①

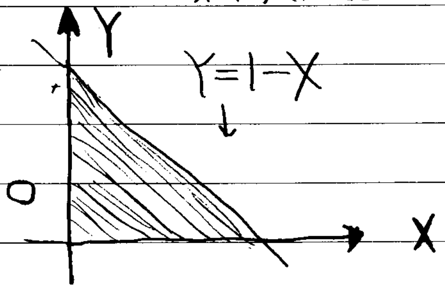
r05246013@ntu.edu.tw

105、

統計學 第1次小考 參考答案 森元

$\boxed{1}$

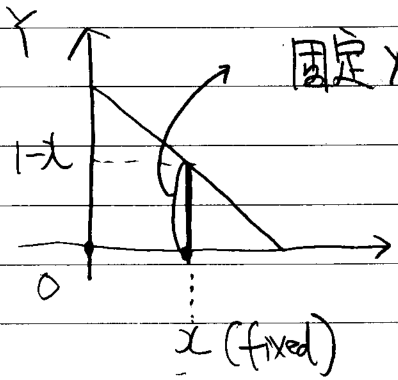
(a)



由於 $\begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \\ 0 \leq x+y \leq 1 \end{cases}$，

除了上圖黑色部份之外

其餘範圍之機率密度皆為0。



固定X=x時，$0 \leq Y \leq 1-x$

X的邊際分布 $f_X(x) = \int_{y=0}^{y=1-x} 24xy\,dx$

$= x \cdot \left[12y^2\right]_0^{1-x} = 12x(1-x)^2$

$(0 \leq x \leq 1)$

②

同樣道理，$f_Y(y) = 12y(1-y)^2$

(註) $X, Y \sim Be(2,3)$ … Beta 分佈

$Z \sim Be(\alpha, \beta)$ $\quad f_Z(z) = \dfrac{z^{\alpha-1}(1-z)^{\beta-1}}{Be(\alpha,\beta)}$ $\quad (0 \leq z \leq 1)$

$Be(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

「if $f_{XY}(x,y) = f_X(x) f_Y(y)$ (as)

$\Rightarrow$ X、Y 為獨立」

$f_{XY}(x,y) = 24xy$

$f_X(x) \cdot f_Y(y) = 12x(1-x)^2 \cdot 12y(1-y)^2$

$f_{XY}(x,y) = f_X(x) f_Y(y)$ (as) 不成立，

(但應該證明 $\exists\, c_1, c_2$ st $Pr(X \leq c_1, Y \leq c_2) \neq Pr(X \leq c_1) Pr(Y \leq c_2)$

(或 (d) $Cov \neq 0 \Rightarrow$ 不獨立)

$\therefore$ X, Y 並非獨立

③

$\boxed{1}$

(b) $f_{X|Y}(x|y) \overset{\text{def}}{=} \dfrac{f_{XY}(x,y)}{f_Y(y)} = \dfrac{24xy}{12y(1-y)^2} = \dfrac{2x}{(1-y)^2}$

$(0 \leq x \leq 1-y)$

$\therefore f_{X|Y}(x|y) = \begin{cases} \dfrac{2x}{(1-y)^2} & (0 \leq x \leq 1-y) \\ 0 & (\text{otherwise}) \end{cases}$

↳ 可以寫成 $\dfrac{2x}{(1-y)^2} \cdot I_{[0,1-y]}(x)$

$I_A(x) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

(c) 根據雙重期望值之原理，

$E[(X-g(Y))^2]$

$= E[E[(X-g(Y))^2 | Y]]$

(待會利用這個公式)

④

- 我們先考慮一個隨機變數的情形
- Z ～ 隨機變數，

Z 服從某種分布 D，而且 D 的參數是 θ

我們可以把這個狀況寫成：

「$Z|\theta \sim D(\theta)$」

（比如說，$Z|\theta \sim Uni(0,\theta)$，～ 均勻分布
$Z|\theta \sim exp(\theta)$，指數分布 etc）

我們平常不會刻意寫成 $Z|\theta \sim D(\theta)$

而只會寫成 $Z \sim D(\theta)$，但其實你可以

想成「在給定 $\Theta = \theta$ 之下，Z 的條件分布

為 $D(\theta)$」

↑ 除了 Z 以外，背後還有一個隨機變數 Θ

現在考慮常數 a 使得 $E[(Z-a)^2]$ 為最小

大部分的人應該知道 $a = \mu_Z$（Z 的期望值）

使得 $E[(Z-a)^2]$ 為最小

⑤

但是 $E[(Z-a)^2]$ 可以想成 $E[(Z-a)^2 \mid \Theta=\theta]$

(背後隱藏的隨機變數)

所以 μ_Z 其實也可以想成 $E[Z \mid \Theta=\theta]$

回歸正題, ...

$E[(X-g(Y))^2] = E[E[(X-g(Y))^2 \mid Y]]$

我們先考慮 $g(Y)$ 使得 $E[\underbrace{E[\sim \mid Y]}_{※}]$ 裡面

※的部分為最小.

$※ = E[(X-g(Y))^2 \mid Y]$

根據先前的討論, $g(Y) = E[X \mid Y]$ 時

$E[(X-g(Y))^2 \mid Y]$ 為最小.

($\because$「給定 $Y=y$」之後, 你可以把 Y 當成「常數」.

或你可以把 $Y=y$ 看成 $\Theta=\theta$

$\Rightarrow E[(X-g(\theta))^2 \mid \Theta=\theta]$

(6)

$\therefore g^*(Y) \overset{def}{=} \underbrace{E[X|Y]}$ $\forall g \cdots$ 任意函數

↳ 這是Y的函數

「 $E[(X-g^*(Y))^2 | Y=y] \leq E[(X-g(Y))^2 | Y=y]$

for all $y \in [0,1]$ 」

$\therefore E[E(X-g^*(Y))^2 | Y=y]] \leq E[E[(X-g(Y))^2 | Y=y]]$

$\left(\begin{array}{l} \text{左} \int_{y=0}^{y=1} E[(X-g^*(Y))^2 | Y=y] \cdot f_Y(y)\, dy \\ \text{右} \leq \int_{y=0}^{y=1} E[(X-g(Y))^2 | Y=y] \cdot f_Y(y)\, dy \end{array} \right.$

求 $E[X|Y] = \int_{x=0}^{x=1-y} x \cdot f_{X|Y}(x|y)\, dy$

$= \int_{x=0}^{x=1-y} x \cdot \frac{2x}{(1-y)^2}\, dx$

$= \frac{1}{(1-y)^2} \cdot \left[\frac{2x^3}{3}\right]_{x=0}^{x=1-y}$

$= \frac{2}{3} \cdot (1-y)$ $\therefore g^*(Y) = \frac{2}{3}(1-Y)$

⑦ hint $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = Be(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$\boxed{1}$ (d)

$$E[XY] = \iint_{\left\{\binom{x}{y} \,\middle|\, \substack{0\le x\le 1\\ 0\le y\le 1\\ 0\le x+y\le 1}\right\}} \underbrace{(24xy)}_{\text{pdf}}\cdot(xy)\,dx\,dy$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} 24x^2y^2\,dy\,dx$$

$$= \int_{x=0}^{x=1} 24x^2 \left[\frac{y^3}{3}\right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 24x^2 \cdot \frac{(1-x)^3}{3}dx = \int_0^1 8x^2(1-x)^3dx$$

$$= 8\cdot Be(3,4) = 8\cdot\frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = 8\cdot\frac{2!\,3!}{6!}$$

$$= 8\cdot\frac{\cancel{6}\cdot\cancel{2}}{\cancel{6}\cdot5\cdot4\cdot3\cdot\cancel{2}\cdot1} = \frac{8}{60} = \frac{2}{15}$$

$$E[X] = E[Y] = \int_0^1 x\cdot 12x(1-x)^2dx$$

$$= \int_0^1 12x^2(1-x)^2dx$$

$$= 12\cdot Be(3,3) = 12\cdot\frac{\Gamma(3)\Gamma(3)}{\Gamma(6)}$$

$$= 12\cdot\frac{2!\,2!}{5!} = \frac{12\cdot4}{120} = \frac{2}{5}$$

$$\therefore \mathrm{cov}[X,Y] = \frac{2}{15} - \left(\frac{2}{5}\right)^2 = \frac{10-12}{75} = \frac{-2}{75}$$

$\boxed{2}$

$\boxed{方法}$ 利用 cdf (累積分佈函數)

$$\underbrace{\frac{d}{dx}\Pr(Z^2 \leq x)}_{Z\text{的 cdf}} = Z\text{的}\ \underbrace{\text{pdf}}_{\text{機率密度函數}}$$

$$\Pr(Z^2 \leq x) = \begin{cases} 0 \cdots & \text{if } x<0 \\ \Pr(-\sqrt{x} \leq Z \leq \sqrt{x}) & \text{if } x \geq 0 \end{cases}$$

$$\Pr(-\sqrt{x} \leq Z \leq \sqrt{x}) = \Phi(\sqrt{x}) - \Phi(-\sqrt{x}) = \Phi(\sqrt{x}) - (1 - \Phi(\sqrt{x})) = 2\Phi(\sqrt{x}) - 1.$$

$$\therefore \Pr(Z^2 \leq x) = \left(2\Phi(\sqrt{x}) - 1\right) \cdot I_{[0,\infty)}(x)$$

註

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

$$\frac{d}{dx}\Pr(Z^2 \leq x) = 2\phi(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} I_{[0,\infty)}(x)$$

$$= \frac{1}{\sqrt{x}}\phi(\sqrt{x}) = \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x}{2}\right) I_{[0,\infty)}(x)$$

$$\wedge\ I_{[0,\infty)}(x)$$

$$= \frac{x^{\frac{1}{2}-1}}{\Gamma(\frac{1}{2}) \cdot 2^{\frac{1}{2}}} \cdot \exp\left(\frac{-x}{2}\right) \cdot I_{[0,\infty)}(x)$$

$$\therefore Z^2 \sim \Gamma\left(\frac{1}{2}, \frac{1}{2}\right) = \chi^2_1.$$

9

$\boxed{\text{方法2}}$ 積分的變數轉換

㊉ 全機率 $= 1 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz$

$X \overset{def}{=} Z^2 \qquad (Z = \pm\sqrt{X})$

$Z: -\infty \to 0 \;;\; 0 \to \infty$

$X: \infty \to 0 \;;\; 0 \to \infty$

$$\begin{cases} ① \; Z \le 0 \cdots \quad \frac{dx}{dz} = 2Z = -2\sqrt{X} \qquad dz = \frac{-1}{2\sqrt{x}} \\ ② \; Z > 0 \cdots \quad \frac{dx}{dz} = 2Z = 2\sqrt{X} \qquad dz = \frac{1}{2\sqrt{x}} \end{cases}$$

$$㊉ = \underbrace{\int_{-\infty}^{0}}_{\substack{Z: -\infty \to 0 \\ X: \infty \to 0}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) \underbrace{dz}_{\frac{-1}{2\sqrt{X}}dx} + \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) \underbrace{dz}_{\frac{1}{2\sqrt{X}}dx} = 1$$

$$= \int_{\infty}^{0} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x}{2}\right) \cdot \frac{(-1)}{2\sqrt{X}} dx + \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-X}{2}\right) \cdot \frac{dx}{2\sqrt{X}}$$

$$= \int_{0}^{\infty} \frac{1}{2\sqrt{X}} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-X}{2}\right) dx + \int_{0}^{\infty} \frac{1}{2\sqrt{X}} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-X}{2}\right) dx.$$

$$= \int_{0}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi X}} \exp\left(\frac{-X}{2}\right)}_{X = Z^2 \text{之 pdf}} dx = 1$$

⑩ $\boxed{\text{方法3}}$ （有一些課本這樣定義）

$\chi^2(1)$ … 自由度 1 的卡方分布的定義 $= Z^2$ $\quad Z \sim N(0,1)$

∴ 自由度 n 之卡方分布 $= \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$ 之 Gamma 分布

$$\therefore \quad f_X(x) = \frac{x^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right) \cdot 2^{\frac{n}{2}}} \exp\left(\frac{-x}{2}\right) \quad (x \geq 0)$$

$(n=1)$

註 $X \sim \Gamma(\alpha, \beta)$ $\quad f_X(x) = \frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta x) \quad (x \geq 0)$

$\boxed{3}$ $\boxed{\text{方法1}}$ 積分的變數轉換

$$\begin{cases} W = X+Y \\ Z = \frac{X}{Y} \end{cases} \Rightarrow \begin{cases} X = \frac{WZ}{1+Z} \\ Y = \frac{W}{1+Z} \end{cases}$$

$$\frac{\partial}{\partial W}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{Z}{1+Z} \\ \frac{1}{1+Z} \end{pmatrix} \qquad \therefore J = \begin{pmatrix} \frac{Z}{1+Z} & \frac{W}{(1+Z)^2} \\ \frac{1}{1+Z} & \frac{-W}{(1+Z)^2} \end{pmatrix}$$

$$\frac{\partial}{\partial Z}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{W}{(1+Z)^2} \\ \frac{-W}{(1+Z)^2} \end{pmatrix}$$

$$\det J = \frac{-WZ-W}{(1+Z)^3} = \frac{-W}{(1+Z)^2}$$

$$\therefore |\det J| = \frac{W}{(1+Z)^2}$$

$$\therefore\ dx\,dy = \frac{w}{(1+z)^2}\,dw\,dz$$

$$\begin{cases} X \geq 0,\ Y \geq 0 \\ \Rightarrow \dfrac{wz}{1+z} \geq 0,\ \dfrac{w}{1+z} \geq 0 \\ \therefore\ z \geq 0,\ w \geq 0 \end{cases}$$
…變數轉換後的積分範圍

$$\text{全機率} = 1 = \int_{X\geq 0}\int_{Y\geq 0} \lambda \exp(-\lambda x)\ \lambda \exp(-\lambda y)\,dx\,dy$$

$$= \int_{\underbrace{x\geq 0}}\int_{\underbrace{y\geq 0}} \lambda^2 \exp(-\lambda \underbrace{(x+y)}_{w})\ \underbrace{dx\,dy}_{\frac{w}{(1+z)^2}dw\,dz}$$

$$\begin{cases} w \geq 0 \\ z \geq 0 \end{cases}$$

$$= \int_{w\geq 0}\int_{z\geq 0} \underbrace{\lambda^2 \exp(-\lambda w)\cdot w \cdot \frac{1}{(1+z)^2}}_{f_{W,Z}(w,z)}\,dw\,dz$$

W, Z 之聯合機率密度函數

$$f_W(w) = \int_{z=0}^{z=\infty} \lambda^2 w \exp(-\lambda w)\cdot \frac{1}{(1+z)^2}\,dz$$

$$= \lambda^2 w \exp(-\lambda w)\left[\frac{-1}{1+z}\right]_{z=0}^{z=\infty} = \lambda^2 \exp(-\lambda w)\quad (w \geq 0)$$

⑫

同樣道理，$f_Z(z) = \int_{w=0}^{w=\infty} \lambda^3 w \exp(-\lambda w) \cdot \frac{1}{(1+z)^2} dw$

$= \frac{1}{(1+z)^2} \int_{w=0}^{w=\infty} \lambda^2 w \exp(-\lambda w)\, dw$

put $\lambda w = t \quad \frac{dt}{dw} = \lambda$

$= \frac{1}{(1+z)^2} \int_{t=0}^{t=\infty} \lambda^2 \left(\frac{t}{\lambda}\right) \exp(-t) \cdot \frac{dt}{\lambda}$

$= \frac{1}{(1+z)^2} \underbrace{\int_{t=0}^{t=\infty} t^{2-1} \exp(-t)\, dt}$

$= \Gamma(2)$ ($\because$ Gamma 函數)

$= 1! = 1$

$= \frac{1}{(1+z)^2} \quad (z \geq 0)$

$\therefore f_{WZ}(w,z) = f_W(w) \cdot f_Z(z)$ for all (w,z) $(w \geq 0,\ z \geq 0)$

$\boxed{\text{方法 2}}$ $X_1, X_2 \overset{iid}{\sim} \exp(\lambda)$ (λ: unknown parameter)

X_1+X_2 為 λ 之完備充份統計量

$\frac{X_2}{X_1} = \frac{(2\lambda X_2)/2}{(2\lambda X_1)/2}$ $\quad 2\lambda X_2 \sim \exp\left(\frac{1}{2}\right) = \Gamma\left(1, \frac{1}{2}\right) = \chi^2_{(2)}$ $\because$ 自由度 2 之卡方分佈

$2\lambda X_1 \sim \exp\left(\frac{1}{2}\right) = \Gamma\left(1, \frac{1}{2}\right) = \chi^2_{(2)}$

$\therefore \frac{X_2}{X_1}$ 之分佈與 λ 無關．$\therefore \frac{X_2}{X_1}$ 為 λ 之輔助統計量

根據 BASU 定理 X_1+X_2 vs $\frac{X_2}{X_1}$ 獨立．∎

證明完成

⑬

PS

- $X_1 + X_2 \sim \Gamma(2, \lambda)$
- $\dfrac{X_2}{X_1} = \dfrac{\left(\frac{2\lambda X_2}{2}\right)}{\left(\frac{2\lambda X_1}{2}\right)} \sim F_{2,2}$

注 $\dfrac{\left(\frac{X}{m}\right)}{\left(\frac{Y}{n}\right)} \sim F_{m,n}$ $\begin{cases} X \sim \chi^2_m, \quad Y \sim \chi^2_n \\ X, Y \text{ 独立} \end{cases}$

注 $X_1, X_2 \ldots X_n \overset{iid}{\sim} exp(\lambda) \quad (= \Gamma(1, \lambda))$

$X_1 + X_2 + \ldots X_n \sim \Gamma(n, \lambda)$

$\boxed{4}$ Taylor展開 (at μ $= E[X]$) $g(X) \approx g'(\mu)(X - \mu) + g(\mu)$ $\quad (g(X) = \sqrt{X})$

$\therefore E[g(X)] \approx g(\mu) = 2$

$V[g(X)] \approx V[g'(\mu)(X-\mu) + g(\mu)]$

$= (g'(\mu))^2 V[X]$

$= \left(\dfrac{1}{2\sqrt{\mu}}\right)^2 \cdot \sigma^2$

$= \dfrac{\sigma^2}{4\mu} = \dfrac{3}{4 \cdot 4} = \dfrac{3}{16}$

NOTE