作業6. 参考答案　by morimoto

10.5

$$\mathrm{cov}[F_n(u), F_n(v)] = \mathrm{cov}\left[\frac{1}{n}\sum_{i=1}^{n} I(X_i \leq u), \frac{1}{n}\sum_{j=1}^{n} I(Y_i \leq v)\right]$$

$$= \frac{1}{n^2}\sum_i\sum_j \mathrm{cov}[I(X_i \leq u), I(X_j \leq v)]$$

($i \neq j$ は独立)

$$= \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{cov}[I(X_i \leq u), I(X_i \leq v)]$$

$$= \frac{1}{n}\mathrm{cov}[I(X_1 \leq u), I(X_1 \leq v)]$$

$$= \frac{1}{n}\Pr(X_1 \leq u, X_1 \leq v) - \frac{1}{n}\Pr(X_1 \leq u)\Pr(X_1 \leq v)$$

$$= \frac{1}{n}\left(F(\min\{u, v\}) - F(u)F(v)\right)$$

10.9.

$E[\log(1-\hat{F}(t))] - \log(1-F(t)) = \text{bias}$

利用 δ-method... $g(x) = \log(1-x)$

$g(x) \approx g(\mu) + g'(\mu)(x-\mu) + \frac{1}{2!} g''(\mu)(x-\mu)^2$ $\quad (x \to \hat{F}(t)$, $\mu \to F(t))$

$$g(\hat{F}(t)) \approx g(F(t)) + g'(F(t))(\hat{F}(t) - F(t)) + \frac{1}{2!} g''(F(t))(\hat{F}(t) - F(t))^2$$

$$g(\hat{F}(t)) - g(F(t)) \approx g'(F(t))(\hat{F}(t) - F(t)) + \frac{1}{2} g''(F(t))(\hat{F}(t) - F(t))^2$$

$\therefore \text{bias} \approx \frac{1}{2} g''(F(t)) V[\hat{F}(t)]$

$\left(g'(x) = \frac{-1}{1-x} \quad g''(x) = \frac{-1}{(1-x)^2} \right)$

$$\text{bias} \approx \frac{1}{2} \cdot \frac{-1}{(1-F(t))^2} \cdot \frac{1}{n} \cdot F(t)(1-F(t))$$

$$= \frac{-F(t)}{2n(1-F(t))}$$

($F(t) \nearrow 1$ 時, bias 會很大)

10.11.

$$\frac{d}{dt}F(t) = f(t) = \alpha\beta t^{\beta-1} e^{-(\alpha t^{\beta})}$$

$$\frac{f(t)}{1-F(t)} = \frac{\alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}}}{e^{-\alpha t^{\beta}}} = \alpha\beta t^{\beta-1}$$

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10.13.

Weibull 分佈 … $\beta > 1$

$h(t) = \alpha\beta t^{\beta-1}$ ($\because$ 10.11) ↗ (t ↗)

Chryo culture

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10.14.

Weibull 分布 $0<\beta<1$

$h(t) = \alpha\beta t^{\beta-1}$ ($\because$ 10.11) $\searrow$ ($t\nearrow$)

Chryv culture

11.13.

①

$X_1, X_2, \cdots X_n \sim N(\mu, 1)$ $(\mu \geqq 0)$ $(\mu = 0.3,\ n = 25)$

$T = \sum_{j=1}^{n} I(X_j > 0) \sim Bin(n, \Phi(\mu))$

$\left(\because Pr(X_j > 0) = Pr(X_j - \mu > -\mu) = 1 - \Phi(-\mu) = \Phi(\mu) \right)$

n很大時 $T \approx N(n\Phi(\mu), n\Phi(\mu)(1-\Phi(\mu)))$

$$\frac{T - n\Phi(\mu)}{\sqrt{n\Phi(\mu)(1-\Phi(\mu))}} \sim N(0,1)$$

棄卻域 $C = \{T \mid T > c\}$

$\mu = 0$時, $Pr(T \in C \mid \mu = 0) = 0.05$

$$\therefore \frac{T - n\Phi(0)}{\sqrt{n\Phi(0)(1-\Phi(0))}} = \frac{T - \frac{n}{2}}{\sqrt{\frac{n}{4}}} \geqq 1.645$$

$$\therefore T \geqq \underbrace{\frac{n}{2} + \frac{\sqrt{n}}{2} \cdot 1.645}_{= c}$$

$$\beta(\mu) = Pr(T > c \mid \mu) = Pr\left(\underbrace{\frac{T - n\Phi(\mu)}{\sqrt{n\Phi(\mu)(1-\Phi(\mu))}}}_{Z} > \frac{c - n\Phi(\mu)}{\sqrt{n\Phi(\mu)(1-\Phi(\mu))}} \,\middle|\, \mu \right)$$

$$Pr\left(Z > \frac{\frac{n}{2} + \frac{\sqrt{n}}{2} \cdot 1.645 - n\Phi(\mu)}{\sqrt{n\Phi(\mu)(1-\Phi(\mu))}} \,\middle|\, Z \sim N(0,1) \right)$$

$$= 1 - \Phi\left(\frac{\frac{n}{2} + \frac{\sqrt{n}}{2} \cdot 1.645 - n\Phi(\mu)}{\sqrt{n\Phi(\mu)(1-\Phi(\mu))}} \right)$$

$\mu = 0.3 \quad n = 25$

$$\beta(0.3) = 1 - \Phi\left(\frac{12.5 + 2.5 \cdot 1.645 - 25 \cdot \Phi(0.3)}{\sqrt{25\,\Phi(0.3)(1-\Phi(0.3))}} \right) \qquad \Phi(0.3) = 0.6179$$

$$= 1 - \Phi(0.4795)$$

$$= 1 - 0.6844 = \underline{0.3156} \qquad (\text{sign-test})$$

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② Norma Theory...

$\sqrt{n}(\bar{X}-\mu) \sim N(0,1)$

$\mu=0$ 時… $\sqrt{n}\bar{X} \geqq 1.645$

$\therefore\ \bar{X} \geqq \frac{1.645}{\sqrt{n}} \Rightarrow$ 棄卻

$\beta(\mu) = Pr\left(\bar{X} \geqq \frac{1.645}{\sqrt{n}} \,\middle|\, \mu\right)$

$= Pr\left(\sqrt{n}(\bar{X}-\mu) \geqq 1.645-\sqrt{n}\mu \,\middle|\, \mu\right)$

$= 1-\Phi(1.645-\sqrt{n}\mu)$

$n=25,\ \mu=0.3$

$\beta(0.3) = 1-\Phi(1.645-1.5)$

$= 1-\Phi(0.145) \fallingdotseq 0.44$

(檢定力較大)

11.15.

某種分佈

$$X_1, X_2, \ldots X_n \sim D_X(\mu_X, \sigma^2) \qquad (\sigma^2 = 100)$$

$$Y_1, Y_2, \ldots Y_n \sim D_Y(\mu_Y, \sigma^2)$$

根據中央極限定理, $\dfrac{\sqrt{n}(\bar{X}-\mu_X)}{\sigma} \xrightarrow{d} N(0,1)$

$$\frac{\sqrt{n}(\bar{Y}-\mu_Y)}{\sigma} \xrightarrow{d} N(0,1)$$

$$\therefore \frac{\sqrt{n}(\bar{X}-\bar{Y}-\mu_X+\mu_Y)}{\sqrt{2}\cdot\sigma} \sim N(0,1) \quad (n\text{很大})$$

$$\therefore -1.96 \leq \frac{\sqrt{n}(\bar{X}-\bar{Y}-\mu_X+\mu_Y)}{10\sqrt{2}} \leq 1.96$$

$$\frac{-1.96\cdot 10\sqrt{2}}{\sqrt{n}} \leq (\bar{X}-\bar{Y})-(\mu_X-\mu_Y) \leq \frac{1.96\cdot 10\sqrt{2}}{\sqrt{n}}$$

$$\bar{X}-\bar{Y} \pm \frac{1.96\cdot 10\sqrt{2}}{\sqrt{n}} \quad (\text{信賴區間})$$

$$\text{寬度} = \frac{3.92\cdot 10\sqrt{2}}{\sqrt{n}} \leq 2$$

$$\therefore n = (1.96\cdot 10\sqrt{2})^2$$

$$\therefore n \geq 769$$

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11.19.

$X_1 \sim X_{25} \sim N(\mu_X, \sigma^2)$

$Y_1 \sim Y_{25} \sim N(\mu_Y, \sigma^2)$

(1) $\bar{Y} - \bar{X} \sim N\left(\mu_Y - \mu_X, \frac{2\sigma^2}{25}\right) = N(\mu_Y - \mu_X, 2)$

$SE(\bar{Y} - \bar{X}) = \sqrt{2}$

(2) $\mu = \mu_Y - \mu_X \quad H_0: \mu = 0 \ \text{vs} \ H_1: \mu > 0$

Neyman-Pearson's lemma

$C: \{(X_1, \dots X_{25}, Y_1 \dots Y_{25}) \mid \bar{Y} - \bar{X} > c\}$

$\mu = 0$ 時 $\Pr(\bar{Y} - \bar{X} > c \mid \mu = 0)$

$= \Pr\left(\frac{\bar{Y} - \bar{X}}{\sqrt{2}} > \frac{c}{\sqrt{2}} \mid \mu = 0\right) = 1 - \Phi\left(\frac{c}{\sqrt{2}}\right) = 0.05$

$c = \sqrt{2} \cdot \Phi^{-1}(0.95) = \sqrt{2} \cdot 1.645$

$\therefore C = \{(X_1, \dots X_{25}, Y_1, \dots, Y_{25}) \mid \bar{Y} - \bar{X} > \sqrt{2} \cdot 1.645\}$

(3) $\mu_Y - \mu_X = 1$ 時

$Pr(\overline{Y}-\overline{X} > \sqrt{2}\cdot 1.645 \mid \mu=1)$

$= Pr\left(\frac{\overline{Y}-\overline{X}-1}{\sqrt{2}} > \frac{\sqrt{2}\cdot 1.645 - 1}{\sqrt{2}} \mid \mu=1\right)$

$= 1-\Phi\left(1.645 - \frac{1}{\sqrt{2}}\right) = 1-\Phi(0.938) = 1-0.8264$

$= 0.174$

(4) YES

(5) $C = \{(X_1, \cdots X_{25}, Y_1 \cdots Y_{25}) \mid |\overline{X}-\overline{Y}| \geq \sqrt{2}\cdot 1.96\}$

$Pr(\overline{X}-\overline{Y} \geq \sqrt{2}\cdot 1.96 \mid \mu=1)$

$+ Pr(\overline{X}-\overline{Y} \leq -\sqrt{2}\cdot 1.96 \mid \mu=1)$

$= Pr\left(\frac{\overline{X}-\overline{Y}-1}{\sqrt{2}} \geq 1.96 - \frac{1}{\sqrt{2}} \mid \mu=1\right)$

$+ Pr\left(\frac{\overline{X}-\overline{Y}-1}{\sqrt{2}} \leq -1.96 - \frac{1}{\sqrt{2}} \mid \mu=1\right)$

$= 1-\Phi\left(1.96-\frac{1}{\sqrt{2}}\right) + \Phi\left(-1.96-\frac{1}{\sqrt{2}}\right)$

$= 1-\Phi(1.25) + \Phi(-2.67)$

$= 1-(0.8944) + (0.0038) \approx 0.11$

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11.26

$$\begin{cases} X_1, X_2, \dots X_n \sim N(0,1) \\ Y_1, Y_2, \dots Y_n \sim N(1,1) \end{cases}$$

(a) $T = \frac{n(n+1)}{2} + \sum_{i=1}^{n}\sum_{j=1}^{n} I(X_i > Y_j)$ (rank sum)

$$E[T] = \frac{n(n+1)}{2} + E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} I(X_i > Y_j)\right]$$

$$= \frac{1}{2}n(n+1) + n^2 \cdot Pr(X_1 > Y_1)$$

$Pr(X_1 > Y_1) = Pr(Y_1 - X_1 < 0)$ $\quad Y_1 - X_1 \sim N(1,2)$

$$= Pr\left(\frac{Y_1 - X_1 - 1}{\sqrt{2}} \le \frac{-1}{\sqrt{2}}\right)$$

$$= \Phi\left(\frac{-1}{\sqrt{2}}\right) = 1 - \Phi\left(\frac{1}{\sqrt{2}}\right)$$

$$\therefore E[T] = \frac{n(n+1)}{2} + n^2 \cdot \left(1 - \Phi\left(\frac{1}{\sqrt{2}}\right)\right)$$

(6) $V\left[\frac{n(n+1)}{2} + \sum_{i=1}^{n}\sum_{j=1}^{n} I(X_i > Y_j)\right]$

$= V\Big[\sum_{i=1}^{n} \underbrace{\sum_{j=1}^{n} I(X_i > Y_j)}_{U_i}\Big]$

$= V\left[\sum_{i=1}^{n} U_i\right] = n V[U_1] + n(n-1)\, cov[U_1, U_2]$

① $V[U_1] = V\left[\sum_{j=1}^{n} I(X_1 > Y_j)\right]$

$= n \underbrace{V[I(X_1 > Y_1)]}_{\text{bernoulli}\left(1-\Phi\left(\frac{1}{\sqrt{2}}\right)\right)} + n(n-1)\, cov[I(X_1 > Y_1), I(X_1 > Y_2)]$

$\therefore \left(1-\Phi\left(\frac{1}{\sqrt{2}}\right)\right)\cdot\Phi\left(\frac{1}{\sqrt{2}}\right)$

② $cov[U_1, U_2] = \sum_{i=1}^{n}\sum_{j=1}^{n} cov[I(X_1 > Y_i), I(X_2 > Y_j)]$

($i \neq j \Rightarrow$ 獨立)

$= \sum_{j=1}^{n} cov[I(X_1 > Y_j), I(X_2 > Y_j)]$

$= n\, cov[I(X_1 > Y_1), I(X_2 > Y_1)]$

$\therefore$ ①+② ... $n^2 V[I(X_1 > Y_1)] + n^2(n-1)\, cov[I(X_1 > Y_1), I(X_1 > Y_2)]$

$+ n^2(n-1)\, cov[I(X_1 > Y_1), I(X_2 > Y_1)]$

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(三) $\mathrm{cov}[I(X_1>Y_1), I(X_1>Y_2)]$

$$= \underbrace{Pr(X_1>Y_1, Y_2)}_{\text{計算有點困難}} - \underbrace{Pr(X_1>Y_1)}_{1-\Phi(\frac{1}{\sqrt{2}})} \underbrace{Pr(X_1>Y_2)}_{1-\Phi(\frac{1}{\sqrt{2}})}$$

$\mathrm{cov}[I(X_1>Y_1), I(X_2>Y_1)]$

$$= \underbrace{Pr(X_1, X_2>Y_1)}_{\text{計算有點困難}} - \underbrace{Pr(X_1>Y_1)}_{} \underbrace{Pr(X_2>Y_1)}_{} $$

$$\left(1-\Phi\left(\tfrac{1}{\sqrt{2}}\right)\right) \cdot \left(1-\Phi\left(\tfrac{1}{\sqrt{2}}\right)\right)$$

11.31.

$$\hat{\pi} = \frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m} Z_{ij}$$

$$E[\hat{\pi}] = E[Z_{11}] = Pr(X_1 < Y_1) = \frac{1}{2}$$

$$V[\hat{\pi}] = \frac{1}{(mn)^2} V\left[\sum_{i=1}^{n}\sum_{j=1}^{m} Z_{ij}\right]$$

$$W_i \overset{def}{=} \sum_{j=1}^{m} Z_{ij}$$

$$= \frac{1}{(mn)^2} V\left[\sum_{i=1}^{n} W_i\right]$$

$$= \left(\frac{1}{mn}\right)^2 \left(nV[W_1] + n(n-1)\, cov[W_1, W_2] \right)$$

① $V[W_1] = V\left[\sum_{j=1}^{m} Z_{1j}\right]$

$$= mV[Z_{11}] + m(m-1)\, cov[Z_{11}, Z_{12}]$$

$$= \frac{m}{4} + m(m-1)\left\{ E[Z_{11}Z_{12}] - \underbrace{E[Z_{11}]}_{\frac{1}{2}}\,\underbrace{E[Z_{12}]}_{\frac{1}{2}} \right\}$$

$Z_{11} \sim bernoulli(\frac{1}{2})$

$$E[Z_{11}Z_{12}] = Pr(X_1 < Y_1, X_1 < Y_2)$$

$$= \frac{1}{3}$$ (∵ 排列 $= 3!$, $X_1 < Y_1 < Y_2$ or $X_1 < Y_2 < Y_1$ ∴ $\frac{2}{3!} = \frac{1}{3}$)

$$= \frac{m}{4} + \frac{m(m-1)}{12}$$

$$\mathrm{cov}[W_1, W_2] = \sum_{k=1}^{m}\sum_{l=1}^{m} \mathrm{cov}[Z_{1,k}\ Z_{2,l}]$$

($k \neq l$, 独立)

$$= \sum_{k=1}^{m} \mathrm{cov}[Z_{1,k}, Z_{2,k}] = m\,\mathrm{cov}[Z_{11}, Z_{21}]$$

$$= m\Big(\underbrace{\Pr(X_1 < Y_1,\ X_2 < Y_1)}_{\frac{1}{3}} - \underbrace{\Pr(X_1 < Y_1)}_{\frac{1}{2}}\underbrace{\Pr(X_2 < Y_1)}_{\frac{1}{2}} \Big)$$

$$= \frac{m}{12}$$

$$\therefore V[\hat{\pi}] = \left(\frac{1}{mn}\right)^2 \left(n \cdot \left(\frac{m}{4} + \frac{m(m-1)}{12}\right) + \frac{mn(n-1)}{12} \right)$$

$$= \frac{1}{(mn)^2}\left(\frac{mn}{4} + \frac{mn(m-1)}{12} + \frac{mn(n-1)}{12} \right)$$

$$= \frac{1}{(mn)^2}\left(\frac{mn}{12}(m+n-2) + \frac{mn}{4} \right)$$

$$= \frac{1}{(mn)^2}\left(\frac{mn}{12}(m+n+1) \right)$$

$$= \frac{1}{12mn}(m+n+1)$$

11.33.

(1) one-sided test

- $H_0: \sigma_X = \sigma_Y$ vs $H_1: \sigma_X > \sigma_Y$ 의 case...

$$C = \left\{ (X_1, \dots X_n, Y_1 \dots Y_m) \;\middle|\; \frac{S_X^2}{S_Y^2} \geq \psi^{-1}(1-\alpha) \right\}$$

ψ.. $F_{n-1,m-1}$ 의 cdf

- $H_0: \sigma_X = \sigma_Y$ vs H_1 $\sigma_X < \sigma_Y$

$$C = \left\{ (X_1, \dots X_n, Y_1, \dots Y_m) \;\middle|\; \frac{S_X^2}{S_Y^2} \leq \psi^{-1}(\alpha) \right\}$$

two-sided test

- $H_0: \sigma_X = \sigma_Y$ vs H_1 $\sigma_X \neq \sigma_Y$

$$C = \left\{ (X_1, \dots X_n, Y_1, \dots Y_m) \;\middle|\; \frac{S_X^2}{S_Y^2} \geq \psi^{-1}\left(1-\frac{\alpha}{2}\right) \text{ or } \frac{S_X^2}{S_Y^2} \leq \psi^{-1}\left(\frac{\alpha}{2}\right) \right\}$$

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(2) $\dfrac{\sigma_Y^2}{\sigma_X^2} \cdot \dfrac{S_X^2}{S_Y^2} \sim F_{n-1,\,m-1}$

$$\therefore \Pr\left(\psi^{-1}\left(\frac{\alpha}{2}\right) \le \frac{\sigma_Y^2}{\sigma_X^2} \cdot \frac{S_X^2}{S_Y^2} \le \psi^{-1}\left(1-\frac{\alpha}{2}\right) \right) = 1-\alpha$$

$$\therefore \Pr\left(\frac{S_X^2}{S_Y^2} \cdot \frac{1}{\psi^{-1}\left(1-\frac{\alpha}{2}\right)} \le \frac{\sigma_X^2}{\sigma_Y^2} \le \frac{S_X^2}{S_Y^2} \cdot \frac{1}{\psi^{-1}\left(\frac{\alpha}{2}\right)} \right)$$

$$\therefore \left[\frac{S_X^2}{S_Y^2} \cdot \frac{1}{\psi^{-1}\left(1-\frac{\alpha}{2}\right)},\ \frac{S_X^2}{S_Y^2} \cdot \frac{1}{\psi^{-1}\left(\frac{\alpha}{2}\right)} \right]$$

(3) $0.13 \sim 2.16$

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11.32

$$X-Y \sim N(\mu_X-\mu_Y,\ \sigma_X^2+\sigma_Y^2)$$

$$\Pr(X-Y<0) = \Pr\left(\frac{(X-Y)-(\mu_X-\mu_Y)}{\sqrt{\sigma_X^2+\sigma_Y^2}} < \frac{-(\mu_X-\mu_Y)}{\sqrt{\sigma_X^2+\sigma_Y^2}} \right)$$

$$= \Pr\left(Z < \frac{\mu_Y-\mu_X}{\sqrt{\sigma_X^2+\sigma_Y^2}} \right) = \Phi\left(\frac{\mu_Y-\mu_X}{\sqrt{\sigma_X^2+\sigma_Y^2}} \right)$$

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