105.2

作業2　統計学

参考答案　森元

5.1

$$\begin{cases} X_1, X_2, \dots, X_n：独立 \quad E[X_j]=\mu \quad (j=1,\dots,n) \\ V[X_j]=\sigma_j^2<\infty \end{cases}$$

$$E[\bar{X}] = E\left[\frac{X_1+X_2+\dots+X_n}{n}\right] = \sum_{j=1}^{n}\frac{1}{n}E(X_j) = \underline{\mu}$$

$$V[\bar{X}] = V\left[\frac{X_1+X_2+\dots+X_n}{n}\right] = \sum_{j=1}^{n}\frac{1}{n^2}V(X_j)$$

$$= \frac{1}{n^2}\sum_{j=1}^{n}\sigma_j^2$$

利用柴比雪夫不等式

$$Pr\left(|\bar{X}-\underbrace{E[\bar{X}]}_{=\mu}| \geq \underbrace{k\cdot\sqrt{V[\bar{X}]}}_{=\varepsilon\ (>0)}\right) \leq \frac{1}{k^2}$$

$$\therefore k = \frac{\varepsilon}{\sqrt{V[\bar{X}]}}$$

$$k^2 = \frac{\varepsilon^2}{V[\bar{X}]}$$

$$\frac{1}{k^2} = \frac{V[\bar{X}]}{\varepsilon^2}$$

$$\therefore Pr\left(|\bar{X}-E[\bar{X}]| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2}V[\bar{X}]$$

$$= \frac{1}{\varepsilon^2}\frac{1}{n^2}\sum_{j=1}^{n}\sigma_j^2$$

$\therefore n\to\infty$ 時，$\lim_{n} Pr\left(|\bar{X}-E[\bar{X}]| \geq \varepsilon\right) = \lim_{n\to\infty}\frac{1}{\varepsilon^2}\frac{1}{n^2}\sum_{j=1}^{n}\sigma_j^2$

$= 0$ (for all $\varepsilon > 0$) $\quad \therefore \bar{X} \xrightarrow{P} E[\bar{X}] = \mu$

5.3 考慮 $I_j \overset{iid}{\sim} Po(1)$ $(j=1\sim n)$ $(n=10000)$

$N = I_1 + I_2 + \cdots I_n \sim Po(n)$ （期望值 n，變異 n）

中央極限定理 可以當成常態分佈

$\left(I_1 + I_2 + \cdots I_n \sim N(n, n) \right)$

$$\frac{I_1 + I_2 + \cdots I_n - n}{\sqrt{n}} \sim N(0,1)$$

$$Z \overset{def}{=} \frac{N - 10000}{100} \sim N(0,1)$$

$$Pr(N \geq 10200) = Pr(Z \geq 2)$$

$$= 1 - \Phi(2)$$

5.5

$q = 1-p$

$X \sim Bin(n, p)$

$$E[e^{tX}] = \sum_{x=0}^{n} e^{tx} \cdot p^{x} (1-p)^{n-x} \cdot {}_nC_x$$

$$= \sum_{x=0}^{n} (pe^{t})^{x} (1-p)^{n-x} \cdot {}_nC_x$$

$$= (pe^{t}+q)^{n} \cdots X \text{之 Moment Generating Function} \underset{M(t)}{\shortparallel}$$

根據題意 $p = \frac{\lambda}{n}$

$$M(t) = (pe^{t}+q)^{n} = (pe^{t}-p+1)^{n} \quad (q = 1-p)$$

$$= \Big(\underbrace{\frac{\lambda}{n}}_{p}(e^{t}-1)+1\Big)^{n}$$

$$= \left\{\left(1+\frac{\lambda}{n}(e^{t}-1)\right)^{\frac{n}{\lambda(e^{t}-1)}}\right\}^{\lambda(e^{t}-1)}$$

$$n \to \infty \quad M(t) \to e^{\lambda(e^{t}-1)}$$

由此可知， $X \sim Bin(n,p)$ $(np = \lambda)$

$$n \to \infty \quad X \xrightarrow{d} Po(\lambda)$$

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5.16

$f(x) = 2x \quad (Be(2,1))$
$(0 \leq x \leq 1)$

① $X_j \ (j=1\sim20)$ 之 期望值 與 變異數

$\mu = E[X_j] = \int_0^1 x \cdot 2x \, dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}$

$E[X_j^2] = \int_0^1 x^2 \cdot 2x \, dx = \left[\frac{x^4}{2}\right]_0^1 = \frac{1}{2}$

$\sigma^2 = V[X_j] = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{9-8}{18} = \frac{1}{18}$

$\therefore Be(2,1) \cdots \mu = \frac{2}{3}$ (期望)

$\sigma^2 = \frac{1}{18}$ (變異)

② 根據中央極限定理 ($\sigma^2 < \infty \rightarrow$ 可以利用中央極限定理)

$$\sqrt{n}\left(\frac{\frac{X_1+X_2+\cdots X_n}{n} - \mu}{\sigma}\right) \xrightarrow{d} N(0,1)$$

↑分佈收斂

$(n=20 \cdots$ 很大)

$\therefore$ 我們將 $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}$ 當 $N(0,1)$ 計算.

標準常態分佈

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③ 計算 $\Pr(S \leq 10)$ $(n=20)$

除以n 減μ

$$\Pr(X_1 + \cdots X_n \leq 10) = \Pr\left(\overline{X} \leq \frac{10}{n}\right)$$

除以σ

$$= \Pr\left(\overline{X} - \mu \leq \frac{10}{n} - \mu\right) = \Pr\left(\frac{\overline{X}-\mu}{\sigma} \leq \frac{\frac{10}{n}-\mu}{\sigma}\right)$$

乘上$\sqrt{n}$

$$= \Pr\left(\sqrt{n}\left(\frac{\overline{X}-\mu}{\sigma}\right) \leq \sqrt{n}\left(\frac{\frac{10}{n}-\mu}{\sigma}\right)\right)$$

$$Z \overset{def}{:=} \sqrt{n}\left(\frac{\overline{X}-\mu}{\sigma}\right)$$

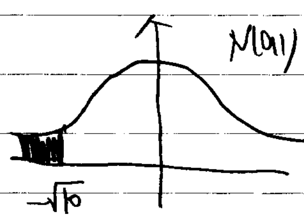
$$= \Pr\left(Z \leq \sqrt{n}\left(\frac{\frac{10}{n}-\mu}{\sigma}\right)\right) \qquad Z \sim N(0,1)$$

$$= \Phi\left(\sqrt{n}\cdot\left(\frac{\frac{10}{n}-\mu}{\sigma}\right)\right)$$

$$n=20 \quad \mu = \frac{2}{3} \quad \sigma^2 = \frac{1}{18}$$

$$= \Phi\left(\sqrt{20}\left(\frac{\frac{1}{2}-\frac{2}{3}}{\left(\frac{1}{3\sqrt{2}}\right)}\right)\right)$$

$$= \Phi\left(6\sqrt{10}\left(-\frac{1}{6}\right)\right) = \Phi(-\sqrt{10})$$



≈ 0.0008

5.17 方法① 利用中央極限定理

$\sigma^2=25<\infty \Rightarrow$ 可以用中央極限定理

n很大時 $\dfrac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \xrightarrow{d} N(0,1)$

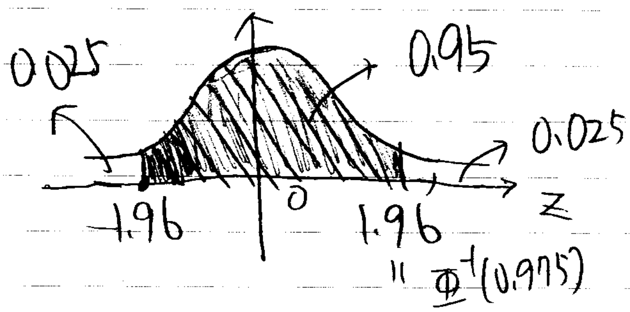
$\therefore \Pr(|\bar{X}-\mu|\leq 1)\geq 0.95$

$\quad\| $

$\Pr\left(\dfrac{\sqrt{n}|\bar{X}-\mu|}{\sigma}\leq \dfrac{\sqrt{n}}{\sigma}\right)\geq 0.95$

$\quad\|$

$\Pr\left(|Z|\leq \dfrac{\sqrt{n}}{5}\right)\geq 0.95 \qquad (Z\sim N(0,1))$



$\therefore \dfrac{\sqrt{n}}{5}=1.96 \qquad n=25\cdot 1.96^2\approx 100$

方法② 柴比雪夫不等式

$\sqrt{V[\bar{X}]}=\sqrt{\dfrac{\sigma^2}{n}}=\sqrt{\dfrac{25}{n}}=\dfrac{5}{\sqrt{n}}$, $E[\bar{X}]=\mu$

$\Pr\left(|\bar{X}-\mu|\geq \underbrace{k\cdot\dfrac{5}{\sqrt{n}}}_{=1}\right)\leq \dfrac{1}{k^2}=0.05=\dfrac{1}{20}$

$\therefore k=\sqrt{20}$ $\therefore$

$\therefore \dfrac{\sqrt{20}\cdot 5}{\sqrt{n}}=1 \qquad \therefore \dfrac{500}{n}=1 \qquad \therefore n=500.$

CHA SHIN

5.17 Note

(為什麼方法①(中央極限定理)
得到的 n 比方法②更小呢?)

① 中央極限定理 假設 X 服從
常態分布

② 柴比雪夫不等式 只要存在 σ^2 $(<\infty)$
即可使用。(任何分布皆可使用)

⇓

① 的方法,你已經知道它的分佈,
(關於 X 的分布,你知道的資訊比②更多)
(也可以說你已經給了很嚴格的假設)

所以①的方法可以提供更精準的預測。

ch6.

6.5 $T \sim \chi^2_n$, $U \sim \chi^2_m$, T, U：獨立

$$X = \frac{\left(\frac{T}{n}\right)}{\left(\frac{U}{m}\right)} \sim F_{n,m}$$ （∵ 自由度 n, m 之 F 分佈的定義）

⇓

$$\frac{1}{X} = \frac{\left(\frac{U}{m}\right)}{\left(\frac{T}{n}\right)} \sim F_{m,n}$$ （ 〃 m, n 〃 ）

6.6 考慮 T 分佈的定義

$$\begin{cases} Z \sim N(0,1), \ T \sim \chi^2_n \\ Z, T \text{：獨立} \end{cases}$$

$$\Rightarrow \quad X = \frac{Z}{\sqrt{\left(\frac{T}{n}\right)}} \sim t_n$$

$$X^2 = \frac{Z^2}{\left(\frac{T}{n}\right)} = \frac{\left(\frac{Z^2}{1}\right)}{\left(\frac{T}{n}\right)}$$ 且 $Z^2 \sim \chi^2(1)$

∴ 根據 F 分佈的定義 $X^2 \sim F_{1,n}$

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6.7. t分佈的機率密度函數(自由度n)

$$f(x|n)=\frac{1}{\sqrt{n}\,Be\left(\frac{1}{2},\frac{n}{2}\right)}\left(1+\frac{x^2}{n}\right)^{-\left(\frac{1}{2}+\frac{n}{2}\right)}$$

$(x\in\mathbb{R})$

$$f(x|n=1)=\frac{1}{Be\left(\frac{1}{2},\frac{1}{2}\right)}\left(1+x^2\right)^{-1}$$

$$Be\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}=\frac{\sqrt{\pi}\,\sqrt{\pi}}{\Gamma(1)}=\pi$$

$$\therefore f(x|n=1)=\frac{1}{\pi(1+x^2)} \quad (x\in\mathbb{R})$$

∴跟 Cauchy 分佈相同 (for all x)

6.8. $X\sim \exp(1)$ $\quad Y\sim \exp(1)$

註 $\exp(\lambda)$.. pdf: $\lambda\exp(-\lambda x)$ $(x\geq 0)$

$$\Rightarrow 2X\sim \exp\left(\frac{1}{2}\right),\quad 2Y\sim \exp\left(\frac{1}{2}\right)$$

$$\exp\left(\frac{1}{2}\right)=\Gamma\left(1,\frac{1}{2}\right)=\chi^2_2$$

$$\therefore \frac{Y}{X}=\frac{(2Y)/2}{(2Y)/2}\sim F_{2,2}$$

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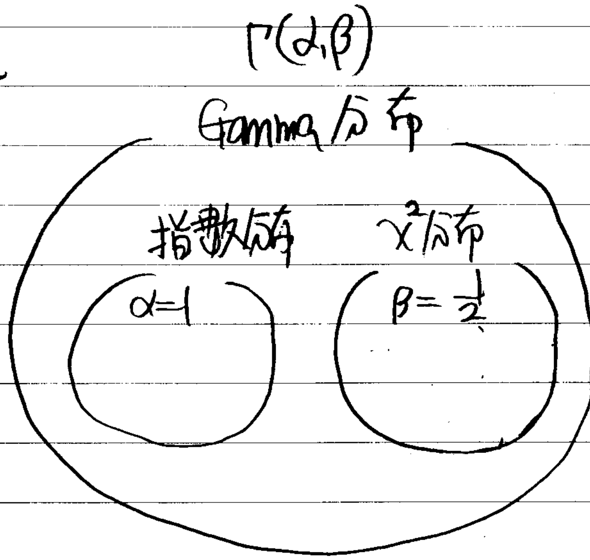

$\boxed{\text{重要性質}}$

① $X \sim \exp(\lambda)$ 時… pdf $\lambda \exp(-\lambda x)$ $(x \geq 0)$

$\Rightarrow \alpha X \sim \exp\left(\frac{\lambda}{\alpha}\right)$
$(\alpha > 0)$

証) Moment Generating Function

$$E[e^{tX}] = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$E[e^{t(\alpha X)}] = E[e^{(\alpha t)X}] = \left(1 - \frac{\alpha t}{\lambda}\right)^{-1}$$

$$= \left(1 - \frac{t}{\left(\frac{\lambda}{\alpha}\right)}\right)^{-1} \quad \therefore \alpha X \sim \exp\left(\frac{\lambda}{\alpha}\right)$$

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② $X_1, X_2 \cdots X_n \overset{iid}{\sim} \exp(\lambda)$

$\Rightarrow X_1 + X_2 + \cdots + X_n \sim \Gamma(n, \lambda)$

証明 Moment Generating Function

$$E\left[e^{t(X_1+\cdots+X_n)}\right] = \left(1-\frac{t}{\lambda}\right)^{-1} \cdot \left(1-\frac{t}{\lambda}\right)^{-1} \cdots\cdots$$
$$= \left(1-\frac{t}{\lambda}\right)^{-n}$$

vs

$$Y \sim \Gamma(n, \lambda) \Rightarrow E\left[e^{tY}\right] = \int_0^\infty e^{ty} \cdot \frac{\lambda^n y^{n-1}}{\Gamma(n)} \exp(-\lambda y)\,dy$$

$$= \int_0^\infty \frac{\lambda^n}{\Gamma(n)} y^{n-1} \exp(-\underbrace{(\lambda-t)y}_{z})\,dy$$

$(\lambda-t)y = z \qquad \frac{dz}{dy} = \lambda - t$

$$= \int_0^\infty \frac{\lambda^n}{\Gamma(n)} \left(\frac{z}{\lambda-t}\right)^{n-1} \exp(-z) \cdot \frac{dz}{\lambda-t}$$

$$= \int_0^\infty \frac{1}{\Gamma(n)} \left(\frac{\lambda}{\lambda-t}\right)^n \cdot z^{n-1} \exp(-z)\,dz = \left(\frac{\lambda}{\lambda-t}\right)^n = \left(\frac{1}{1-\frac{t}{\lambda}}\right)^n$$

$$= \left(1-\frac{t}{\lambda}\right)^{-n}.$$

■

③ $\chi^2_n = \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$ （$\Rightarrow$ 請自己確認）

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6.9. 求 $E[S^2]$ 和 $V[S^2]$ $X_i \sim N(\mu, \sigma^2)$ (iid)

Fact: $\sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma}\right)^2 \sim \chi^2_{n-1}$

首先利用這個事實；

$$T \overset{def}{=} \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma}\right)^2 \sim \chi^2_{n-1}$$ (自由度 $n-1$ 的 χ^2 分佈)

$$S^2 = \frac{\sigma^2}{n-1} T$$

$$\therefore E[S^2] = E\left[\frac{\sigma^2}{n-1} T\right] = \frac{\sigma^2}{n-1} E[T]$$

$$V[S^2] = V\left[\frac{\sigma^2}{n-1} T\right] = \left(\frac{\sigma^2}{n-1}\right)^2 V[T]$$

$\therefore$ 求 $E[T]$ 和 $V[T]$ 即可

$$T \sim \chi^2_{n-1} = \Gamma\left(\frac{n-1}{2}, \frac{1}{2}\right) \dots$$

由 $\begin{cases} \text{平均} \quad 2 \times \frac{n-1}{2} = n-1 = E[T] \\ \text{變異} \quad 2^2 \times \frac{n-1}{2} = 2(n-1) = V[T] \end{cases}$

$\therefore$

$$\begin{cases} \therefore E[S^2] = \frac{\sigma^2}{n-1} \cdot (n-1) = \sigma^2 \\ V[S^2] = \frac{\sigma^4}{(n-1)^2} \cdot 2(n-1) = \frac{2\sigma^4}{(n-1)} \end{cases}$$

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(注) χ^2_{n-1} 之期望值和變異數

$$\chi^2_{n-1} = \Gamma\left(\frac{n-1}{2}, \frac{1}{2}\right)$$

Gamma 分布 $\Gamma(\alpha, \beta)$ 的期望值與變異數...

$X_i \overset{iid}{\sim} \mathrm{Exp}(\beta)$ (指數分佈的期望... $\frac{1}{\beta}$

變異... $\frac{1}{\beta^2}$)

根據「重要性質②」

$$X_1 + X_2 + \dots + X_\alpha \sim \Gamma(\alpha, \beta)$$

$$\therefore E[X_1 + X_2 + \dots + X_\alpha] = \alpha \times \frac{1}{\beta} = \frac{\alpha}{\beta}$$

$$V[X_1 + X_2 + \dots X_\alpha] = \sum_{i=1}^{\alpha} V[X_i] = \frac{\alpha}{\beta^2}$$

$\therefore \Gamma(\alpha, \beta)$ 的期望值... $\frac{\alpha}{\beta}$，變異數 $\frac{\alpha}{\beta^2}$

$\left(\alpha = \frac{n-1}{2}, \beta = \frac{1}{2} \Rightarrow \chi^2_{n-1}\right)$

(注) α 不一定是自然數，但 $\alpha \neq$ 自然數的情況也會成立。

☆ 你只要背 $\mathrm{Exp}(\beta)$ 的期望值，變異數就可以推導。

$\boxed{\text{Note}}$.. $X_i \sim N(\mu, \sigma^2) \Rightarrow \sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2 \sim \chi^2_{n-1}$

的證明。(用跟課本不同的方法)

$\frac{X_i-\mu}{\sigma} \sim N(0,1)$ $\quad Z_i \overset{def}{=} \frac{X_i-\mu}{\sigma}$

$$\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2 = \sum_{i=1}^{n}(Z_i-\bar{Z})^2$$

$\therefore$ 證明 $\sum_{i=1}^{n}(Z_i-\bar{Z})^2 \sim \chi^2_{n-1}$ 即可

$$\sum_{i=1}^{n}(Z_i-\bar{Z})^2 = (Z_1-\bar{Z}, \cdots Z_n-\bar{Z})\begin{pmatrix} Z_1-\bar{Z} \\ \vdots \\ Z_n-\bar{Z}\end{pmatrix}$$

$$\begin{pmatrix} Z_1-\bar{Z} \\ \vdots \\ Z_n-\bar{Z}\end{pmatrix} = \left(I-\frac{J}{n}\right)\begin{pmatrix} Z_1 \\ \vdots \\ Z_n\end{pmatrix}$$

註 $I=\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1\end{pmatrix}$ $\quad J=\begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ 1 & \cdots & 1\end{pmatrix}$ $\quad J^t=J$

$$\therefore (Z_1-\bar{Z}, \cdots Z_n-\bar{Z})\begin{pmatrix} Z_1-\bar{Z} \\ \vdots \\ Z_n-\bar{Z}\end{pmatrix} = (Z_1, \cdots Z_n)^t\underbrace{\left(I-\frac{J}{n}\right)^t\left(I-\frac{J}{n}\right)}\begin{pmatrix} Z_1 \\ \vdots \\ Z_n\end{pmatrix}$$

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$$\vec{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

$$\sum_{i=1}^{n} (z_i - \bar{z})^2 = \vec{z}^t \left(I - \frac{J}{n}\right)^t \left(I - \frac{J}{n}\right) \vec{z}$$

$$= \vec{z}^t \left(I - \frac{J}{n} - \frac{J}{n} + \frac{J^2}{n^2}\right) \vec{z}$$

$$= \vec{z}^t \left(I - \frac{J}{n}\right) \vec{z} \qquad \text{註 } J^2 = nJ$$

$\because A = \left(I - \frac{J}{n}\right)$ 為 Idempotent（$\Leftarrow A^2 = A$ 成立）

且為 對稱矩陣

$\therefore \exists U^t \; (U^tU = I)$

st $U^tAU = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$（對角化）

特徵值 0 or 1

$\Downarrow$

$\therefore$

$\exists U \; (U^tU = I)$ st $U^tAU = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & & & 0 \end{pmatrix}$? / n?

但 $\operatorname{tr} U^tAU = \operatorname{tr} AUU^t = \operatorname{tr} A = \operatorname{tr} \begin{pmatrix} 1-\frac{1}{n} & & -\frac{1}{n} \\ -\frac{1}{n} & \ddots & \\ & & 1-\frac{1}{n} \end{pmatrix}$

$= n-1 \quad \therefore$

由此可知 A的特徵值 $\begin{cases} 1 \cdots n-1 \text{個} \\ 0 \cdots 1 \text{個} \end{cases}$

$\therefore\ U^t A U = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \begin{array}{l} \updownarrow n-1 \\ \updownarrow 1 \end{array}$

$\therefore$ 现在 $\vec{y} \overset{def}{:=} U^t \vec{z}$

$\vec{z}^t A \vec{z} = \vec{y}^t U^t A U \vec{y} = \vec{y}^t \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \vec{y}$

而且 $\vec{z} \sim N(\vec{0}, I_{n \times n})$

$\vec{y} = U^t \vec{z} = U^t \vec{z} \sim N(U^t \vec{0}, U^t I_{n\times n} U)$

$(U^t U = I) \quad = N(\vec{0}, U^t U) = N(0, I_{n\times n})$

$\therefore\ \vec{y} \sim N(0, I_{n\times n})$ $\left(\begin{array}{l} \text{由 } X \sim N(\mu, \Sigma) \\ \Rightarrow AX \sim N(A\mu, A\Sigma A^t) \end{array} \right)$

$\therefore\ \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ $\quad y_1 \sim y_n \sim N(0,1)$ (iid)

$\therefore\ \vec{y}^t \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \vec{y} = y_1^2 + y_2^2 + \cdots + y_{n-1}^2 \sim \underline{\chi^2_{n-1}}$

$\therefore$ 證明完成

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6.10. $Pr\left(a < \frac{S^2}{\sigma^2} < b\right)$

$= Pr\left(a < \frac{T}{n-1} < b\right)$

$= Pr\left(a(n-1) < T < b(n-1)\right)$

$T \sim \chi^2_{n-1}$

$= \underbrace{F_T(b(n-1))}_{} - F_T(a(n-1))$

$\chi^2_{n-1}(b(n-1)) - \chi^2_{n-1}(a(n-1))$

6.11. $S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

$S_Y^2 = \frac{1}{m-1}\sum_{i=1}^{m}(Y_i - \bar{Y})^2$

$$\begin{cases} \frac{(n-1)}{\sigma^2} S_X^2 = \sum_{i=1}^{n}\left(\frac{X_i - \bar{X}}{\sigma}\right)^2 \sim \chi^2_{n-1} \\ \frac{m-1}{\sigma^2} S_Y^2 = \sum_{i=1}^{m}\left(\frac{Y_i - \bar{Y}}{\sigma}\right)^2 \sim \chi^2_{m-1} \end{cases}$$ 独立

$$\therefore \frac{\frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{X_i - \bar{X}}{\sigma}\right)^2}{\frac{1}{m-1}\sum_{i=1}^{m}\left(\frac{Y_i - \bar{Y}}{\sigma}\right)^2} \sim F_{n,m}$$

(F分布的定義)

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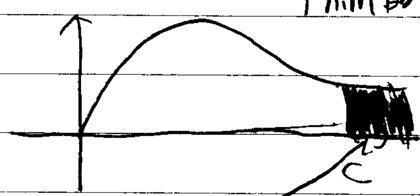
$$\frac{\frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2}{\frac{1}{m-1}\sum_{j=1}^{m}\left(\frac{Y_j-\bar{Y}}{\sigma}\right)^2} = \frac{S_X^2}{S_Y^2}$$

$$\Rightarrow F \stackrel{\text{def}}{=} \frac{S_X^2}{S_Y^2} \sim F_{n,m}$$

$$\therefore Pr\left(\frac{S_X^2}{S_Y^2} > c\right) = Pr(F > c)$$

$$\Rightarrow 1 - \underbrace{F_{F_{n,m}}(c)}_{F_{n,m}\text{之 cdf}}$$

$F_{n,m}$ 的 pdf



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5.6. ① 求Gamma分布的 Moment Generating Function…

$X \sim \Gamma(\alpha, \beta)$

$$f_X(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} \cdot \exp\left(\frac{x}{\beta}\right) \quad (x \geq 0)$$

$$E[e^{tX}] = \int_{x=0}^{x=\infty} \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{x}{\beta} + tx\right) dx$$

$$= \int_0^\infty \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\underbrace{\left(\frac{1}{\beta} - t\right)x}_{\lambda x}\right) dx \quad (\lambda > 0)$$

「-」放外面

$$y \overset{def}{=} \lambda x$$

$$\frac{dy}{dx} = \lambda$$

$$= \int_{y=0}^{y=\infty} \frac{\left(\frac{y}{\lambda}\right)^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} \exp(-y) \frac{dy}{\lambda}$$

$$= \int_{y=0}^{y=\infty} \frac{y^{\alpha-1}\exp(-y)}{\Gamma(\alpha)(\beta\lambda)^\alpha} dy = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \cdot \frac{1}{(\beta\lambda)^\alpha}$$

$$= \left(\frac{1}{\beta\lambda}\right)^\alpha = \left(\frac{1}{1-\beta t}\right)^\alpha$$

∴ $M_X(t) = (1-\beta t)^{-\alpha}$

② $X \sim \Gamma(\alpha,\beta)$ ∴ 期望值∴ $\alpha\beta$

變異數∴ $\alpha\beta^2$

$$Z \stackrel{def}{=} \frac{X-\alpha\beta}{\sqrt{\alpha\beta^2}}$$ (standardize: $\dfrac{X - X\text{的期望值}}{\sqrt{X\text{的變異數}}}$

證明 $Z \xrightarrow{d} N(0,1)$

$$E[e^{tZ}] = E\left[e^{t\left(\frac{X-\alpha\beta}{\sqrt{\alpha\beta^2}}\right)}\right]$$

$$= E\left[e^{\frac{t}{\beta}\frac{X}{\sqrt{\alpha}}} \cdot e^{-t\sqrt{\alpha}}\right]$$

$$= e^{-t\sqrt{\alpha}} \cdot \underbrace{E\left[e^{\left(\frac{t}{\sqrt{\alpha}\beta}\right)X}\right]}_{\left(1-\frac{t}{\sqrt{\alpha}}\right)^{-\alpha}}$$

$$M_Z(t) = E[e^{tZ}] = e^{-t\sqrt{\alpha}} \cdot \left(1-\frac{t}{\sqrt{\alpha}}\right)^{-\alpha}$$

$$\log M_Z(t) = -t\sqrt{\alpha} - \alpha \log\left(1-\frac{t}{\sqrt{\alpha}}\right)$$

$$\log M_Z(t) = -t\sqrt{\alpha} - \alpha \underbrace{\log\left(1-\frac{t}{\sqrt{\alpha}}\right)}_{\text{Taylor展開}}$$

$$\begin{cases} (注)\ 1+r+r^2+r^3+\cdots \approx \dfrac{1}{1-r} & (|r|<1) \\ -\log(1-r) \approx r+\dfrac{r^2}{2}+\dfrac{r^3}{3}+\cdots & (積分) \end{cases}$$

$$r = \left|\frac{t}{\sqrt{\alpha}}\right| \ll 1 \quad (\alpha \to \infty)$$

$$\therefore -\log\left(1-\frac{t}{\sqrt{\alpha}}\right) \approx \frac{t}{\sqrt{\alpha}} + \frac{1}{2}\left(\frac{t}{\sqrt{\alpha}}\right)^2 + \frac{1}{3}\cdot\left(\frac{t}{\sqrt{\alpha}}\right)^3 + \cdots$$

$$-\alpha\log\left(1-\frac{t}{\sqrt{\alpha}}\right) \approx t\sqrt{\alpha} + \frac{t^2}{2} + \underbrace{\frac{t^3}{3\sqrt{\alpha}}}_{\to 0} + \cdots$$

由此可知 $\alpha \to \infty$

$$\log M_Z(t) \approx \frac{t^2}{2}$$

$$M_Z(t) \approx e^{\frac{t^2}{2}}$$

$$(注\ N(0,1)\ の\ mgf = e^{\frac{t^2}{2}})$$