{j=1}^{m} Z_j^2}{\frac{1}{n}\sum_{j=m+1}^{m+n} Z_j^2} \sim F_{m,n}$

(4) 方法1 利用動差母函數

$$a \overset{\text{put}}{=} \begin{pmatrix} s \\ t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix}$$

$$E\left[\exp\left(a^t \begin{pmatrix} \bar{X} \\ X_1-\bar{X} \\ \vdots \\ X_n-\bar{X} \end{pmatrix}\right)\right] = E\left[\exp\left(s\bar{X} + t_1(X_1-\bar{X}) + \cdots + t_n(X_n-\bar{X})\right)\right]$$

$$= E\left[\exp\left((s-t_1-t_2-t_3\cdots-t_n)\bar{X} + t_1X_1 + t_2X_2 + \cdots t_nX_n\right)\right]$$

$$= E\left[\exp\left(\left(\frac{s}{n} - \underbrace{\frac{t_1+t_2+\cdots+t_n}{n}}_{\bar{t}}\right)(X_1+X_2+\cdots+X_n) + t_1X_1 + t_2X_2 + \cdots t_nX_n\right)\right]$$

$$= E\left[\exp\left(\left(\frac{s}{n} - \bar{t}\right)(X_1+X_2+\cdots+X_n) + t_1X_1 + \cdots t_nX_n\right)\right]$$

$$= E\left[\exp\left(\sum_{j=1}^{n}\left(t_j - \bar{t} + \frac{s}{n}\right)X_j\right)\right]$$

$\boxed{2}$

(4) $X_1 \sim X_j \cdots$ 獨立

$$\therefore \prod_{j=1}^{n} \underbrace{E\left[\exp\left(t_j - \bar{t} + \frac{s}{n}\right) X_j\right]}_{\exp\left(\frac{1}{2}\left(t_j - \bar{t} + \frac{s}{n}\right)^2\right)}$$

由 $N(\mu, \sigma^2)$ $\therefore$ mgf $= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$

$$= \prod_{j=1}^{d} \exp\left(\frac{1}{2}\left(t_j - \bar{t} + \frac{s}{n}\right)^2\right)$$

$$= \exp\left(\sum_{j=1}^{d} \frac{1}{2}\left(t_j - \bar{t} + \frac{s}{n}\right)^2\right)$$

$$= \exp\left(\sum_{j=1}^{d} \frac{1}{2}\left\{\left(t_j - \bar{t}\right)^2 + 2\left(t_j - \bar{t}\right)\left(\frac{s}{n}\right) + \frac{s^2}{n^2}\right\}\right)$$

$$= \exp\left(\sum_{j=1}^{d} \frac{1}{2}\left(t_j - \bar{t}\right)^2 + \frac{s^2}{n}\right)$$

$$= \exp\left(\frac{s^2}{n}\right) \cdot \exp\left(\sum_{j=1}^{d} \frac{1}{2}\left(t_j - \bar{t}\right)^2\right)$$

$\therefore$ 可寫成 s 的函數與 t 的函數之相乘

$\therefore \bar{X}$ vs $(X_1 - \bar{X}, X_2 - \bar{X}, \ldots, X_n - \bar{X})$ 為獨立

$\boxed{2}$

(4) $\boxed{\text{方法2}}$ 利用 Basu 定理

考慮 $X_1 \cdots X_n \overset{iid}{\sim} N(\mu, 1)$　μ: 未知

$$f(x_1 \cdots x_n | \mu) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2}\sum_{j=1}^{n}(x_j-\mu)^2\right)$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2}\sum_{j=1}^{n}x_j^2 + \mu(n\bar{x}) - \frac{n\mu^2}{2}\right)$$

$$= \underbrace{\left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2}\sum_{j=1}^{n}x_j^2\right)}_{h(\vec{x})} \exp\left(\underbrace{n\mu\bar{x}}_{c(\mu)T(\vec{x})} - \frac{n\mu^2}{2}\right)$$

由此可知 $N(\mu,1)$ 為指數族 (1-parameter)

$\exp(\sim)$ 內 $\bar{X}$ 之係數為 $n\mu$.

集合 $\{n\mu \mid \mu \in (-\infty, \infty)\}$ 可內含 $\mathbb{R}$ 上的開集合

由此可知 $\bar{X}$ 為 μ 之完備充份統計量

$X_j - \bar{X}$ 的分佈與 μ 無關

$(\because (X_j-\mu) - (\bar{X}-\mu))$

$X_j - \bar{X}$ 為 μ 之輔助統計量

$\therefore$ $\bar{X}$ vs $X_j - \bar{X}$ 獨立 ($\because$ Basu 定理)

$\boxed{2}$

(4) $\boxed{\text{方法3}}$ 利用矩陣與無相關性

$$\bar{Z} = \frac{1}{n}(1\ 1\ 1\cdots 1)\begin{pmatrix} Z_1\\ \vdots \\ Z_n\end{pmatrix}$$

$$Z_1-\bar{Z} = (1\ 0\ 0\cdots 0)\begin{pmatrix} Z_1\\ \vdots \\ Z_n\end{pmatrix} - \frac{1}{n}(1\ 1\cdots 1)\begin{pmatrix} Z_1\\ \vdots \\ Z_n\end{pmatrix}$$

$$Z_2-\bar{Z} = (0\ 1\ 0\cdots 0)\begin{pmatrix} Z_1\\ \vdots \\ Z_n\end{pmatrix} - \frac{1}{n}(1\cdots 1)\begin{pmatrix} Z_1\\ \vdots \\ Z_n\end{pmatrix}$$

$$\vdots$$

$$Z_n-\bar{Z} = (0\ 0\cdots 1)\begin{pmatrix} Z_1\\ \vdots \\ Z_n\end{pmatrix} - \frac{1}{n}(1\cdots 1)\begin{pmatrix} Z_1\\ \vdots \\ Z_n\end{pmatrix}$$

$$\therefore\quad n+1\ \begin{pmatrix} \bar{Z}\\ Z_1-\bar{Z}\\ \vdots \\ Z_n-\bar{Z}\end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n}\\ 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n}\\ -\frac{1}{n} & 1-\frac{1}{n} & & -\frac{1}{n}\\ & & \ddots & \\ -\frac{1}{n} & \cdots & & 1-\frac{1}{n}\end{pmatrix}}_{A_{(n+1)\times n}}\begin{pmatrix} Z_1\\ \vdots \\ Z_n\end{pmatrix}$$

$$V\left[\begin{pmatrix} \bar{Z}\\ Z_1-\bar{Z}\\ \vdots \\ Z_n-\bar{Z}\end{pmatrix}\right] = A\underbrace{V\left[\begin{pmatrix} Z_1\\ \vdots \\ Z_n\end{pmatrix}\right]}_{I_{n\times n}}A^t = AA^t$$

為了簡單 將A寫成 Block 矩陣.

$$1_n = \begin{pmatrix} 1\\ \vdots \\ 1\end{pmatrix}_{n\times 1}\qquad 1_n^t = (1\ 1\cdots 1)_{1\times n}$$

$$J = 1_n 1_n^t = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1\\ \vdots & & \cdots & & 1\\ 1 & \cdots & & & 1\end{pmatrix}_{n\times n}$$

$\boxed{2}$ (4)

$$A = \begin{bmatrix} \mathbb{1}_n^t \\ I - \frac{1}{n}J \end{bmatrix}$$

$$\therefore AA^t = \begin{bmatrix} \mathbb{1}_n^t \\ I - \frac{J}{n} \end{bmatrix} \begin{bmatrix} \mathbb{1}_n & I - \frac{1}{n}J \end{bmatrix} \quad (I^t = I,\ J^t = J)$$

$$= \begin{bmatrix} \mathbb{1}_n^t \mathbb{1}_n & \mathbb{1}_n^t - \frac{\mathbb{1}_n^t J}{n} \\ \mathbb{1}_n - \frac{J}{n}\mathbb{1}_n & I - \frac{2J}{n} + \frac{J^2}{n^2} \end{bmatrix}$$

㊟ $J\mathbb{1}_n = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ n \\ \vdots \\ n \end{pmatrix} = n\mathbb{1}_n$

$$\Rightarrow (J\mathbb{1}_n)^t = \mathbb{1}_n^t J^t = \mathbb{1}_n^t J = n\mathbb{1}_n^t$$

- $J^2 = \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \\ 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \\ 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} n & \cdots & n \\ \vdots & & \\ n & \cdots & n \end{pmatrix} = nJ$

$$\therefore AA^t = \begin{bmatrix} n & O \\ O & I - \frac{J}{n} \end{bmatrix}$$

$$\therefore V\begin{bmatrix} \bar{Z} \\ Z_1 - \bar{Z} \\ \vdots \\ Z_n - \bar{Z} \end{bmatrix} = \begin{array}{c} \begin{bmatrix} n & 0 \cdots 0 \\ 0 & \\ \vdots & \left[I - \frac{J}{n} \right] \\ 0 & \end{bmatrix} \updownarrow n \\ \overleftrightarrow{\quad n \quad} \end{array}$$

→ 由此可知 $\bar{X}$ vs $(X_i - \bar{X})$

∴無相關

㊟ 常態分佈 ⇒「獨立 ⇔ 無相關」

CHA SHIN

設 $\begin{array}{c} p_1 \updownarrow \\ p_2 \updownarrow \end{array} \begin{pmatrix} \vec{X_1} \\ \vec{X_2} \end{pmatrix} \sim N\left(\begin{pmatrix} \vec{\mu_1} \\ \vec{\mu_2} \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right)$

接著證明「$\operatorname{cov}[\vec{X_1}, \vec{X_2}] = 0 \iff \vec{X_1}$ 與 $\vec{X_2}$ 獨立」

$\begin{pmatrix} \vec{X_1} \\ \vec{X_2} \end{pmatrix}$ 之 動差母函數 $\quad a = \begin{pmatrix} \vec{a_1} \\ \vec{a_2} \end{pmatrix} \begin{array}{l} \updownarrow p_1 \\ \updownarrow p_2 \end{array}$

$$\| $$

$$E\left[\exp\left(a^t \begin{pmatrix} \vec{X_1} \\ \vec{X_2} \end{pmatrix} \right) \right] = E\left[\exp\left(\vec{a_1}^t \vec{X_1} + \vec{a_2}^t \vec{X_2} \right) \right]$$

$$= \exp\left(a^t \begin{pmatrix} \vec{\mu_1} \\ \vec{\mu_2} \end{pmatrix} + \frac{a^t}{2} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} a \right)$$

$$= \exp\left(\vec{a_1}^t \vec{\mu_1} + \vec{a_2}^t \vec{\mu_2} + \frac{1}{2} \left(\vec{a_1}^t \ \vec{a_2}^t \right) \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} \vec{a_1} \\ \vec{a_2} \end{pmatrix} \right)$$

$$= \exp\left(\vec{a_1}^t \vec{\mu_1} + \vec{a_2}^t \vec{\mu_2} + \frac{1}{2} \left(\vec{a_1}^t \ \vec{a_2}^t \right) \begin{pmatrix} \Sigma_{11} \vec{a_1} + \Sigma_{12} \vec{a_2} \\ \Sigma_{21} \vec{a_1} + \Sigma_{22} \vec{a_2} \end{pmatrix} \right)$$

$$= \exp\left(\vec{a_1}^t \vec{\mu_1} + \vec{a_2}^t \vec{\mu_2} + \frac{1}{2} \left(\vec{a_1}^t \Sigma_{11} \vec{a_1} + \vec{a_1}^t \Sigma_{12} \vec{a_2} + \vec{a_2}^t \Sigma_{21} \vec{a_1} + \vec{a_2}^t \Sigma_{22} \vec{a_2} \right) \right) \Bigg]$$

$\cdots$ ⊛ 」

• $\vec{a_2} = \vec{0} \cdots$ 得 $\vec{X_1}$ 之 mgf $= \exp\left(\vec{a_1}^t \vec{\mu_1} + \frac{1}{2} \vec{a_1}^t \Sigma_{11} \vec{a_1} \right) \cdots$ ①

• $\vec{a_1} = \vec{0} \cdots$ 得 $\vec{X_2}$ 之 mgf $= \exp\left(\vec{a_2}^t \vec{\mu_2} + \frac{1}{2} \vec{a_2}^t \Sigma_{22} \vec{a_2} \right) \cdots$ ②

$\vec{X_1}$ 與 $\vec{X_2}$ 獨立 $\iff$ ①·② = ⊛ $\iff \Sigma_{12} = 0$ 且 $\Sigma_{21} = 0$

$\Downarrow$

$\begin{pmatrix} \operatorname{cov}[\vec{X_1}, \vec{X_2}] = 0 \\ \operatorname{cov}[\vec{X_2}, \vec{X_1}] = 0 \end{pmatrix}$

∴「無相關 $\iff$ 獨立」

（常態分佈）

$\boxed{3}$ (1) 中央極限定理… $\sqrt{n}\left(\dfrac{\bar{X}-\mu}{\sigma}\right) \sim N(0,1)$

$$\Pr(|\bar{X}-\mu|<1) = \Pr\Big(\underbrace{\frac{\sqrt{n}}{\sigma}|\bar{X}-\mu|}_{= Z} < \frac{\sqrt{n}}{\sigma}\Big)$$

$$\Pr\left(|Z|<\frac{\sqrt{n}}{\sigma}\right) \geq 0.95 \qquad (Z \sim N(0,1))$$

$$\Pr(|Z|<1.96)=0.95$$

$$\therefore \frac{\sqrt{n}}{\sigma} \geq 1.96 \qquad (\sigma^2=25)$$

$$\therefore \sqrt{n} \geq 5\cdot 1.96$$

$$n \geq 25\cdot 1.96^2 \approx 100$$

(2) 柴比雪夫不等式… $V[\bar{X}] = \frac{1}{n}\cdot\sigma^2 \left(=\frac{25}{n}\right)$

$E[\bar{X}] = \mu$

$$\Pr\left(|\bar{X}-\mu| \geq k\sqrt{V[\bar{X}]}\right) \leq \frac{1}{k^2}$$

$$\Leftrightarrow \Pr\left(|\bar{X}-\mu| < k\sqrt{V[\bar{X}]}\right) \geq 1-\frac{1}{k^2} = 0.95 \qquad \therefore k=\sqrt{20}$$

$\downarrow$

$$\therefore \sqrt{20}\cdot\sqrt{V[\bar{X}]} = \frac{\sqrt{20}\cdot 5}{\sqrt{n}} = 1 \qquad \therefore n = 20\cdot 25 = \underline{500}$$

柴比雪夫不等式，只要存在變異數，任何分佈都可以使用。
但中央極限定理假設它的分佈，也就是說
關於它的分佈，你給了一個很詳細的條件，
所以預測的精準度會更高。

CHA SHIN

注 $\mu = \sum_{j=1}^{N} \frac{1}{N} S_j$, $\sigma^2 = \sum_{j=1}^{N} \frac{1}{N}(S_j-\mu)^2$

(平均與變異數之定義)

4

(1) 假設有限母體 $= \{S_1, S_2, \dots, S_N\}$

$$E[X_1] = \sum_{j=1}^{N} S_j \cdot \underbrace{Pr(X_1 = S_j)}_{=\frac{1}{N}} = \sum_{j=1}^{N} \frac{1}{N} S_j$$

$$= \frac{1}{N}(S_1+S_2+\dots+S_N) = \mu$$

注 為什麼 $Pr(X_1=S_j)=\frac{1}{N}$ 呢?

「取後不放回」,你可以想成「將 $S_1 \sim S_N$ 隨機排列」

$\{S_1, S_2, S_3, S_4, S_5, S_6\}$ ($N=6$ 的 case)

↓ 隨機排列

例

$S_3\ S_1\ S_4\ S_2\ S_6\ S_5$

$n=3$

$X_1\ X_2\ X_3$ 未抽出

排列方式 $= 6!$ $(N!)$

$X_1=S_3$ 之排列方式 $= 5!$ $((N-1)!)$

$\therefore Pr(X_1=S_3) = \frac{5!}{6!} = \frac{1}{6}$ $\left(\frac{1}{N}\right)$

$$E[X_1X_2] = \sum_{\substack{i=1 \\ (i\neq j)}}^{N} \sum_{j=1}^{N} S_i \cdot S_j \cdot Pr(X_1=S_i, X_2=S_j)$$

· 所有排列方式 $= N!$ 種

· 固定 $X_1=S_i$ 且 $X_2=S_j$ 時

排列方式 $= (N-2)!$ 種

$\therefore \frac{(N-2)!}{N!} = \frac{1}{N(N-1)}$

$\boxed{4}$

(1) $$E[X_1X_2] = \sum_{\substack{i=1 \\ (i\neq j)}}^{N}\sum_{j=1}^{N} S_i \cdot S_j \cdot \Pr(X_1 = S_i, X_2 = S_j)$$

$$= \frac{1}{N(N-1)} \underbrace{\sum_{\substack{i=1 \\ (i\neq j)}}^{N}\sum_{j=1}^{N} S_i S_j}$$

↓ 全部寫出來

$$(S_1S_2 + S_1S_3 + \cdots S_1S_N) \quad (i=1 \ \ j=2\sim N)$$

$$+ (S_2S_1 + S_2S_3 + \cdots S_2S_N) \quad (i=2 \ \ j=1,3\sim N)$$

$$\vdots$$

$$+ (S_NS_1 + S_NS_2 + \cdots S_NS_{N-1}) \quad (i=N, \ j=1\sim N-1)$$

$$= S_1(S_1+S_2+\cdots+S_N) - S_1^2$$

$$+$$

$$S_2(S_1+S_2+\cdots S_N) - S_2^2$$

$$+$$

$$\vdots$$

$$S_N(S_1+\cdots S_N) - S_N^2$$

$$= (S_1+S_2+\cdots S_N)^2 - (S_1^2+S_2^2+\cdots+S_N^2)$$

$$= N^2\mu^2 - \underbrace{\sum_{j=1}^{N} S_j^2}_{N(\mu^2+\sigma^2)}$$

$$\left(\begin{array}{l} \text{注1} \\ \sigma^2 = \sum_{j=1}^{N} \frac{1}{N}(S_j-\mu)^2 = \sum_{j=1}^{N}\frac{1}{N}S_j^2 - \mu^2 \\ \Rightarrow N(\mu^2+\sigma^2) = \sum_{j=1}^{N} S_j^2 \end{array} \right)$$

$$\therefore E[X_1X_2] = \frac{1}{N(N-1)}\left(N^2\mu^2 - N(\mu^2+\sigma^2)\right) = \mu^2 - \frac{1}{N-1}\sigma^2$$

$$\therefore \mathrm{Cov}[X_1, X_2] = E[X_1X_2] - E[X_1]E[X_2] = \mu^2 - \frac{\sigma^2}{N-1} - \mu^2 = \underline{\frac{-\sigma^2}{N-1}}$$

(2) $$E\left[\frac{X_1+X_2+\cdots X_n}{n}\right] = \sum_{j=1}^{n} \frac{1}{n} \underbrace{E[X_j]}_{\mu} = \mu$$

(3) $$V[\bar{X}] = V\left[\frac{X_1+X_2+\cdots X_n}{n}\right] = \frac{1}{n^2} E\left[\left((X_1-\mu)+\cdots(X_n-\mu)\right)^2\right]$$

$$= \frac{1}{n^2}\left\{ \underbrace{\sum_{j=1}^{n} V(X_j)}_{n\sigma^2} + \underbrace{\sum\sum_{i\neq j} \operatorname{cov}[X_i, X_j]}_{\frac{-\sigma^2}{N-1}\cdot n(n-1)} \right\} \quad (\because (1))$$

$$\Downarrow$$

$$\text{注}\ V[X_1] = E[(X_1-\mu)^2]$$

$$= \sum_{j=1}^{N} (S_j-\mu)^2 \cdot \underbrace{\Pr(X_1=S_j)}_{\frac{1}{N}}$$

$$= \frac{1}{N}\sum_{j=1}^{N} (S_j-\mu)^2 = \sigma^2$$

$$\therefore = \frac{1}{n^2}\left(n\sigma^2 - \frac{n(n-1)\sigma^2}{N-1} \right)$$

$$= \frac{1}{n^2}\left(\frac{nN-\cancel{n}-n^2+\cancel{n}}{N-1}\sigma^2 \right)$$

$$= \frac{1}{n^2}\left(\frac{n(N-n)}{N-1}\sigma^2 \right) = \frac{N-n}{N-1}\cdot\frac{\sigma^2}{n}$$