2017. 6. 26 : updated

Homework 5. chapter 9 參考答案 森天俊成

9.4

(a) $\Lambda(x_1) = \dfrac{Pr(x_1 \mid H_0)}{Pr(x_1 \mid H_A)} = \dfrac{0.2}{0.1} = 2$

$\Lambda(x_2) = \dfrac{Pr(x_2 \mid H_0)}{Pr(x_2 \mid H_A)} = \dfrac{0.3}{0.4} = 0.75$

$\Lambda(x_3) = \dfrac{Pr(x_3 \mid H_0)}{Pr(x_3 \mid H_A)} = \dfrac{0.3}{0.1} = 3$

$\Lambda(x_4) = \dfrac{Pr(x_4 \mid H_0)}{Pr(x_4 \mid H_A)} = \dfrac{0.2}{0.4} = 0.5$

(b) $C = \{x \mid \Lambda(x) \leq k\}$ 為棄卻域 (但 k 還不確定)

if $k = 0.5$ … $C = \{x \mid \Lambda(x) \leq 0.5\} = \{x_4\}$

if $k = 0.75$ … $C = \{x \mid \Lambda(x) \leq 0.75\} = \{x_2, x_4\}$

if $k = 2$ … $C = \{x \mid \Lambda(x) \leq 2\} = \{x_1, x_2, x_4\}$

if $k = 3$ … $C = \{x \mid \Lambda(x) \leq 3\} = \{x_1, x_2, x_3, x_4\}$

• $\alpha = 0.2$ … H_0為真時 $Pr(X \in C \mid H_0) = 0.2$ → 找出這樣的k

if $k = 0.5 \Rightarrow Pr(X \in C \mid H_0) = Pr(X \in \{x_4\} \mid H_0)$

$= Pr(X = x_4 \mid H_0) = 0.2$ … ok

∴ 因此可取 $k = 0.5$ (or $0.5 \leq k < 0.75$)

$\{x \mid \Lambda(x) \le 0.5\}$

$\therefore$ $C=\{x_4\}$ 為 $\alpha=0.2$ 下的 MP 檢定
(最強)

同樣 $\alpha=0.5$ 的 case...

$k=0.75 \Rightarrow C=\{x_2, x_4\}$ (or $0.75 \le k < 2$)

$Pr(X \in \{x_2, x_4\} \mid H_0) = 0.3+0.2=0.5$

$\therefore$ $C=\{x_2, x_4\}$ 為 $\alpha=0.5$ 下的 MP 檢定.

(c) 考慮 x 使得 $Pr(H_0 \mid x) > Pr(H_1 \mid x)$

$\Leftrightarrow$ $Pr(X=x, H_0) > Pr(X=x, H_1)$ (兩邊乘上 $Pr(X=x)$)

$\Leftrightarrow$ $\dfrac{Pr(X=x, H_0)}{Pr(H_0)} > \dfrac{Pr(X=x, H_1)}{Pr(H_1)}$ ($\because$ $Pr(H_0)=Pr(H_1)$)

$\Leftrightarrow$ $Pr(X=x \mid H_0) > Pr(X=x \mid H_1)$ (Bayes)

X	H_0		H_A	
x_1	0.2	$>$	0.1	✓
x_2	0.3	$<$	0.4	
x_3	0.3	$>$	0.1	✓
x_4	0.2	$<$	0.4	

$\therefore$ x_1 和 x_3

※ 你觀測到數據 X (例如: 體重), 然後要判斷 它來自 H_0 還是 H_1.
(男性) (女性)

若 H_1 的機率較高 (女性) 則棄卻 H_0 (男性)

(d) $p \overset{def}{=} Pr(H_0) \quad (\therefore 1-p = Pr(H_1))$

$$C = \{x \mid Pr(H_1|x) \geq Pr(H_0|x)\}$$

$$= \{x \mid Pr(H_1, x) \geq Pr(H_0, x)\}$$

$$= \{x \mid Pr(x|H_1) \cdot Pr(H_1) \geq Pr(x|H_0) \cdot Pr(H_0)\}$$

$$= \{x \mid \frac{Pr(H_1)}{Pr(H_0)} \geq \Lambda(x)\}$$

$$= \{x \mid \underbrace{\frac{1-p}{p}}_{k} \geq \Lambda(x)\}$$

$$\left(\text{or } 0.5 \leq k = \frac{1-p}{p} < 0.75 \iff \frac{4}{7} < p \leq \frac{2}{3}\right)$$

根據 (b) 的結果, $\alpha = 0.2 \therefore \frac{1-p}{p} = k = 0.5$

$$\Rightarrow p = \frac{2}{3} \quad \left(Pr(H_0) = \frac{2}{3}\right)$$

$\alpha = 0.5 \therefore \frac{1-p}{p} = k = 0.75$

$$\Rightarrow p = \frac{4}{7} \quad \left(Pr(H_1) = \frac{4}{7}\right)$$

$\frac{1-p}{p}$
$\parallel$
or $0.75 \leq k < 2$

$\therefore \frac{1}{3} < p \leq \frac{4}{7}$

※

(a)(b) vs (d) ⋯ {
(a),(b) 沒有關於 $Pr(H_0)$, $Pr(H_1)$ 的資訊, 所以利用 Neyman-Pearson's lemma

(d) ⋯ 有關於 $Pr(H_0)$, $Pr(H_1)$ 的資訊 $\therefore$ 比較 $Pr(H_0|x)$, $Pr(H_1|x)$ 的大小
}

$H_0: \theta \in \Theta_0$ vs $H_1: \theta \in \Theta_1$

9.5

$$\alpha = \sup_{\theta \in \Theta_0} Pr(X \in C \mid \theta)$$

C: 棄卻域

(a) False.

顯著水準為「H_0為真時，棄卻H_0的機率（之上限）」

(b) False.

$\alpha_1 \geq \alpha_2$

$$\begin{cases} \alpha_1 = \sup_{\theta \in \Theta_0} Pr(X \in C_1 \mid \theta) \\ \alpha_2 = \sup_{\theta \in \Theta_0} Pr(X \in C_2 \mid \theta) \end{cases}$$

$C_1 \supseteq C_2$

∴ β_1 (power) $= Pr(X \in C_1 \mid \theta)$ $(\theta \in \Theta_1)$

β_2 (power) $= Pr(X \in C_2 \mid \theta)$ (〃)

$\beta_1 \geq \beta_2$

(c) False

- 顯著水準的定義 → (a)
- 這跟 $Pr(H_0$ true$)$ 無關，

而且我們通常不知道 $Pr(H_0$: true$)$ 的機率

(d) False

"power of test" → "the probability of type I error."

(e) False

少了" when H_0 actually is true.

(f) False

type I error 通常比較嚴重

$\left\{\begin{array}{l}\text{type I error: 冤枉無辜的人} \\ \text{type II error: 放過兇手}\end{array}\right.$

(g) False

$$檢定力 = Pr(\underbrace{X}_{\text{檢定統計量}} \in C \mid H_1)$$

∴ 跟 H_1 下的分佈有關

(h) True.

9.7.

① $$\Lambda = \frac{\Pr(X_1=x_1, X_2=x_2, \ldots, X_n=x_n \mid H_0)}{\Pr(X_1=x_1, X_2=x_2, \ldots, X_n=x_n \mid H_A)}$$

$$= \frac{e^{-\lambda_0}\cdot\frac{\lambda_0^{x_1}}{x_1!}\cdots e^{-\lambda_0}\cdot\frac{\lambda_0^{x_n}}{x_n!}}{\left(e^{-\lambda_1}\cdot\frac{\lambda_1^{x_1}}{x_1!}\right)\cdots\left(e^{-\lambda_1}\cdot\frac{\lambda_1^{x_n}}{x_n!}\right)}$$

$$= e^{-n(\lambda_0-\lambda_1)}\cdot\left(\frac{\lambda_0}{\lambda_1}\right)^{(X_1+X_2+\cdots+X_n)}$$

$\frac{\lambda_0}{\lambda_1}<1$ $(\because 0<\lambda_0<\lambda_1)$

∴ 我們應該注意 $X_1+X_2+\cdots+X_n$ 增加 $\Leftrightarrow$ Λ: 減少
(∵ λ之充份統計量)

② $\{(X_1, X_2, \ldots X_n) \mid \Lambda \leq k\}$

$= \{(X_1, X_2, \ldots X_n) \mid X_1+X_2+\cdots X_n \geq m\}$

$\therefore C = \{(X_1, X_2, \ldots X_n) \mid X_1+X_2+\cdots+X_n \geq m\}$

③ 決定 m…

$X_1+X_2+\dots X_n \sim Po(n\lambda_0)$ （H_0為真時）

$\alpha = Pr(X_1+X_2+\dots+X_n \geq m \mid H_0)$

但 X_i 為離散性的隨機變數.

所以不一定剛好存在 m 使得 $Pr(X_1+X_2+\dots X_n \geq m \mid H_0) = \alpha$.

case 1… 若存在 m 使得 $Pr(X_1+X_2+\dots+X_n \geq m \mid H_0) = \alpha$

$$\Rightarrow \varphi(X) = \begin{cases} 1 \dots & X_1+X_2+\dots X_n \geq m \\ 0 \dots & \text{else} \end{cases}$$

（1… 代表棄卻
0… 代表不棄卻）

case 2… 若不存在 m 使得 $Pr(X_1+X_2+\dots X_n \geq m \mid H_0) = \alpha$

先找出 m 使得 $Pr(X_1+X_2+\dots X_n \geq m-1) > \alpha > Pr(X_1+\dots X_n \geq m)$

$$\varphi(X) = \begin{cases} 1 \dots & X_1+X_2+\dots X_n \geq m \\ r & X_1+X_2+\dots X_n = m-1 \\ 0 \dots & X_1+X_2+\dots X_n \leq m-2 \end{cases}$$ （隨機化）

丟銅板決定是否該棄卻 H_0.

$* \; r = \alpha - Pr(X_1+X_2+\dots X_n \geq m \mid H_0)$

↳ 銅板出現正面的機率

9.8.

根據 Neyman-Pearson's lemma, 9.7 所求的 $\varphi(X)$ 為
顯著水準 α 下, 對於 $H_0: \lambda=\lambda_0$ vs $H_1: \lambda=\lambda_1$
之最強力檢定 $(\lambda_1 > \lambda_0)$

$\Rightarrow$ $E[\varphi(X) \mid \lambda=\lambda_0]=\alpha$

$\beta(\lambda_1) \overset{def}{=} E[\varphi(X) \mid \lambda=\lambda_1]$ (檢定力)

令 $\varphi^*(X)$ 為任意檢定, 滿足 $E[\varphi^*(X) \mid \lambda=\lambda_0]=\alpha$ $(\leq \alpha)$.

$\beta^*(\lambda_1) \overset{def}{=} E[\varphi^*(X) \mid \lambda=\lambda_1]$. (檢定力)

對於任意 $\varphi^*(X)$, $\beta(\lambda_1) \geq \beta^*(\lambda_1)$ (最強力檢定)

- 我們現在考慮 $H_0: \lambda=\lambda_0$, vs $H_1: \lambda=\lambda_2$ $(\lambda_2 > \lambda_0)$
  (λ_1 變成 λ_2 而已)
  同樣道理, $\varphi(X)$ 為最強力檢定.
- 接著考慮 $H_0: \lambda=\lambda_0$ vs $H_1: \lambda \in \{\lambda_1, \lambda_2\}$ $(\lambda_1, \lambda_2 > \lambda_0)$
  $\varphi(X)$ 為顯著水準 α 下的均勻最強力檢定.
  ($\because$ $\varphi(X)$ 依然滿足 $E[\varphi(X) \mid \lambda_0]=\alpha$, $\beta(\lambda_1) \geq \beta^*(\lambda_1)$
  $\beta(\lambda_2) \geq \beta^*(\lambda_2)$)
- 最後考慮 $H_0: \lambda=\lambda_0$ vs $H_1: \lambda > \lambda_0$
  同樣道理, $\varphi(X)$ 為均勻最強力檢定.
  ($\because$ $E[\varphi(X) \mid \lambda_0]=\alpha$, $\forall \lambda > \lambda_0$ $\beta(\lambda) \geq \beta^*(\lambda)$)

9.17.

(a) 樣本只有一個

$$\Lambda(x) = \frac{f(x \mid H_0)}{f(x \mid H_A)} = \frac{\frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{1}{2\sigma_0^2} x^2\right)}{\frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{1}{2\sigma_1^2} x^2\right)}$$

$$= \left(\frac{\sigma_1}{\sigma_0}\right) \exp\left(\underbrace{\left(\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2}\right)}_{\text{負}} x^2\right)$$

$\therefore$ x^2 增加 $\Leftrightarrow$ Λ 減少

$\therefore$ $C = \{x \mid x^2 \geq c\}$

$$\Pr(x \in C \mid H_0) = \alpha$$

$$= \Pr\left(\left(\frac{X}{\sigma_0}\right)^2 \geq \frac{c}{\sigma_0^2} \,\Big|\, X \sim N(0, \sigma_0^2)\right) \qquad \left(\frac{X}{\sigma_0}\right)^2 \sim \chi_1^2$$

$$= 1 - \psi\left(\frac{c}{\sigma_0^2}\right) = \alpha \qquad \psi : \chi_1^2 \text{ 之 cdf.}$$

$\therefore$ $c = \sigma_0^2 \cdot \psi^{-1}(1-\alpha)$

$\therefore$ $C = \{x \mid x^2 \geq \sigma_0^2 \cdot \psi^{-1}(1-\alpha)\}$

ψ: χ_1^2 之 cdf

(b) 同樣道理, $\mathbf{C}=\{(x_1,x_2,\dots x_n) \mid X_1^2+X_2^2+\dots X_n^2 \geq c\}$

$Pr((X_1 \dots X_n) \in \mathbf{C} \mid H_0) = \alpha$

$Pr\left(\frac{X_1^2+X_2^2+\dots+X_n^2}{\sigma_0^2} \geq \frac{c}{\sigma_0^2} \mid H_0\right) = \alpha \quad \left(\frac{X_1^2+\dots+X_n^2}{\sigma_0^2} \sim \chi_n^2\right)$

$c = \sigma_0^2 \cdot \psi_n^{-1}(1-\alpha) \quad (\psi_n = \chi_n^2 \text{ 之 cdf})$

$\therefore \mathbf{C}=\{(X_1,X_2,\dots,X_n) \mid X_1^2+X_2^2+\dots+X_n^2 \geq \sigma_0^2 \cdot \psi_n^{-1}(1-\alpha)\}$

(c) YES.

(跟 9.8 一樣道理)

9.18.

$$\Lambda(x) = \frac{f(x_1, x_2, \dots x_n \mid H_0)}{f(x_1, x_2, \dots x_n \mid H_1)}$$

$$= \frac{\frac{1}{2^n} \cdot \lambda_0^n \exp\left(-\lambda_0 \left(|x_1| + |x_2| + \dots + |x_n|\right)\right)}{\frac{1}{2^n} \lambda_1^n \exp\left(-\lambda_1 \left(|x_1| + |x_2| + \dots |x_n|\right)\right)}$$

$$= \left(\frac{\lambda_0}{\lambda_1}\right)^n \cdot \exp\left(\underbrace{(\lambda_1 - \lambda_0)}_{>0} \underbrace{\{|x_1| + \dots |x_n|\}}_{\text{檢定統計量}}\right)$$

$|x_1| + |x_2| + \dots + |x_n|$ 增加 $\Leftrightarrow$ Λ 增加

$$\Lambda(x) \le k \Leftrightarrow |x_1| + |x_2| + \dots |x_n| \le c$$

$$T \overset{\text{def}}{=} |X_1| + |X_2| + \dots |X_n|$$

$|X_j| \sim \exp(\lambda_0)$ (under H_0) $(j = 1 \sim n)$

$\therefore T \sim \Gamma(n, \lambda_0)$

注 Gamma 函數與 χ^2 的關係

$$2\lambda_0 T \sim \Gamma\left(n, \frac{1}{2}\right) = \chi^2_{2n}$$

$$\Pr(T \le c \mid H_0) = \Pr(2\lambda_0 T \le 2\lambda_0 c \mid H_0) = \Psi_{2n}(2\lambda_0 c) = \alpha$$

$$\therefore \alpha = \Psi_{2n}^{-1}(\alpha) \cdot \frac{1}{2\lambda_0}$$

$\therefore C = \frac{1}{2\lambda_0} \Psi_{2n}^{-1}(\alpha)$

$\therefore C = \left\{ (x_1, x_2, \ldots x_n) \;\middle|\; |x_1| + |x_2| + \cdots |x_n| \leq \frac{\Psi_{2n}^{-1}(\alpha)}{2\lambda_0} \right\}$

註 Ψ_{2n} 為 χ^2_{2n} 之cdf

$\therefore \varphi(X) = \begin{cases} 1 & \text{if } (x_1, \ldots x_n) \in C \\ 0 & \text{if } \quad " \quad \notin C \end{cases}$

根據 Neyman-Pearson's lemma, $\varphi(X)$ 為 對 $H_0: \lambda = \lambda_0$ $H_1: \lambda = \lambda_1$ $(\lambda_1 > \lambda_0)$ 之 最強力 檢定 (顯著水準 $=\alpha$)

跟 9.8 一樣道理, $\varphi(X)$ 為 對 $H_0: \lambda = \lambda_0$ vs $H_1: \lambda > \lambda_0$ 之 均勻最強力 檢定.

9.19.

(a) $Pr(H_0) = Pr(H_1) = \frac{1}{2}$

考慮 $Pr(H_0|X) \geq Pr(H_1|X)$ 兩邊乘上 $f_X(x)$

$\Leftrightarrow f(x, H_0) \geq f(x, H_1)$

$\Leftrightarrow \dfrac{f(x,H_0)}{Pr(H_0)} \geq \dfrac{f(x,H_1)}{Pr(H_1)}$ $\quad Pr(H_0) = Pr(H_1)$

$\Leftrightarrow f_0(x) \geq f_1(x) \quad (f_0 = \frac{dF_0}{dx}, \ f_1 = \frac{dF_1}{dx})$

$\Leftrightarrow 2x \geq 3x^2$

$\Leftrightarrow x \leq \frac{2}{3}$

$\therefore [0, \frac{2}{3}]$

(b)
(c) $\Lambda(x) = \dfrac{f(x|H_0)}{f(x|H_1)} = \dfrac{f_0(x)}{f_1(x)} = \dfrac{2x}{3x^2} = \dfrac{2}{3x} \leq k$

$\Leftrightarrow x \geq \frac{2}{3k}$

$\Leftrightarrow x \geq k' \quad (k' \stackrel{def}{=} \frac{2}{3k})$

$\alpha = Pr(x \geq k' \mid H_0) = \int_{k'}^{1} f_0(x)dx = [x^2]_{k'}^{1} = \alpha$

$\therefore 1 - k'^2 = \alpha \quad \therefore k' = \sqrt{1-\alpha}$

$\therefore \; C = \{ x \mid x \geq \sqrt{1-\alpha} \}$

検定力

(d) power $= Pr(X \in C \mid H_1)$

$= \int_{\sqrt{1-\alpha}}^{1} f(x)\,dx = \int_{\sqrt{1-\alpha}}^{1} 3x^2\,dx$

$= [x^3]_{\sqrt{1-\alpha}}^{1} = \underline{1-(1-\alpha)^{\frac{3}{2}}}$

9.21 $X \sim U(0,\theta)$

$H_0: \theta=1$ vs $H_1: \theta=2$

(a) $\alpha=0$ 表示 H_0 為真時，也就是說 $X\sim U(0,1)$

時，$Pr(X\in C \mid H_0)=0$。

∴ 取 $C=\{X \mid X>1\}$

(∵ H_0 時，X 絕對不會超過 1)

$Power=Pr(X\in C \mid H_1) = Pr(X>1 \mid \theta=2)$

$=\int_1^2 \frac{1}{2}dx = \left[\frac{X}{2}\right]_1^2 = \frac{1}{2}$

∴ $Power=\frac{1}{2}$

(b) $C=\{X \mid 0\leq X\leq \alpha\}$

顯著水準 $=Pr(X\in C \mid H_0)=\int_0^\alpha 1\,dx = [x]_0^\alpha=\alpha$

$Power= Pr(X\in C \mid H_1) = \int_0^\alpha \frac{1}{2}dx = \frac{\alpha}{2}$

(c) $C = \{X \mid 1-\alpha \leqq X \leqq 1\}$

顕著水準 $= \int_{1-\alpha}^{1} 1\, dx = [x]_{1-\alpha}^{1} = \alpha$

Power $= \int_{1-\alpha}^{1} \frac{1}{2} dx = \left[\frac{x}{2}\right]_{1-\alpha}^{1} = \frac{\alpha}{2}$

(d) $C = \{X \mid k-\alpha \leqq X \leqq k\}$

$(\forall k \in [\alpha, 1])$

(e) $\Lambda(X) = \dfrac{f(X \mid H_0)}{f(X \mid H_1)} = \dfrac{I_{[0,1]}(X)}{\frac{1}{2} I_{[0,2]}(X)} = \dfrac{2 I_{[0,1]}(X)}{I_{[0,2]}(X)}$

$= \begin{cases} 2 \cdots & 0 \leqq X \leqq 1 \\ 0 \cdots & 1 < X \leqq 2 \end{cases}$

X: 増加 → Λ: 減少

$\therefore C = \{X \mid X \geqq c\}$ (棄却域)

$\varphi(X) = \begin{cases} 1 \cdots & c \leqq X \leqq 2 \\ 0 \cdots & \text{else} \end{cases}$

$E[\varphi(X) \mid H_0] = \int_c^2 I_{[0,1]}(x)\, dx = \int_c^1 dx = 1 - c = \alpha$

$\therefore c = 1 - \alpha$

$\therefore \varphi(X) = \begin{cases} 1 & \text{if: } 1-\alpha \leq X \leq 2 \\ 0 & \text{otherwise} \end{cases}$

為最強力檢定. ($\because$ Neyman-Pearson's lemma)

但 $\varphi^* = \begin{cases} 1 & \text{if } [0,\alpha] \cup [1,2] \\ 0 & \text{else} \end{cases}$

亦為顯著水準 α 下的最強力檢定.

(其 power $= \int_{[0,\alpha]\cup[1,2]} \frac{1}{2} dx = \frac{1+\alpha}{2}$.) $\therefore$ not unique

(f) $\Lambda(X) = \dfrac{I_{[0,2]}(X)}{2 I_{[0,1]}(X)}$ $\quad X: \begin{cases} 0 \leq X \leq 1 & : \frac{1}{2} \\ 1 < X \leq 2 & : \infty \end{cases}$

$\because X$: 增加 $\therefore \Lambda$: 增加

$C = \{X \mid 0 \leq X \leq c\}$

$\int_0^c \frac{1}{2} dx = \alpha \quad \therefore c = 2\alpha$

$\varphi(X) = \begin{cases} 1 & x \in [0, 2\alpha] \\ 0 & x \notin [0, 2\alpha] \end{cases}$ 為最強力檢定

但 $\varphi^*(x) = \begin{cases} 1 & x \in [1-2\alpha, 1] \\ 0 & x \notin [1-2\alpha, 1] \end{cases}$ 亦為最強力檢定 $(\alpha < 0.5)$

$\therefore$ not unique

9.23.

「$-3 \notin (-2,3) \to$ 棄卻 H_0」

• 假設 σ^2 已知… μ的信賴區間 (99%)

$$\left(\bar{X}-\frac{2.57}{\sqrt{n}}\sigma,\ \bar{X}+\frac{2.57}{\sqrt{n}}\sigma\right)$$

$$=(-2,3)=(0.5-2.5,\ 0.5+2.5)$$

$\because \bar{X}=0.5$; $\frac{2.57}{\sqrt{n}}\sigma = 2.5$

• 對於 $H_0: \mu=-3$ vs $H_1: \mu\neq-3$ 的檢定:

$$\frac{\sqrt{n}(\bar{X}+3)}{\sigma} \in [-2.57,\ 2.57]^c \Rightarrow \text{棄卻}$$

$$\|$$

$$(0.5+3)\cdot\frac{2.57}{2.5} \in [-2.57,\ 2.57]^c$$

$\therefore$ 棄卻 H_0

9.26.

(a) YES. $\Lambda(x) = \dfrac{\sup_{\theta\in\Theta_0} L(\theta|x)}{\sup_{\theta\in\Theta_0\cup\Theta_1} L(\theta|x)}$

↓

分母 θ 的範圍包含分子 θ 的範圍

∴ 分母 ≥ 分子

(b) NO. 先決定顯著水準 α

⇒ 如果 p 值小於 α，就棄卻 H_0

(c) NO. 你觀測到的 p 值跟顯著水準沒有關係

or 、p 值 $\overset{def}{=} \inf_{\alpha}\{x\in C_\alpha\}$

YES. 如果這句英文的意思是：

「你已經得到一筆資料，然後

在顯著水準 0.06 下，棄卻了虛無假設，

p 值是否不超過 0.06。」

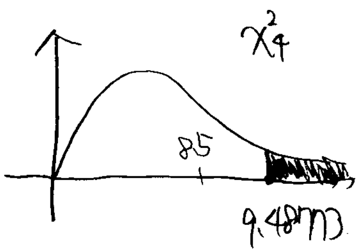
⇒ YES.

(d) NO

(e) NO

(f) NO. (p值大於0.05)

$T \sim \chi_4^2$ $\quad Pr(T \geq 9.48773 \mid \chi_4^2) = 0.05$



9.34. 適合度檢定

(流程) $i=1\sim k$ (i: category)

- O_i … 實際上觀測到的量
- E_i … 理論上會觀測到的量 (H_0下)

檢定統計量 $T = \sum_{i=1}^{k} \frac{(E_i - O_i)^2}{E_i}$

T的分布… χ^2_{k-1} (H_0下)

(如果計算 E_i 時，你估計 r 個參數的話，T的分布… χ^2_{k-r-1})

$T \geq t_0 \Rightarrow$ 棄卻

接下來，我們按照上述流程求T的值。

	Type		Count	
$i=1$	Starchy	green	1997	$\rightarrow O_1$
$i=2$	Starchy	white	906	$\rightarrow O_2$
$i=3$	Sugary	green	904	$\rightarrow O_3$
$i=4$	Sugary	white	32	$\rightarrow O_4$

$\therefore k=4$ (一共有4個 category)

· 根據 8.55，$\hat{\theta}_{MLE} = 0.0357$。
↓
我們估計了一個參數 ∴ $r=1$

· 求 $E_1 \sim E_4$.

※ 我們不知道真實的 θ，所以以 $\hat{\theta}$ 來代替。

E_1 … 理論上出現的 "Starchy green" 的數量

$$= \underbrace{(1997+906+904+32)}_{=3839} \times \frac{1}{4}(2+\underbrace{\hat{\theta}}_{0.0357})$$

（= 整體個數 × 理論上的比例）

E_2, E_3, E_4 … 同樣方法

· $T = \sum_{i=1}^{4} \frac{(E_i - O_i)^2}{E_i} = \frac{(E_1-O_1)^2}{E_1} + \frac{(E_2-O_2)^2}{E_2} + \frac{(E_3-O_3)^2}{E_3} + \frac{(E_4-O_4)^2}{E_4}$

$T \sim \chi^2_{4-1-1} = \chi^2_2$ （$h=4, r=1$）

$T = 2.0155$ $\chi_2^{2^{-1}}(0.95) = 5.9915$

· $T < 5.9915$ … 不棄卻 H_0

9.41 $n_1, n_2, \dots n_m$ 應為已知參數

$$\Lambda(X) = \frac{\sup_{p_1=p_2=\dots=p_m} L(p_1, p_2, \dots p_m)}{\sup_{p_1, p_2, \dots, p_m} L(p_1, p_2, \dots p_m)}$$

$$L(p_1, p_2, \dots p_m) = \prod_{i=1}^{m} \Pr(X_i = x_i \mid n_i, p_i)$$

$$= \prod_{i=1}^{m} {}_{n_i}C_{x_i}\, p_i^{x_i} (1-p_i)^{n_i - x_i}$$

① $\sup_{p_1=p_2=\dots=p_m} L(p_1, p_2, \dots, p_m)$ ...

$$p \overset{\text{def}}{=} p_1 = p_2 = \dots = p_m$$

$$L(p) = \prod_{i=1}^{m} {}_{n_i}C_{x_i}\, p^{x_i} (1-p)^{n_i - x_i}$$

$$\ell(p) = \ln L(p) = \sum_{i=1}^{m} \left\{ \ln({}_{n_i}C_{x_i}) + x_i \ln(p) + (n_i - x_i)\ln(1-p) \right\}$$

$$\ell'(p) = \frac{1}{p}(x_1 + x_2 + \dots + x_m) - \frac{1}{1-p}(n_1 + n_2 + \dots + n_m - x_1 - x_2 \dots - x_m) = 0$$

$$\Leftrightarrow \quad (1-p)(x_1 + \dots x_m) - p(n_1 + n_2 + \dots + n_m - x_1 - x_2 \dots - x_m) = 0$$

$$\Leftrightarrow \quad x_1 + x_2 + \dots x_m = p(n_1 + n_2 + \dots n_m)$$

$$\therefore \hat{p}_{MLE} = \frac{x_1 + x_2 + \dots x_m}{n_1 + n_2 + \dots n_m}$$

$\therefore$ $p_1 = p_2 = \cdots = p_m$ 時

$$p_1 = p_2 = \cdots = p_m = \frac{\sum_{i=1}^{m} x_i}{\sum_{i=1}^{m} n_i} \text{ 為 } p \text{ 之 MLE}$$

$$\therefore \hat{p_0} \stackrel{def}{=} \frac{\sum_{i=1}^{m} x_i}{\sum_{i=1}^{m} n_i}$$

$$\therefore \sup_{p_1 = p_2 = \cdots = p_m} L(p_1, \cdots p_m) = L(\hat{p_0}, \hat{p_0}, \cdots, \hat{p_0})$$

② $\sup_{p_1, p_2, \cdots p_m} L(p_1, p_2, \cdots p_m) \cdots$

$$L(p_1, p_2, \cdots p_m) = \prod_{i=1}^{m} {}_{n_i}C_{x_i}\, p_i^{x_i} (1-p_i)^{n_i - x_i}$$

$$\begin{aligned} l(p_1, p_2, \cdots p_m) &= \ln L(p_1, \cdots p_m) \\ &= \sum_{i=1}^{m} \ln({}_{n_i}C_{x_i}) + x_i \ln p_i + (n_i - x_i)\ln(1-p_i) \end{aligned}$$

$$\frac{\partial l}{\partial p_i} = \frac{x_i}{p_i} - \frac{n_i - x_i}{1-p_i} = 0$$

$$\Leftrightarrow x_i(1-p_i) - p_i(n_i - x_i) = 0$$

$$\Leftrightarrow p_i = \frac{x_i}{n_i}$$

$\therefore \hat{p}_i = \frac{x_i}{n_i} \quad (i=1\sim m)$

$\therefore \sup\limits_{p_1,p_2,\cdots p_m} L(p_1,p_2,\cdots p_m) = L(\hat{p}_1,\hat{p}_2,\cdots \hat{p}_m)$

③ $\Lambda(X) = \dfrac{L(\hat{p}_0,\hat{p}_0,\cdots \hat{p}_0)}{L(\hat{p}_1,\hat{p}_2,\cdots \hat{p}_m)} \leq k$

$\Rightarrow$ 棄却

另外，$H_0: p_1=p_2=\cdots=p_m$ 時，参數只有1個

H_1 時、参數有 m 個

$\therefore$ 差 $= m-1$ 個

$\therefore -2\ln\Lambda(X) \xrightarrow{d} \chi^2_{m-1}$

（大樣本時）