Statistics Midterm Examination 27 April 2017

1. Let X be a random variable with cdf $F(x) = 1-\exp(-\alpha x^{\beta})$ for $x \geq 0$ and $F(x) = 0$ for $x < 0$, where $\alpha > 0$ and $\beta > 0$.
   (a) (3 pts.) Show that F is a cdf.
   (b) (3 pts.) Find the density of X.
   (c) (4 pts.) Find the probabilities $P(1 \leq X < 2)$ and $P(X = 3)$.

2. Let X and Y have the joint density

$$f(x,y) = \frac{6}{7}(x+y)^2, \; 0 \leq x \leq 1, \; 0 \leq y \leq 1.$$

   (a) (8 pts.) Find the two conditional densities.
   (b) (12 pts.) Find $\mathrm{E}(Y|X = x)$ and $\mathrm{Var}(Y|X = x)$, $0 \leq x \leq 1$, and verify that $\mathrm{E}[\mathrm{Var}(Y|X)] < \mathrm{Var}(Y)$.

3. A six-sided die is rolled 240 times.
   (a) (6 pts.) Approximate the probability that the face showing a three turns up between 30 and 60 times.
   (b) (6 pts.) Approximate the probability that the sum of the face values is greater than 400.

4. (8 pts.) If X and Y are independent exponential random variables with $\lambda = 1$, find the distribution of X/Y.

5. (10 pts.) Let $X_1, \ldots, X_n$ be a simple random sample without replacement from a finite population $\{x_1, \ldots, x_N\}$. Is $\bar{X}^2$ an unbiased estimator of μ^2, where μ is the population mean? If not, what is the bias?

6. Let $X_1, \ldots, X_n$ be i.i.d. uniform on $[0, \theta]$.
   (a) (6 pts.) Find the method of moments estimate of θ and its mean and variance.
   (b) (8 pts.) Find the MLE of θ and its mean and variance.
   (c) (6 pts.) Compare the bias, variance and mean squared error of the MLE to those of the method of moments estimate.

7. Let $X_1, \ldots, X_n$ be i.i.d. random variables with the density function

$$f(x|\theta) = (\theta+1)x^{\theta}, \quad 0 \leq x \leq 1.$$

   (a) (5 pts.) Find the MLE of θ.
   (b) (5 pts.) Find the asymptotic variance of the MLE.
   (c) (5 pts.) Find a sufficient statistic for θ.
   (d) (5 pts.) Is the MLE a UMVUE? Why or why not?