105- 統計學(台大、數學系) 4/21 期中考 參考答案、 by 森元徐誠

$\boxed{1}$

(a) $$F(x)=\begin{cases}0 & (x<0)\\ 1-\exp(-\alpha x^{\beta}) & (x\geq 0)\end{cases}$$

・$\lim_{x\to-\infty}F(x)=0$，$\lim_{x\to+\infty}F(x)=1$

・$\frac{dF}{dx}=\alpha\beta x^{\beta-1}\exp(-\alpha x^{\beta})\geq 0$ ； $\frac{dF}{dx}(0)=0\geq 0$

・$\lim_{x\to x_0-0}F(x)=F(x_0)$ (右連續)

$\therefore$ F為 cdf.

(b) $$\frac{dF}{dx}=\begin{cases}0 & (x<0)\\ \alpha\beta x^{\beta-1}\exp(-\alpha x^{\beta}) & (x\geq 0)\end{cases}$$

(c) $Pr(1\leq X<2)=Pr(X<2)-Pr(X<1)$

・$Pr(X<2)=\lim_{\varepsilon\to+0}Pr(X\leq 2-\varepsilon)=\lim_{\varepsilon\searrow 0}F(2-\varepsilon)$

$=F(2)$

同理，$Pr(X<1)=F(1)$

$\therefore F(2)-F(1)=\exp(-\alpha)-\exp(-\alpha 2^{\beta})$

$Pr(X=3)=Pr(X\leqq 3)-Pr(X<3)=F(3)-F(3)=0$

②

$\boxed{2}$

(a) $f_X(x) = \int_{y=0}^{y=1} \frac{6}{7}(x+y)^2 dy = \left[\frac{2}{7}(x+y)^3\right]_{y=0}^{y=1}$

$= \frac{2}{7}\{(x+1)^3 - x^3\} \quad (0 \leq x \leq 1)$

$f_Y(y) = \frac{2}{7}\{(y+1)^3 - y^3\}$

$\therefore f_{X|Y} = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{6}{7}(x+y)^2}{\frac{2}{7}\{(y+1)^3 - y^3\}} \quad (0 \leq x \leq 1)$

$f_{Y|X} = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{6}{7}(x+y)^2}{\frac{2}{7}\{(x+1)^3 - x^3\}} \quad (0 \leq y \leq 1)$

(b) $E[Y|X=x] = \int_{y=0}^{y=1} y \cdot f_{Y|X}(y|x)\, dy$

$= \frac{3}{(x+1)^3 - x^3} \int_{y=0}^{y=1} y(x+y)^2 dy$

$= \frac{3}{(x+1)^3 - x^3} \int_{y=0}^{y=1} (y^3 + 2xy^2 + x^2 y)\, dy$

$= \frac{3}{(x+1)^3 - x^3} \left[\frac{y^4}{4} + \frac{2x}{3}y^3 + \frac{x^2 y^2}{2}\right]_{y=0}^{y=1}$

$= \frac{3}{(x+1)^3 - x^3} \left[\frac{1}{4} + \frac{2x}{3} + \frac{x^2}{2}\right]$

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$\boxed{2}$

(b) $$\frac{1}{(X+1)^3-X^3}\cdot\frac{1}{4}\left(6X^2+8X+3\right)$$

$$=\frac{1}{4}\cdot\frac{1}{3X^2+3X+1}\left(6X^2+8X+3\right)$$

$$\therefore E[Y|X=x]=\frac{1}{4}\cdot\frac{1}{3X^2+3X+1}\left(6X^2+8X+3\right)$$

接著求 $E[Y^2|X=x]$

$$=\int_{y=0}^{y=1} y^2\cdot 3\cdot\frac{(x+y)^2}{(X+1)^3-X^3}\,dy$$

$$=\frac{3}{(X+1)^3-X^3}\cdot\underbrace{\int_{y=0}^{y=1}\left(y^4+2xy^3+x^2y^2\right)dy}_{\parallel}$$

$$\left[\frac{y^5}{5}+\frac{xy^4}{2}+\frac{x^2y^3}{3}\right]_{y=0}^{y=1}$$

$$=\frac{X^2}{3}+\frac{X}{2}+\frac{1}{5}$$

$$=\frac{1}{10}\cdot\frac{1}{(X+1)^3-X^3}\left(10X^2+15X+6\right)$$

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④

$$V[Y|X=x] = \underbrace{E[Y^2|X=x]}_{\parallel} - E[Y|X=x]^2$$

$$\frac{1}{10}\cdot\frac{1}{3x^2+3x+1}(10x^2+15x+6)$$

$$-\frac{1}{16}\cdot\frac{1}{(3x^2+3x+1)^2}(6x^2+8x+3)^2$$

$$=\frac{1}{80}\cdot\frac{1}{(3x^2+3x+1)^2}\cdot\underbrace{8(10x^2+15x+6)(3x^2+3x+1)}_{①}$$

$$-\frac{1}{80}\cdot\frac{1}{(3x^2+3x+1)^2}\cdot\underbrace{5(6x^2+8x+3)^2}_{②}$$

$$①\cdots 8(30x^4+45x^3+18x^2+30x^3+45x^2+18x+10x^2+15x+6)$$

$$=8(30x^4+75x^3+73x^2+33x+6)$$

$$②\cdots 5(36x^4+64x^2+9+96x^3+48x+36x^2)$$

$$=5(36x^4+96x^3+100x^2+48x+9)$$

$$\therefore ①-② = (240x^4+600x^3+584x^2+264x+48)$$

$$-(180x^4+480x^3+500x^2+240x+45)$$

$$=60x^4+120x^3+84x^2+24x+3$$

$$\therefore \frac{3}{80}\cdot\frac{1}{(3x^2+3x+1)^2}(20x^4+40x^3+28x^2+8x+1)=V[Y|X]$$

⑤

$\boxed{2}$

(b) $V[Y] = E[V[Y|X]] + V[E[Y|X]]$

$\therefore V[Y] - E[V[Y|X]] = V[E[Y|X]]$

$E[Y|X] = \frac{1}{4} \cdot \frac{1}{3X^2+3X+1}(6X^2+8X+3)$

$E[Y|X]$ 並非常數.

$(\nexists\, c \in \mathbb{R} \text{ s.t. } Pr(E[Y|X] = c) = 1)$

$\therefore \underline{V[E[Y|X]] > 0}$

$\therefore V[Y] > E[V[Y|X]]$

證明完成

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⑦

$\boxed{3}$

(a) $I_j = \begin{cases} 1 & \text{if } i\text{th 為 } 3 \\ 0 & \text{else} \end{cases}$

$I_j \overset{iid}{\sim} \text{bernoulli}\left(\frac{1}{6}\right) \quad E[I_j] = \frac{1}{6} \quad V[I_j] = \frac{5}{36} < \infty$

根據中央極限定理,

$$\sqrt{n}\left(\frac{I_1 + \dots I_n}{n} - \frac{1}{6}\right) \xrightarrow{d} N\left(0, \frac{5}{36}\right)$$

$$\therefore \sqrt{n}\,\frac{6}{\sqrt{5}}\left(\frac{I_1+I_2+\dots I_n}{n} - \frac{1}{6}\right) \xrightarrow{d} N(0,1)$$

$$Z \overset{\text{let}}{=} \sqrt{n}\cdot\frac{6}{\sqrt{5}}\left(\frac{I_1+\dots I_n}{n} - \frac{1}{6}\right) \xrightarrow{d} N(0,1)$$

$$30 \leq I_1 + I_2 + \dots I_{240} \leq 60$$

$$\Updownarrow$$

$$\sqrt{240}\cdot\frac{6}{\sqrt{5}}\left(\frac{-1}{24}\right) \leq Z \leq \sqrt{240}\cdot\frac{6}{\sqrt{5}}\left(\frac{1}{12}\right)$$

$$\sqrt{240}\cdot\frac{6}{\sqrt{5}}\left(\frac{-1}{24}\right) = -\sqrt{3} \qquad \sqrt{240}\cdot\frac{6}{\sqrt{5}}\left(\frac{1}{12}\right) = 2\sqrt{3}$$

$$\therefore \approx \Phi(2\sqrt{3}) - \Phi(-\sqrt{3})$$

⑧

(b) $\Pr(X_j = a) = \frac{1}{6} \quad a \in \{1,2,3,4,5,6\}$

$E[X_j] = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$

$E[X_j^2] = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$

$V[X_j] = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \frac{35}{12} < \infty$

根據中央極限定理 ($\because X_1 \sim X_n$: iid $\quad V[X_j] < \infty$)

$\sqrt{n}\left(\bar{X} - \frac{7}{2}\right) \xrightarrow{d} N\left(0, \frac{35}{12}\right)$

$$\therefore \underbrace{\sqrt{n} \cdot \frac{2\sqrt{3}}{\sqrt{35}}\left(\bar{X} - \frac{7}{2}\right)}_{= Z} \xrightarrow{d} N(0,1)$$

$\bar{X} \geq \frac{400}{240} \qquad \bar{X} - \frac{7}{2} \geq \frac{-440}{240} = \frac{-11}{6}$

$\therefore Z \geq \sqrt{240} \cdot \frac{2\sqrt{3}}{\sqrt{35}} \cdot \frac{-11}{6} = \frac{4\sqrt{3}}{\sqrt{7}} \cdot \frac{\sqrt{3}}{3} \cdot (-11)$

$= \frac{-44}{\sqrt{7}}$

$\therefore \Pr\left(Z \geq \frac{-44}{\sqrt{7}}\right) = 1 - \Phi\left(\frac{-44}{\sqrt{7}}\right) = \Phi\left(\frac{44}{\sqrt{7}}\right) \approx 1$

$\boxed{4}$ $X \sim \exp(1)$ $Y \sim \exp(1)$

$$2X \sim \exp(2) = \Gamma(1,2) = \chi^2_2 \qquad 2Y \sim \exp(2) = \Gamma(1,2) = \chi^2_2$$

$$\therefore \frac{X}{Y} = \frac{(2X)/2}{(2Y)/2} \sim F_{2,2}$$

⑩

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$$E\left[\left\{\frac{(X_1+X_2+\cdots X_n)}{n}\right\}^2\right] = \frac{1}{n^2} E\left[\sum_{j=1}^{n} X_j^2 + \sum_{(i \neq j)}\sum X_i X_j\right]$$

$$= \frac{1}{n^2} \cdot \left\{ n E[X_1^2] + n(n-1) E[X_1 X_2] \right\}$$

$$= \frac{1}{n} \left\{ E[X_1^2] + (n-1) E[X_1 X_2] \right\}$$

① $E[X_1^2] = \frac{1}{N} \sum_{j=1}^{N} x_j^2 = \frac{1}{N} \left\{ \sum_{j=1}^{N} (x_j - \bar{x})^2 + N\bar{X}^2 \right\}$

$$= \sigma^2 + \mu^2$$

② $E[X_1 X_2] = \frac{1}{N(N-1)} \sum_{(i \neq j)}\sum x_i x_j$

$$= \frac{1}{N(N-1)} \left\{ x_1(x_2 + \cdots x_N) + x_2(x_1 + x_3 + \cdots x_N) + \cdots + x_N(x_1 + \cdots x_{N-1}) \right\}$$

$$= \frac{1}{N(N-1)} \left\{ (x_1 + \cdots + x_N)^2 - (x_1^2 + x_2^2 + \cdots + x_N^2) \right\}$$

$$= \frac{1}{N(N-1)} \left\{ N^2\mu^2 - N(\sigma^2 + \mu^2) \right\}$$

$$= \mu^2 - \frac{\sigma^2}{N-1}$$

∴ 根據①,②, $E[\bar{X}^2] = \frac{1}{n}\left(\sigma^2 + \mu^2 + (n-1).\left(\mu^2 - \frac{\sigma^2}{N-1}\right)\right)$

$$= \frac{1}{n}\left(n\mu^2 + \sigma^2\left(1 - \frac{n-1}{N-1}\right)\right)$$

$$= \mu^2 + \sigma^2 \cdot \frac{1}{n} \frac{N-n}{N-1} \neq \mu^2$$

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(11)

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(a)

$$E[X_j] = \int_0^\theta \frac{x}{\theta}\,dx = \frac{1}{\theta}\left[\frac{x^2}{2}\right]_0^\theta = \frac{\theta}{2}$$

$$E[X_j^2] = \int_0^\theta \frac{x^2}{\theta}\,dx = \frac{1}{\theta}\left[\frac{x^3}{3}\right]_0^\theta = \frac{\theta^2}{3} \qquad \therefore V[X_j] = \frac{\theta^2}{12}$$

$$\therefore \frac{X_1+\cdots+X_n}{n} \xrightarrow{p} \frac{\theta}{2} \qquad \therefore 2\overline{X} \xrightarrow{p} \theta$$

$$\therefore \hat{\theta}_{MME} = 2\overline{X} \qquad E[2\overline{X}] = \theta \qquad V[2\overline{X}] = \frac{4}{n}V[X_j] = \frac{\theta^2}{3n}$$

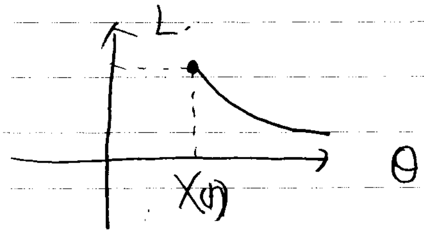
(b)

$$\underbrace{f_X(x_1 \ldots x_n \mid \theta)}_{\text{聯合pdf}} = \prod_{j=1}^{n}\left(\frac{1}{\theta}\right) I_{[0,\theta]}(x_j) \qquad 0 \leq x_j \leq \theta$$

$$= \prod_{j=1}^{n} \frac{1}{\theta} I_{[x_j,\infty]}(\theta)$$

$$= \frac{1}{\theta^n} I_{[X_{(n)},\infty]}(\theta) \qquad X_{(n)} = \max\{X_1 \ldots X_n\}$$

$$\therefore L(\theta \mid X_1 \ldots X_n) = \frac{1}{\theta^n} I_{[X_{(n)},\infty]}(\theta) \cdots \text{Liklihood Function.}$$



$\therefore \theta = X_{(n)} = \max\{X_1 \ldots X_n\}$ 使得 $L(\theta)$ 最大.

$$\therefore \hat{\theta}_{MLE} = \max\{X_1, \ldots, X_n\}$$

$$Pr(\hat{\theta}_{MLE} \leq t) = \begin{cases} \left(\frac{t}{\theta}\right)^n & (0 \leq t \leq \theta) \\ 0 & (t < 0) \\ 1 & (t \geq \theta) \end{cases}$$

$$\therefore \frac{d}{dt}\Pr(\hat{\theta}_{MLE} \leq t) = \begin{cases} \dfrac{nt^{n-1}}{\theta^n} & (0 \leq t \leq \theta) \\ 0 & (\text{elsewhere}) \end{cases}$$

$$f_{\hat{\theta}}(t) = \frac{nt^{n-1}}{\theta^n} \quad (0 \leq t \leq \theta)$$

$$E[\hat{\theta}] = \int_0^\theta \frac{nt^n}{\theta^n}\,dt = \frac{n}{n+1}\cdot\theta$$

$$E[\hat{\theta}^2] = \int_0^\theta \frac{nt^{n+1}}{\theta^n}\,dt = \frac{n}{n+2}\theta^2$$

$$V[\hat{\theta}] = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 = \frac{n(n+1)^2 - n^2(n+2)}{(n+1)^2(n+2)}\theta^2$$

$$= \frac{n\theta^2}{(n+1)^2(n+2)}$$

(c) ① bias … $E[\hat{\theta}_{MME}] - \theta = 0$ (better)

$$E[\hat{\theta}_{MLE}] - \theta = \frac{-1}{n+1}\theta$$

② variance $V[\hat{\theta}_{MME}] = \dfrac{\theta^2}{3n}$

$$V[\hat{\theta}_{MLE}] = \frac{n\theta^2}{(n+1)^2(n+2)} \quad \text{… (better)}$$

③ MSE … $MSE[\hat{\theta}_{MME}] = V[\hat{\theta}_{MLE}] = \dfrac{\theta^2}{3n}$

$$MSE[\hat{\theta}_{MLE}] = \frac{n\theta^2}{(n+1)^2(n+2)} + \frac{\theta^2}{(n+1)^2} \quad \text{(better)}$$

$\left(\because \hat{\theta}_{MME} \text{ … } O\left(\frac{1}{n}\right) ;\ \hat{\theta}_{MLE} \text{ … } O\left(\frac{1}{n^2}\right)\right)$ $\quad \frac{1}{n^2} \ll \frac{1}{n}$

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$\boxed{7}$

(a) $f_X(x_1,\dots,x_n|\theta) = (\theta+1)^n (x_1\cdots x_n)^\theta = L(\theta)$

$$\frac{\partial}{\partial\theta}\log L(\theta) = \frac{\partial}{\partial\theta}\left\{ n\log(\theta+1) + \theta\sum_{j=1}^{n}\log X_j \right\}$$

$$= \frac{n}{\theta+1} + \sum_{j=1}^{n}\log X_j = 0$$

$$\Rightarrow \theta+1 = \frac{-n}{\sum_{j=1}^{n}\log X_j} \Rightarrow \theta = \frac{-n}{\sum_{j=1}^{n}\log X_j} - 1 \Rightarrow L\text{最大}$$

$$\therefore \hat{\theta}_{MLE} = \frac{-n}{\sum_{j=1}^{n}\log X_j} - 1.$$

(b) 先求 $-\log X_j$ $(j=1\sim n)$ 之分佈 $(Y_j = -\log X_j)$

$1 = \int_0^1 (\theta+1)x^\theta dx$ (全機率)

$$\left(\begin{array}{ll} Y = -\log X & (\because X: 0\to 1 \\ \frac{dY}{dX} = \frac{-1}{X} = -e^Y & \ \ \ \ Y: \infty\to 0) \end{array} \right)$$

$$= \int_\infty^0 (\theta+1)(e^{-\theta y})\cdot(-e^{-y})dy$$

$$= \int_0^\infty (\theta+1)e^{-(\theta+1)y}dy$$

由此可知 $Y_j \sim \exp(\theta+1)$ (mean $\frac{1}{\theta+1}$)

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接著 $Y_j \overset{def}{=} -\log X_j$ $\quad E[Y_j]=\frac{1}{\theta+1}$ $\quad V[Y_j]=\left(\frac{1}{\theta+1}\right)^2$

根據中央極限定理，$\sqrt{n}\left(\bar{Y}-\frac{1}{\theta+1}\right)\xrightarrow{d} N\left(0,\frac{1}{(\theta+1)^2}\right)$

利用 δ-method $\quad g(x)=\frac{1}{x} \quad g'(x)=-\frac{1}{x^2}$

$$\sqrt{n}\left(g(\bar{Y})-g\left(\frac{1}{\theta+1}\right)\right)\xrightarrow{d} N\left(0,\frac{1}{(\theta+1)^2}\cdot\left\{g'\left(\frac{1}{\theta+1}\right)\right\}^2\right)$$

$$\therefore \sqrt{n}\left(\frac{1}{\bar{Y}}-(\theta+1)\right)\xrightarrow{d} N\left(0,\frac{1}{(\theta+1)^2}\cdot(\theta+1)^4\right)$$

$$= N\left(0,(\theta+1)^2\right)$$

注意 $\frac{1}{\bar{Y}}=\frac{n}{Y_1+Y_2+\cdots+Y_n}=\frac{-n}{\sum_{j=1}^{n}\log X_j}$

故此 $\sqrt{n}\left(\underbrace{\frac{-n}{\sum\log X_j}-1}_{\hat{\theta}_{MLE}}-\theta\right)\xrightarrow{d} N\left(0,(\theta+1)^2\right)$

$$\therefore \sqrt{n}\left(\hat{\theta}_{MLE}-\theta\right)\xrightarrow{d} N\left(0,(\theta+1)^2\right)$$

(c) $\prod_{j=1}^{n} f(x_j|\theta)=(\theta+1)^n(x_1x_2\cdots x_n)^\theta$

put $T=X_1X_2\cdots X_n$

$$=(\theta+1)T^\theta$$

根據 Neyman-Fisher 分解定理，

T 為 θ 之充份統計量.

四

註

(c) $T = X_1 X_2 X_3 \cdots X_n = \exp\left(\sum_{j=1}^{n} \log X_j\right)$

$\exp(x)$ 為 one-to-one 的函數，∴ $\sum_{j=1}^{n} \log X_j$ 亦為 θ 的充份統計量

(d) MLE 並非不偏估計量，∴ 並非 UMVUE

$$MLE = \frac{-n}{\sum_{j=1}^{n} \log X_j} - 1.$$

$$-\log X_j \sim \exp\left(\frac{1}{\theta+1}\right)$$

$$\therefore W \overset{def}{=} \sum_{j=1}^{n} -\log X_j \sim \Gamma\left(n, \frac{1}{\theta+1}\right)$$

$$\hat{\theta}_{MLE} = \frac{n}{W} - 1$$

$$E\left[\frac{1}{W}\right] = \int_0^\infty \frac{1}{w} \cdot (\theta+1)^n \cdot \frac{w^{n-1}}{\Gamma(n)} \exp(-(\theta+1)w)\, dw$$

$$z = (\theta+1) w \qquad \frac{dz}{dw} = (\theta+1)$$

$$= \int_0^\infty (\theta+1)^n \cdot \left(\frac{z}{\theta+1}\right)^{n-2} \cdot \frac{1}{\Gamma(n)} \exp(-z) \cdot \frac{dz}{\theta+1}$$

$$= \int_0^\infty (\theta+1) \cdot \frac{z^{n-2}}{\Gamma(n)} \exp(-z)\, dz$$

$$= (\theta+1) \cdot \frac{\Gamma(n-1)}{\Gamma(n)} = \frac{\theta+1}{(n-1)}$$

$$\therefore E\Big[\underbrace{\frac{n}{W}-1}_{\hat{\theta}_{MLE}}\Big] = \frac{n}{n-1}(\theta+1)-1 = \frac{n\theta+n-n+1}{n-1} = \frac{n\theta+1}{n-1} \neq \theta$$

$\therefore E[\hat{\theta}_{MLE}] \neq \theta$ $\therefore$ $\hat{\theta}_{MLE}$ 並非 UMVUE.