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第2章 統計學・作業1.(參考答案) (TA. 森元俊成)

40.

(a) 「全機率=1」

$\int_{x\in[0,1]} cx^2 dx = 1$

$\left[\frac{cx^3}{3}\right]_0^1 = \frac{c}{3}$

$\therefore c=3$

$\int_{x\in A}$ 機率密度函數 dx

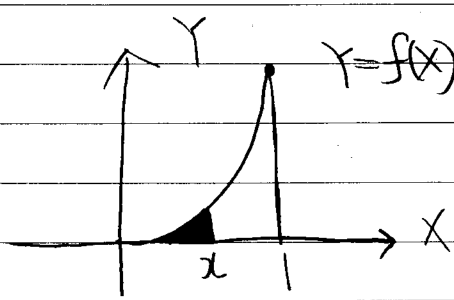
$= \Pr(X\in A)$

(b) $y=f(x)$ (機率密度函數) $= \begin{cases} 3x^2 & (x\in[0,1]) \\ 0 & (x\notin[0,1]) \end{cases}$

$\Downarrow$

(可以寫成 $=3x^2 \cdot I_{(0,1)}(x)$)

(註) $I_A(x) = \begin{cases} 1 & (\text{if } x\in A) \\ 0 & (\text{if } x\notin A) \end{cases}$



cdf = cumulative distribution function

cdf ⇔ 黑色部分的面積

$F(x) = \int_{-\infty}^{x} f(t)\,dt$

= 累積分配函數

②

if: $-\infty < x < 0$ … $F(x) = \int_{-\infty}^{x} \underbrace{f(t)}_{\substack{\shortparallel \\ 0}} dt$ $(\because x \notin [0,1])$

$= 0$

if: $0 \leq x \leq 1$ … $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$= \int_{-\infty}^{0} \underbrace{f(t)}_{\substack{\shortparallel \\ f(t)=0}} dt + \int_{0}^{x} \underbrace{f(t)}_{\substack{\shortparallel \\ f(t)=3t^2}} dt$

$= \left[t^3\right]_0^x = x^3$

if $x > 1$ … $F(x) = \int_{-\infty}^{0} \underbrace{f(t)}_{\substack{\shortparallel \\ 0 \\ \because (x \notin [0,1])}} dt + \int_{0}^{1} \underbrace{f(t)}_{\substack{\shortparallel \\ \int_0^1 3t^2\,dt \\ \shortparallel \\ [t^3]_0^1 \\ \shortparallel \\ 1}} dt + \int_{1}^{x} \underbrace{f(t)}_{\substack{\shortparallel \\ 0 \\ \because (x \notin [0,1])}} dt$

$= 1$

$\therefore \underbrace{F(x)}_{\substack{\shortparallel \\ \text{cdf}}} = \begin{cases} 0 & (-\infty < x < 0) \\ x^3 & (0 \leq x \leq 1) \\ 1 & (x > 1) \end{cases}$

③

第2章、

40. $\Pr(0.1 \leq X < 0.5)$

$$\| \quad \int_{0.1}^{0.5} f(x)\,dx = \left[x^3\right]_{0.1}^{0.5} = 0.125 - 0.001$$

$$= \underline{0.124}$$

58. $Z \sim N(\mu, \sigma^2)$ 時 $Y = e^Z$ 之分佈為何？

$$\begin{cases} Z\text{的範圍}: -\infty \to \infty \\ \qquad\qquad \Updownarrow \text{vs} \\ Y = e^Z \text{的}: \ 0 \to \infty \end{cases}$$

$$\frac{dy}{dz} = \frac{d}{dz}(e^z) = e^z = y \quad \therefore \ dz = \frac{1}{y}dy$$

<u>全機率 = 1</u>

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(z-\mu)^2\right) dz = 1$$

$Z: -\infty \to \infty$ ↓ $\log y$ ↓ $\frac{dy}{y}$

$y: 0 \to \infty$

※ 改為 y 的積分

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(\log y - \mu)^2\right) \cdot \frac{1}{y}\,dy$$

④

$$\therefore 1=\int_0^{\infty} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(\log y-\mu)^2\right)\frac{1}{y}}_{\text{這是Y的pdf.}}\,dy$$

由此可見，Y的機率密度函數為...

$$f(y)=\begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}}\cdot \exp\left(\frac{-1}{2\sigma^2}(\log y-\mu)^2\right)\cdot\frac{1}{y} & (0\le y<\infty) \\ 0 & (-\infty< y<0) \end{cases}$$

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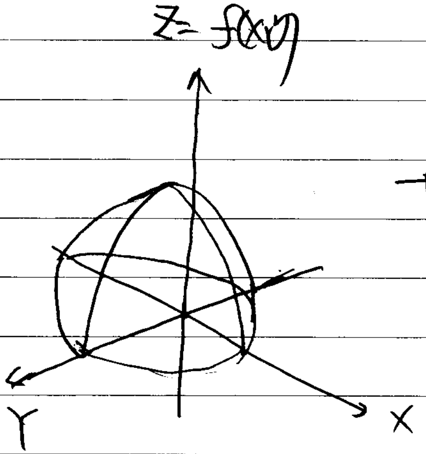
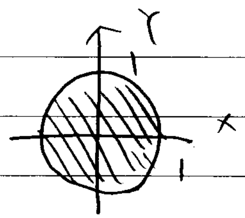
15.

(a) $$\text{全機率}=1=\iint_{\underbrace{\{(x,y)\mid x^2+y^2\le 1\}}_{\text{積分範圍}}} c\sqrt{1-x^2-y^2}\,dx\,dy$$





∴ 可以想成這個立體圖形的$\underset{\text{volume}}{\text{體積}}$=1

⑤

第4章

15.

(a) $\begin{pmatrix} X \\ Y \end{pmatrix}$ 寫成極坐標：

$$\begin{cases} X = r\cos\theta & (0 \le \theta < 2\pi) \\ Y = r\sin\theta & (0 \le r < \infty) \end{cases}$$

但 $X^2+Y^2 \le 1$

所以 $r^2\cos^2\theta + r^2\sin^2\theta = r^2 \le 1$

(但 $r \ge 0$)

$$\therefore \begin{cases} 0 \le r \le 1 \\ 0 \le \theta < 2\pi \end{cases}$$

Jacobian:

$$J = \begin{pmatrix} \frac{\partial X}{\partial r} & \frac{\partial X}{\partial \theta} \\ \frac{\partial Y}{\partial r} & \frac{\partial Y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\det J = r\cos^2\theta + r\sin^2\theta = \underline{r}$$

$$\Rightarrow \quad dx\,dy = r\,dr\,d\theta$$

$$\iint_{\{(x,y) \mid x^2+y^2 \le 1\}} c\sqrt{1-x^2-y^2}\,\underline{dx\,dy} \qquad \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} R \\ \theta \end{pmatrix}$$

$$= \iint_{\substack{0 \le r \le 1 \\ 0 \le \theta < 2\pi}} c\sqrt{1-r^2}\,\underline{r\,dr\,d\theta}$$

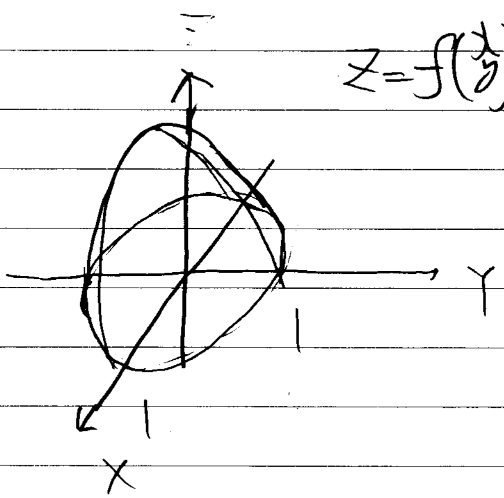
⑥

$$= \underbrace{\int_0^1 c\sqrt{1-r^2}\cdot r\,dr}_{=\left[-\frac{c}{3}(1-r^2)^{\frac{3}{2}}\right]_0^1 = \frac{c}{3}} \cdot \underbrace{\int_0^{2\pi} d\theta}_{=\int_0^{2\pi} 1\,d\theta = [\theta]_0^{2\pi} = 2\pi}$$

$$= \frac{2\pi}{3}\cdot c = \underbrace{1}_{\text{全確率}}$$

$$\therefore c = \frac{3}{2\pi}$$

(b)



$$z = f\binom{x}{y} = \frac{3}{2\pi}\sqrt{1-x^2-y^2}$$

$$(x^2+y^2 \leq 1)$$

⑦

第3章

15.

(c) $\Pr(\underbrace{X^2+Y^2}_{\parallel} \le \frac{1}{2})$

$r^2 \le 1 \quad (r \ge 0)$

$\parallel$

$= \Pr(0 \le r \le \frac{1}{\sqrt{2}})$

欲求之機率:

$$= \iint_{\{(x,y)\,|\,x^2+y^2 \le \frac{1}{2}\}} \frac{3}{2\pi}\sqrt{1-x^2-y^2}\,dx\,dy$$

$\parallel$

$$\iint_{\substack{0 \le r \le \frac{1}{\sqrt{2}} \\ 0 \le \theta < 2\pi}} \frac{3}{2\pi}\sqrt{1-r^2}\cdot r\,dr\,d\theta$$

$$= \frac{3}{2\pi}\cdot\left[-\frac{(1-r^2)^{\frac{3}{2}}}{3}\right]_{r=0}^{r=\frac{1}{\sqrt{2}}}\cdot\left[\theta\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(1-\left(\frac{1}{2}\right)^{\frac{3}{2}}\right)\cdot 2\pi = 1-\frac{1}{2\sqrt{2}}$$

⑧

(d) X的邊際分佈… $f_X(x) =$

$= \int_y f(x,y)\, dy$ (→這個變成 x 的函數)

$= \int_{x^2+y^2 \leq 1} \frac{3}{2\pi} \sqrt{1-x^2-y^2}\, dy$

$\| $

$-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$

(註 在這裡, X可視為常數)

為了方便起見 $t \overset{\text{put}}{=} \frac{1}{\sqrt{1-x^2}} y$

$\begin{cases} y: -\sqrt{1-x^2} \to \sqrt{1-x^2} \\ \text{vs} \\ t: -1 \to 1 \end{cases}$

Y的積分轉變為 t 的積分

$\frac{dt}{dy} = \frac{1}{\sqrt{1-x^2}}$

⇔

$(\therefore dy = \sqrt{1-x^2}\, dt$

$\int_{\underbrace{-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}}_{-1 \leq t \leq 1}} \frac{3}{2\pi} \underbrace{\sqrt{(1-x^2)-y^2}}_{\sqrt{1-x^2}\cdot\sqrt{1-t^2}} \cdot \underbrace{dy}_{\sqrt{1-x^2}\, dt}$

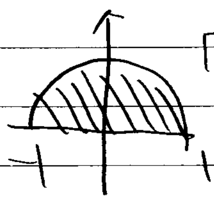
⑨

15 (d)

$$\int_{-1}^{1} \frac{3}{2\pi}\cdot(1-x^2)\sqrt{1-t^2}\,dt$$

$$\parallel$$

$$\frac{3}{2\pi}(1-x^2)\int_{-1}^{1}\sqrt{1-t^2}\,dt$$



$\boxed{圓}$ $= \frac{\pi}{2}$

$$= \frac{3}{4}(1-x^2) \quad (-1 \leqq x \leqq 1)$$

注

$\int_{-1}^{1} \frac{3}{4}(1-x^2)\,dx$

$= \int_0^1 \frac{3}{2}(1-x^2)\,dx$

$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_0^1$

$= 1$ (全機率)

$f_X(x)$ 確實也是 pdf

X的邊際機率密度函數

$$\leftarrow \quad f_X(x) = \begin{cases} \frac{3}{4}(1-x^2) & (-1 \leqq x \leqq 1) \\ 0 & (\text{else}) \end{cases}$$

同樣道理，$f_Y(y) = \begin{cases} \frac{3}{4}(1-y^2) & (-1 \leqq y \leqq 1) \\ 0 & (\text{else}) \end{cases}$

$\boxed{重要}$ $f_{XY}(x,y) = f_X(x)\,f_Y(y) \Leftrightarrow$ X, Y 獨立

$\parallel$

$f(x,y)$ (聯合分佈)

⑩

但 $\frac{3}{2\pi}\sqrt{1-x^2-y^2} \neq \frac{3}{4}(1-x^2)\frac{3}{4}(1-y^2)$

∴ X、Y 並非獨立!!

(e) 求 $X|Y=y$ (or $Y|X=x$)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{3}{2\pi}\sqrt{1-x^2-y^2}}{\frac{3}{4}(1-y^2)}$$

(重要)

$(x^2+y^2\leq 1)$

$$= \begin{cases} \frac{2}{\pi}\cdot\frac{\sqrt{1-x^2-y^2}}{(1-y^2)} & (x^2+y^2\leq 1) \\ 0 & (\text{else}) \end{cases}$$

($Y|X=x$ … 替換 X 與 Y 即可)

(∵ X、Y … 對稱)

⑪

3章

46. $\begin{cases} T_1 \sim \exp(\lambda_1) \\ T_2 \sim \exp(\lambda_2) \end{cases}$

$\begin{cases} Z = T_1 + T_2 \\ W = T_1 \end{cases}$

(註 為了將 t_1, t_2 的雙重積分變成別的變數的積分，我們需要另一個變數 $W = T_1$ （$W = T_2$ 亦可）)

全機率 $= 1 = \int_{t_1 \geq 0} \int_{t_2 \geq 0} \lambda_1 \exp(-\lambda_1 t_1) \lambda_2 \exp(-\lambda_2 t_2) \, dt_1 dt_2$

註 (T_1, T_2 獨立。所以它們的聯合密度函數 $=$ T_1 的密度函數 $\times$ T_2 的密度函數)

接著 $\iint \sim dt_1 dt_2 \rightarrow \iint \sim dw dz$

① Jacobian $\begin{cases} T_1 = W \\ T_2 = Z - W \end{cases}$

$$J = \begin{pmatrix} \frac{\partial t_1}{\partial w} & \frac{\partial t_1}{\partial z} \\ \frac{\partial t_2}{\partial w} & \frac{\partial t_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$|J| = 1 \quad \therefore \underline{dt_1 dt_2 = dw dz}$

② 積分範囲 $T_1 \geqq 0$

$T_2 \geqq 0$

$\Downarrow$

$$\begin{cases} T_1 = W \geqq 0 \\ T_2 = Z - W \geqq 0 \end{cases}$$

$\therefore \ 0 \leqq W \leqq Z$ $\perp$

③

$$\iint_{\substack{t_1 \geqq 0 \\ t_2 \geqq 0}} \lambda_1 \exp(-\lambda_1 t_1)\, \lambda_2 \exp(-\lambda_2 t_2)\, \underbrace{dt_1 dt_2}_{\substack{\| \ (\because ①) \\ dwdz}}$$

$t_1 \to w$, $t_2 \to (z-w)$, w, z

$$= \iint_{0 \leqq w \leqq z \ (\because ②)} \lambda_1 \lambda_2 \exp(-\lambda_1 w - \lambda_2 (z-w))\, dwdz$$

$$= \iint_{0 \leqq w \leqq z} \lambda_1 \lambda_2 \exp(-(\lambda_1 - \lambda_2) w - \lambda_2 z)\, dwdz$$

$$\therefore \ f_{W,Z}(w,z) = \begin{cases} \lambda_1 \lambda_2 \exp(-(\lambda_1 - \lambda_2) w - \lambda_2 z) & (0 \leqq w \leqq z) \\ 0 & (\text{else}) \end{cases}$$

$\|$

W,Z的 聯合 pdf

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我們想知道 Z 的邊際分布

$\Downarrow$

$$\int_{0\leq w\leq z} f_{W,Z}(w)\,dw \to (變成\ Z的\ pdf)$$

$$\|$$

$$\int_{w=0}^{w=z} \lambda_1\lambda_2 \exp(-(\lambda_1-\lambda_2)w-\lambda_2 z)\,dw$$

(看 Z = 常數)

$$= \lambda_1\lambda_2\exp(-\lambda_2 z)\underbrace{\int_{w=0}^{w=z}\exp(-(\lambda_1-\lambda_2)(w))\,dw}$$

$$\|$$

$$\left[\frac{-1}{\lambda_1-\lambda_2}\exp(-(\lambda_1-\lambda_2)(w))\right]_0^z$$

$$= \frac{1}{\lambda_1-\lambda_2}\left(1-\exp(-(\lambda_1-\lambda_2)z)\right)$$

$$= \lambda_1\lambda_2\exp(-\lambda_2 z)\cdot\frac{1}{\lambda_1-\lambda_2}\left(1-\exp(-\lambda_1 z)\exp(\lambda_2 z)\right)$$

$$= \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}\left(\exp(-\lambda_2 z)-\exp(-\lambda_1 z)\right)$$

$$\therefore f_Z(z) = \begin{cases} \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}\left(\exp(-\lambda_2 z)-\exp(-\lambda_1 z)\right) & (z\geq 0) \\ 0 & (\text{else}) \end{cases}$$

(14)

64.

方法1 跟46同樣方法：

$$\begin{cases} W = X+Y \\ Z = \dfrac{X}{Y} \end{cases}$$ 求W, Z之聯合密度函數

$\downarrow$

$$\begin{cases} X = \dfrac{WZ}{1+Z} \\ Y = \dfrac{W}{1+Z} \end{cases}$$

(1) $X, Y \overset{iid}{\sim} exp(\lambda)$

X, Y的聯合密度函數 $f_{XY}(x,y) = \lambda exp(-\lambda x)\, \lambda exp(-\lambda y)$
$(x \geq 0, y \geq 0)$

全機率 $\iint_{\substack{x \geq 0 \\ y \geq 0}} \lambda exp(-\lambda x)\, \lambda exp(-\lambda y)\, dx\, dy = 1$
$= 1$

(2) 先考慮 Jacobian ...

⑮

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$$J=\begin{pmatrix}\frac{\partial x}{\partial w} & \frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial z}\end{pmatrix}=\begin{pmatrix}\frac{z}{1+z} & \frac{w}{(1+z)^2}\\ \frac{1}{1+z} & \frac{-w}{(1+z)^2}\end{pmatrix}$$

$$\det J=\frac{-wz}{(1+z)^3}-\frac{w}{(1+z)^3}=\frac{-w(1+z)}{(1+z)^3}=\frac{-w}{(1+z)^2}$$

$$\therefore |\det J|=\frac{w}{(1+z)^2}$$

$$\therefore dxdy=\frac{w}{(1+z)^2}dwdz$$

③ 積分範囲 $\begin{cases}x\geq 0\\ y\geq 0\end{cases}$

$$\Leftrightarrow\begin{cases}\frac{wz}{1+z}\geq 0\\ \frac{w}{1+z}\geq 0\end{cases}$$

$$\therefore\begin{cases}\therefore w\geq 0\\ z\geq 0\end{cases}$$

④ $\iint\sim dxdy$ 写成 $\iint\sim dwdz$

⑩

$$1 = \iint_{\substack{x\geq 0\\ y\geq 0}} \lambda \exp(-\lambda x)\, \lambda \exp(-\lambda y)\, dx\, dy$$
(全機率)

$$= \iint_{\substack{x\geq 0\\ y\geq 0}} \lambda^2 \exp(-\lambda(x+y))\, dx\, dy$$

$$\underbrace{\substack{x\geq 0\\ y\geq 0}}_{\substack{w\geq 0\\ z\geq 0}} \qquad \underbrace{(x+y)}_{w} \qquad \underbrace{dx\,dy}_{=\frac{w\,dw\,dz}{(1+z)^2}}$$

$$= \iint_{\substack{w\geq 0\\ z\geq 0}} \frac{\lambda^2}{(1+z)^2}\, w \exp(-\lambda w)\, dz\, dw$$

$$\therefore f_{W,Z}(w,z) = \frac{\lambda^2}{(1+z)^2} w \exp(-\lambda w)$$

$$(w\geq 0,\ z\geq 0)$$

接著證明 W, Z 為獨立的隨機變數 (證明 $f_{W,Z} = f_W \cdot f_Z$)

$$f_W(w) = \int_{z\geq 0} \frac{\lambda^2}{(1+z)^2} w \exp(-\lambda w)\, dz$$

$$= \left[\frac{-1}{1+z}\right]_{z=0}^{z=\infty} \lambda^2 w \exp(-\lambda w)$$

$$= \lambda^2 w \exp(-\lambda w) \quad (w\geq 0)$$

$$f_Z(z) = \int_{w\geq 0} \frac{\lambda^2}{(1+z)^2} w \exp(-\lambda w)\, dw$$

⑰

3章
64

$$\int_{w\geq 0} \frac{\lambda^2}{(1+z)^2} w \exp(-\lambda w)\, dw$$

$v \overset{\text{置換}}{=} \lambda w \qquad w \geq 0 \Leftrightarrow v \geq 0$

$(\because \lambda > 0)$

$\frac{dv}{dw} = \lambda$

$$= \int_{v\geq 0} \frac{\lambda^2}{(1+z)^2} \left(\frac{v}{\lambda}\right) \exp(-v) \cdot \frac{dv}{\lambda}$$

$$= \int_{v\geq 0} \frac{1}{(1+z)^2} v \exp(-v)\, dv$$

$$= \frac{1}{(1+z)^2} \underbrace{\int_{v\geq 0} v^{2-1} \exp(-v)\, dv}_{\text{Gamma 函數}}$$

㊟ $\int_{x\geq 0} x^{n-1} \exp(-x)\, dx = \Gamma(n) = (n-1)!$

$(\because$ if n 為整數$)$

$$= \frac{1}{(1+z)^2} \Gamma(2) = \frac{1}{(1+z)^2}$$

$\therefore f_Z(z) = \frac{1}{(1+z)^2} \quad (z \geq 0)$ ⊥

$\therefore f_{W,Z}(w,z) = f_W(w) f_Z(z) \quad$ (for all (w,z))

$\therefore W, Z$ 為獨立

⑱

64. 定理2 我們以後會學到「完備充份統計量」

$$\begin{cases} X_1, X_2 \overset{iid}{\sim} \exp(\lambda) \quad (\text{mean } \tfrac{1}{\lambda}) \\ \lambda: \text{unknown parameter} \end{cases}$$

(指數分布 = Γ分布的 α=1 的情形)

觀測 X_1, X_2 時，X_1+X_2 為 λ 之完備充份統計量

$$\frac{X_1}{X_2} = \frac{\lambda X_1}{\lambda X_2}, \quad \begin{matrix} \lambda X_1 \sim \exp(1) \\ \lambda X_2 \sim \exp(1) \end{matrix} \quad (\text{跟}\lambda\text{無關})$$

(一般來說 $X_1 \sim X_n \overset{iid}{\sim} e(\beta)$, $X_1+\cdots+X_n \sim \Gamma(n,\beta)$)

∴ 由此可知 $\frac{X_1}{X_2}$ 的分佈與 λ 無關

($\frac{X_1}{X_2}$ 為 λ 之輔助統計量)

根據 BASU 定理，X_1+X_2 (λ 之完備充份統計量)

vs

X_1/X_2 (λ 之輔助統計量)

為獨立

另外 $X_1+X_2 \sim \Gamma(2,\lambda)$ (Gamma 分佈) → 可以用動差母函數證明！

$$\frac{X_1}{X_2} = \frac{(2\lambda X_1)/2}{(2\lambda X_2)/2} \quad \begin{matrix} 2\lambda X_1 \sim \Gamma(1,\tfrac{1}{2}) = \chi^2(2) \\ 2\lambda X_2 \sim \Gamma(1,\tfrac{1}{2}) = \chi^2(2) \end{matrix}$$

自由度 2 之 χ^2 分佈

$$\therefore \frac{X_1}{X_2} \sim F_{2,2} \quad \left(\text{定義 } \frac{\left(\frac{\chi^2_m}{m}\right)}{\left(\frac{\chi^2_n}{n}\right)} \sim F_{m,n} \text{ 定義}\right)$$

(20)

$U_{(1)}, U_{(2)}, U_{(3)}$ 的 (聯合) 機率密度函數

$$= \begin{cases} 1\cdot1\cdot1\cdot 3! = 6 & (0 \le U_{(1)} < U_{(2)} < U_{(3)} \le 1) \\ 0 & \end{cases}$$

原因 $U_1 \sim U_3$ 之間沒有大小關係，所以可能

$$\underset{\text{個組合}}{3!} = \begin{cases} \text{、} U_1 < U_2 < U_3 & \text{or} \\ \text{、} U_1 < U_3 < U_2 & \text{or} \\ \text{、} U_2 < U_1 < U_3 & \text{or} \\ \text{、} U_2 < U_3 < U_1 & \text{or} \\ \text{、} U_3 < U_1 < U_2 & \text{or} \\ \text{、} U_3 < U_2 < U_1 & . \end{cases}$$

但 $U_{(1)}, U_{(2)}, U_{(3)}$ 已規定 $U_{(1)} < U_{(2)} < U_{(3)}$

所以 $U_{(1)} \sim U_{(3)}$ 的密度要原本的 $3!$ 倍。

$$\therefore f_{U_{(1)}, U_{(2)}, U_{(3)}}(U_{(1)}, U_{(2)}, U_{(3)}) = \begin{cases} 6 & (0 \le U_{(1)} < U_{(2)} < U_{(3)} \le 1) \\ 0 & (\text{else}) \end{cases}$$

(請確認 ⇒ $\iiint_{0 \le U_{(1)} < U_{(2)} < U_{(3)} \le 1} 6\, dU_{(1)}\, dU_{(2)}\, dU_{(3)} = 1$)

$$= \iint_{0 \le U_{(2)} \le U_{(3)} \le 1} (6U_{(2)})\, dU_{(2)}\, dU_{(3)}$$

$$= \int_{0 \le U_{(3)} \le 1} (3U_{(3)}^2)\, dU_{(3)} = \left[U_{(3)}^3\right]_0^1 = 1$$

(19)

第3章

68. • 此題並未寫到 $U_1 \sim U_3$ 之分佈（只寫到均勻分佈）

(a) 在此假設 $U_1, U_2, U_3 \overset{iid}{\sim} U(0,1)$

• 另外，先說明一下 $U_{(1)}\ U_{(2)}\ U_{(3)}$ vs $U_1\ U_2\ U_3$

的差別。

「$U_{(1)} < U_{(2)} < U_{(3)}$」

可以說是，觀測到 U_1, U_2, U_3 後，依照大小順序（由小往大）

將 $U_1 \sim U_3$ 排列的東西 $= U_{(1)}, U_{(2)}, U_{(3)}$

（所以 $U_{(1)} = \min\{U_1, U_2, U_3\}$

$U_{(3)} = \max\{U_1, U_2, U_3\}$）

這叫做「順序統計量」(order statistics)

• U_1, U_2, U_3 的聯合機率密度函數

$$\begin{cases} = 1\cdot1\cdot1=1 & (0\leq u_1\leq1,\ 0\leq u_2\leq1,\ 0\leq u_3\leq1) \\ 0 & (\text{else}) \end{cases}$$

㉑

第3章

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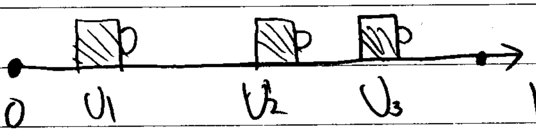
(b) $U_1, U_2, U_3 \overset{iid}{\sim} U(0,1)$ 分別代表加油站的位置。



現在求「任何兩間加油站的距離 $\geq \frac{1}{3}$」之機率

$$\Vert$$

$$Pr\left(|U_1-U_2| \geq \frac{1}{3} \cap |U_2-U_3| \geq \frac{1}{3} \cap |U_3-U_1| \geq \frac{1}{3}\right)$$

$$\downarrow$$

要考慮 U_1, U_2, U_3 的大小關係

$$\Downarrow$$

有點複雜⋯⋯。

$$\Downarrow$$

考慮 (a) 的 順序統計量!

$$\Vert$$

$$Pr\left(U_{(2)}-U_{(1)} \geq \frac{1}{3} \cap U_{(3)}-U_{(2)} \geq \frac{1}{3}\right)$$ 即可!

$$\Vert$$

$$\int\int_{\begin{cases} 0 \leq u_{(1)} < u_{(2)} < u_{(3)} \leq 1 \\ u_{(2)}-u_{(1)} \geq \frac{1}{3},\ u_{(3)}-u_{(2)} \geq \frac{1}{3} \end{cases}} 6\, du_{(1)}\, du_{(2)}\, du_{(3)}$$

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$$= \int_{\substack{0 \leq u_{(1)} \leq u_{(2)} - \frac{1}{3} \\ \frac{1}{3} \leq u_{(2)} \leq u_{(3)} - \frac{1}{3} \\ \frac{2}{3} \leq u_{(3)} \leq 1}} 6 \, du_{(1)} du_{(2)} du_{(3)}$$

$$= \int_{\substack{\frac{1}{3} \leq u_{(2)} \leq u_{(3)} - \frac{1}{3} \\ \frac{2}{3} \leq u_{(3)} \leq 1}} \underbrace{\Big[6u_{(1)} \Big]_{0}^{u_{(2)} - \frac{1}{3}}}_{6u_{(2)} - 2} du_{(2)} du_{(3)}$$

$$= \int_{\frac{2}{3} \leq u_{(3)} \leq 1} \underbrace{\Big[3u_{(2)}^2 - 2u_{(2)} \Big]_{\frac{1}{3}}^{u_{(3)} - \frac{1}{3}}}_{3\left(u_{(3)} - \frac{1}{3}\right)^2 - 2\left(u_{(3)} - \frac{1}{3}\right) + \frac{1}{3}}$$

$$= \Big[\left(u_{(3)} - \frac{1}{3}\right)^3 - \left(u_{(3)} - \frac{1}{3}\right)^2 + \frac{1}{3} u_{(3)} \Big]_{\frac{2}{3}}^{1}$$

$$= \underbrace{\left(\frac{8}{27} - \frac{4}{9} + \frac{1}{3} \right)}_{\frac{5}{27}} - \underbrace{\left(\frac{1}{27} - \frac{1}{9} + \frac{2}{9} \right)}_{\frac{4}{27}} = \underline{\frac{1}{27}}$$

$$\therefore \frac{1}{27}$$

㉓

第4章

20.

$X \sim Po(\lambda)$

$$Pr(X=x|\lambda) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$$

$$E\left[\frac{1}{X+1}\right] = \sum_{x=0}^{\infty} \frac{e^{-\lambda}}{(x+1)} \cdot \frac{\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}}{(x+1)!} \cdot \lambda^x \quad (z = x+1)$$

$$= \sum_{z=1}^{\infty} \frac{e^{-\lambda}}{z!} \cdot \lambda^{(z-1)}$$

$$= \frac{e^{-\lambda}}{\lambda} \cdot \sum_{z=1}^{\infty} \frac{\lambda^z}{z!}$$

注 $e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots$ (Taylor展開)

$e^{\lambda} - 1 = \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots$

$$e^{\lambda} - 1 = \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots = \sum_{z=1}^{\infty} \frac{\lambda^z}{z!}$$

$$\therefore = \frac{e^{-\lambda}}{\lambda} \cdot (e^{\lambda} - 1) = \frac{1 - e^{-\lambda}}{\lambda}$$

49.

(a) $E[Z] = \int_{(x,y)\in R^2} (\alpha x + (1-\alpha) y) \cdot f_{XY}(x,y)\,dx\,dy$

$= \alpha \int_{(x,y)\in R^2} x\, f_{XY}(x,y)\,dx\,dy$

$+ (1-\alpha) \int_{(x,y)\in R^2} y\, f_{XY}(x,y)\,dx\,dy$

$= \alpha E[X] + (1-\alpha) E[Y]$

$= \alpha\mu + (1-\alpha)\mu = \underset{\sim}{\mu}$

(b) $V[Z] = V[\alpha X + (1-\alpha) Y]$

$= E[(\alpha X + (1-\alpha) Y - \mu)^2]$

$= E[\{\alpha(X-\mu) + (1-\alpha)(Y-\mu)\}^2]$

$= \alpha^2 V[X] + 2\alpha(1-\alpha)\underbrace{\mathrm{cov}[X,Y]}_{=0} + (1-\alpha)^2 V[Y]$

$= \alpha^2 \cdot \sigma_X^2 + (1-\alpha)^2 \sigma_Y^2$ (∵ X, Y = 独立)

$\therefore \underset{\sim}{V[Z] = \alpha^2 \sigma_X^2 + (1-\alpha)^2 \sigma_Y^2}$

(or $V[\alpha X + (1-\alpha)Y] \xrightarrow{\text{直接}} = \alpha^2 V(X) + (1-\alpha)^2 V(Y)$)

∵ X, Y 独立

(25)

㊟ (高中數學唔! 2次函數之最小值)

49. (b)

$V[Z]=f(\alpha)=\alpha^2\sigma_X^2+(1-\alpha)^2\sigma_Y^2$ (當成 α 的函數)

$$=(\sigma_X^2+\sigma_Y^2)\alpha^2-2\alpha\sigma_Y^2+\sigma_Y^2$$

$$=(\sigma_X^2+\sigma_Y^2)\left(\alpha^2-\frac{2\sigma_Y^2}{\sigma_X^2+\sigma_Y^2}\alpha\right)+\sigma_Y^2$$

$$=(\sigma_X^2+\sigma_Y^2)\left(\alpha-\frac{\sigma_Y^2}{\sigma_X^2+\sigma_Y^2}\right)^2+\ldots$$ (略)

$\therefore \alpha=\frac{\sigma_Y^2}{\sigma_X^2+\sigma_Y^2}$ 時 $V[Z]$ 最小

(c) 變異數小 ⇒ 得到 μ 附近的機率比較高 ⇒ Happy

why?

$\therefore V\left[\frac{X+Y}{2}\right]=\frac{1}{4}(V(X)+V(Y))=\frac{1}{4}(\sigma_X^2+\sigma_Y^2)$

$V[X]=\sigma_X^2$

$V[Y]=\sigma_Y^2$

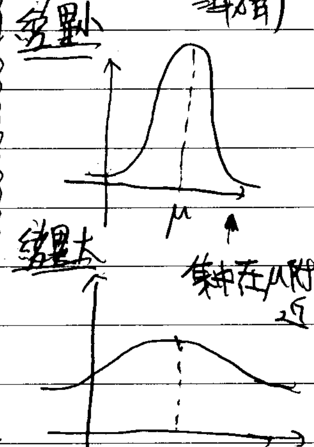
$V\left[\frac{X+Y}{2}\right]\le V(X)$ 且 $V\left[\frac{X+Y}{2}\right]\le V(Y)$

$\Rightarrow 3\sigma_X^2\ge\sigma_Y^2$ $\Rightarrow 3\sigma_Y^2\ge\sigma_X^2$

$3\ge\frac{\sigma_Y^2}{\sigma_X^2}$ $\frac{\sigma_Y^2}{\sigma_X^2}\ge\frac{1}{3}$

$\therefore \frac{1}{3}\le\frac{\sigma_Y^2}{\sigma_X^2}\le 3 \Rightarrow V\left[\frac{X+Y}{2}\right]$ 較佳

(預測比較準確)



可能得到離 μ 很遠的 data

50.

- $E[\bar{X}] = E\left[\frac{1}{n}(X_1+X_2+\cdots+X_n)\right]$

$$= \int \frac{1}{n}(x_1+x_2+\cdots+x_n)\, f_X(x_1,x_2,\ldots,x_n)\, dx_1dx_2\cdots dx_n$$

$$= \sum_{j=1}^{n} \frac{1}{n} \underbrace{\int x_j\, f_X(x_1,x_2,\ldots,x_n)\, dx_1dx_2\cdots dx_n}_{\substack{\| \\ E[X_j] \\ \| \\ \mu}}$$

$$= \sum_{j=1}^{n} \frac{\mu}{n} = \mu$$

- $V[\bar{X}] = E[(\bar{X}-\mu)^2] = E\left[\left(\frac{X_1+\cdots+X_n}{n}-\mu\right)^2\right]$

$$= E\left[\left(\frac{1}{n}\{(X_1-\mu)+(X_2-\mu)+\cdots+(X_n-\mu)\}\right)^2\right]$$

$$= \int \frac{1}{n^2}\sum_{(i,j)}\sum (x_i-\mu)(x_j-\mu)\, f(x_1,x_2,\ldots,x_n)\, dx_1\cdots dx_n$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int (x_i-\mu)(x_j-\mu)\, f(x_1,x_2,\ldots,x_n)\, dx_1dx_2\cdots dx_n$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E[(X_i-\mu)(X_j-\mu)]$$

$$\left(\begin{array}{l} i\neq j \Rightarrow E[(X_i-\mu)(X_j-\mu)] = E[(X_i-\mu)]E[(X_j-\mu)] = 0 \\ i=j \Rightarrow E[(X_i-\mu)^2] = V[X_i] = \sigma^2 \end{array}\right.$$

hint $\int_0^\infty x^{\alpha-1}e^{-x}dx = \Gamma(\alpha) = (\alpha-1)!$ (if α=整数)

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$\downarrow$ $\left(注\ \Gamma(\tfrac{1}{2})=\sqrt{\pi}\right)$

4章

50. $= \frac{1}{n^2}\sum\sum_{1\le i=j\le n}\sigma^2 = \frac{\sigma^2}{n}$

77.

(a)

$$\rho = \frac{Cov[X,Y]}{\sqrt{V(X)}\sqrt{V(Y)}} = \frac{1}{\sqrt{1}\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$Cov[X,Y] = \underbrace{E[XY]}_{3} - \underbrace{E[X]}_{1}\underbrace{E[Y]}_{2} = 1$$

$$E[XY] = \iint_{0\le x\le y<\infty} xy\,e^{-y}\,dx\,dy$$

$$= \int_{y=0}^{y=\infty}\left[\frac{x^2}{2}\right]_0^y y\,e^{-y}\,dy$$

$$= \int_0^\infty \frac{y^3}{2}e^{-y}\,dy$$

$$= \frac{1}{2}\Gamma(4) \quad \left(注\ \int_0^\infty x^{n-1}\exp(-x)\,dx = \Gamma(n)\right)$$

$$= \frac{1}{2}\cdot 3! = \frac{6}{2} = 3$$

$$E[X] = \int_{0\le x\le y<\infty} x\,e^{-y}\,dx\,dy = \int_0^\infty \frac{y^2}{2}e^{-y}\,dy = \frac{\Gamma(3)}{2} = 1$$

$$E[X^2] = \int_{0\le x\le y<\infty} x^2 e^{-y}\,dx\,dy = \int_0^\infty \frac{y^3}{3}e^{-y}\,dy = \frac{\Gamma(4)}{3} = 2$$

$$E[Y] = \int_{0\le x\le y<\infty} 1\cdot y\,e^{-y}\,dx\,dy =$$

$$= \int_0^\infty [x]_0^y\, y\,e^{-y}\,dy = \int_0^\infty y^2 e^{-y}\,dy = \Gamma(3) = 2$$

同様,

$$E[Y^2] = \int_0^\infty y^3 e^{-y}\,dy = \Gamma(4) = 6\ (=3!)$$

$V[X] = 2-1^2 = 1$ $V[Y] = 6-4 = 2$

(22)

(b) 先求 $X|Y=y$ 之分佈

$$f_Y(y)=\int_{0\le x\le y} e^{-y}\,dx = [x]_0^y e^{-y} = ye^{-y}$$

$$\therefore f_{X|Y}(x|y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}=\frac{e^{-y}}{ye^{-y}}=\frac{1}{y}\quad(0\le x\le y)$$

由此可知，$X|Y=y \sim \mathrm{Uni}(0,y)$ $(0\le x\le y)$
(均勻分佈)

$$E[X|Y=y]=\int_0^y x\cdot\frac{1}{y}\,dx$$
$$=\frac{1}{y}\left[\frac{x^2}{2}\right]_0^y=\frac{y}{2}$$

接下來求 $Y|X=x$ 之分佈

$$f_X(x)=\int_{y\ge x} e^{-y}\,dy = \left[-e^{-y}\right]_x^\infty = e^{-x}$$

$$\therefore f_{Y|X}(Y|X=x)=\frac{f_{X,Y}(x,y)}{f_X(x)}=\frac{e^{-y}}{e^{-x}}=e^{-(y-x)}\quad(y\ge x)$$

$$\therefore f_{Y|X}(Y|X)=\begin{cases} e^{-(y-x)} & (y\ge x)\\ 0 & (\text{其他})\end{cases}$$

NOTE: $X \sim \Gamma(\alpha, \beta)$ $f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x)$

X's mgf: $\left(1-\frac{t}{\beta}\right)^{-\alpha}$

29

$E\left[e^{\frac{tX}{c}}\right] = \left(1-\frac{t}{c\beta}\right)^{-\alpha}$ ∴ $\frac{X}{c} \sim \Gamma(\alpha, \beta c)$

77. (b) $E[Y|X=\lambda] = \int y \cdot f_{Y|X}(y|x)\,dy$

$= \int_{y \geq x} y e^{-(y-x)}\,dy$

$\left(z = y-\lambda \quad \frac{dz}{dy} = 1\right)$

$= \int_{z \geq 0} (z+\lambda) e^{-z} \cdot dz$

$= \Gamma(2) + \lambda \int_{z \geq 0} e^{-z}\,dz$

$= \lambda + 1$

∴ $E[Y|X=\lambda] = X+1$

(c) ① $E[X|Y] = \frac{Y}{2}$

∴ 求 $\frac{Y}{2}$ 之分布 ∴

(註 $Z \sim \Gamma(\alpha, \beta)$
$f_Z(z) = \frac{\beta^\alpha z^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta z)$)

※ 可以用動差母函數證明

Y的邊際分布: $f_Y(y) = y e^{-y}$ ∴ $Y \sim \Gamma(2, 1)$ (Gamma分布)

∴ $\frac{Y}{2}$ 的分布 $\sim \Gamma(2, 2)$

(註 $T \sim \Gamma(2, 2)$, $f_T(t) = \frac{2^2 t}{\Gamma(2)} \exp(-2t)$ $(t>0)$

$= 4t e^{-2t}$ $(t>0)$

177.

(c) $E[Y|X] = X+1$. $X \sim e(1)$ ($\because f_X(x) = e^{-x}$)

$$Z = X+1 \quad \left(\begin{array}{l} X: 0 \to \infty \\ Z: 1 \to \infty \end{array} \right., \ \frac{dz}{dx} = 1$$

$$\int_0^\infty e^{-x}\,dx = \int_1^\infty e^{-(z-1)}\,dz$$

$\therefore$ $Z = X+1$ ($= E[Y|X]$)

$$f_Z(z) = \begin{cases} e^{-(z-1)} & (z \geq 1) \\ 0 & (\text{else}) \end{cases}$$

89.

hint) (1) $X \sim N(\mu, \sigma^2) \Leftrightarrow X$之 Moment Generating Function

$$M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

or (2) 若X為多變量 ($X = \begin{pmatrix} X_1 \\ \vdots \\ X_p \end{pmatrix}$) 隨機變數 $\Leftrightarrow$

$$\vec{X} \sim N(\vec{\mu}, \Sigma_{pp})$$

$\vec{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_p \end{pmatrix}$之 Moment Generating Function

$$M_X(\vec{t}) = M_X\begin{pmatrix} t_1 \\ \vdots \\ t_p \end{pmatrix} = \exp\left(\vec{\mu}^t t + \frac{\vec{t}^t \Sigma \vec{t}}{2}\right)$$

$\vec{t} = \begin{pmatrix} t_1 \\ \vdots \\ t_p \end{pmatrix}$

(31)

89. ① Y服從常態分佈：…… 證明

Y之 Moment Generating Function

$$E[e^{tY}] = E[e^{t(\alpha_1 X_1 + \cdots + \alpha_n X_n)}]$$

$$= E[e^{(t\alpha_1)X_1} \cdot e^{(t\alpha_2)X_2} \cdots e^{(t\alpha_n)X_n}]$$

($X_1 \sim X_n$ 獨立)

$$= \underbrace{E[e^{(t\alpha_1)X_1}]}_{M_{X_1}(\alpha_1 t)} \underbrace{E[e^{(t\alpha_2)X_2}]}_{M_{X_2}(\alpha_2 t)} \cdots \underbrace{E[e^{(t\alpha_n)X_n}]}_{M_{X_n}(\alpha_n t)}$$

$$\exp\left(\mu(\alpha_1 t) + \frac{\sigma_1^2 \alpha_1^2 t^2}{2}\right)$$

$$= \exp\left(\underbrace{\mu(\alpha_1 + \alpha_2 + \cdots + \alpha_n)}_{\mu_Y} t + \frac{\sigma_1^2\alpha_1^2 + \cdots + \sigma_n^2\alpha_n^2}{2} t^2\right)$$

σ_Y^2

Y的 Moment Generating Function 亦為

$$M_Y(t) = \exp\left((\sim)t + \frac{(\sim)}{2}t^2\right)$$

期望值　變異數

由此可知 Y服從常態分佈 （不用算 $M_Y'(0)$, $M_Y''(0)$ 喔）

② 而且 其期望值 $= \mu(\alpha_1 + \alpha_2 + \cdots \alpha_n)$，變異數 $= \sigma_1^2\alpha_1^2 + \cdots \sigma_n^2\alpha_n^2$

Note 若你習慣多變量的情況，可以這麼處理↓

$$E\left[e^{tY}\right] = E\left[e^{t\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}^t \vec{x}}\right]$$

$$= E\left[e^{\begin{pmatrix}t\alpha_1\\ \vdots\\ t\alpha_n\end{pmatrix}^t \vec{x}}\right] \qquad \vec{\theta} \overset{put}{=} \begin{pmatrix}t\alpha_1\\ \vdots\\ t\alpha_n\end{pmatrix}$$

$$= E\left[e^{\vec{\theta}^t \vec{x}}\right] = M_{\vec{X}}(\vec{\theta}) = \exp\left(\vec{\theta}^t \vec{\mu} + \frac{\vec{\theta}^t \Sigma \vec{\theta}}{2}\right)$$

$$\begin{pmatrix}t\alpha_1\\ \vdots\\ t\alpha_n\end{pmatrix}^t \begin{pmatrix}\mu_1\\ \vdots\\ \mu_n\end{pmatrix} = (\alpha_1\mu_1 + \cdots + \alpha_n\mu_n)\, t$$

$$(t\alpha_1\ t\alpha_2\ \cdots\ t\alpha_n)\underbrace{\begin{pmatrix}\sigma_1^2 & & 0\\ & \ddots & \\ 0 & & \sigma_n^2\end{pmatrix}}_{\Sigma}\begin{pmatrix}t\alpha_1\\ \vdots\\ t\alpha_n\end{pmatrix}\cdot\frac{1}{2}$$

$$= (\alpha_1^2\sigma_1^2 + \cdots + \alpha_n^2\sigma_n^2)\, t^2 \cdot \frac{1}{2}$$

($\because$ X_i, X_j $(i \neq j)$ 獨立)

註 $$\Sigma = \begin{pmatrix} V(X_1) & \cdots & cov(X_1, X_n)\\ cov(X_2, X_1) & V(X_2) & \\ \vdots & & \ddots \\ cov(X_n, X_1) & & V(X_n)\end{pmatrix}$$

第4章

100. ① exact mean & variance

$$E[Y] = E\left[\frac{1}{X}\right] = \int_{10}^{20} \frac{1}{10} \cdot \frac{1}{x}\, dx$$

$$= \frac{1}{10}\left[\log x\right]_{10}^{20} = \frac{1}{10}(\log 20 - \log 10)$$

$$= \frac{1}{10}\log 2 = 0.06931$$

$$E[Y^2] = E\left[\left(\frac{1}{X}\right)^2\right] = \int_{10}^{20} \frac{1}{10} \cdot \frac{1}{x^2}\, dx$$

$$= \frac{1}{10}\left[-\frac{1}{x}\right]_{10}^{20} = \frac{1}{200}$$

$$V[Y] = E(Y^2) - E(Y)^2 = \frac{1}{200} - \frac{1}{100}(\log 2)^2$$

$$= 0.0001954\ldots$$

② $g(x) = \frac{1}{x}$ 於 $x = \mu\ (E(X))$ Taylor 展開

$$g(X) \fallingdotseq g'(\mu)(x-\mu) + g(\mu)$$

$$\therefore E[g(X)] \fallingdotseq E[g'(\mu)(X-\mu) + g(\mu)] = g(\mu) = \frac{1}{\mu}$$

$$= \frac{1}{15}$$

這是常數, 可以無視掉,

$$V[g(X)] \simeq V[g'(\mu)(X-\mu) + g(\mu)] = (g'(\mu))^2 V(X)$$

注 $(V[ax+b] = a^2 V(X))$

$$= \frac{1}{225^2} \cdot \frac{25}{3} = \frac{1}{3 \cdot 45^2}$$

注 $\frac{1}{Y}$ … 並非服從均匀分布 $X = \frac{1}{Y}$

Y $10 \to 20$ $\quad x = \frac{1}{y}$

X: $\frac{1}{10} \to \frac{1}{20}$ $\quad \frac{dx}{dy} = \frac{-1}{y^2} = -x^2 \quad dy = \frac{-1}{x^2}\,dx$

$$1 = \int_{10}^{20} \frac{1}{10}\,dy = \int_{\frac{1}{10}}^{\frac{1}{20}} \frac{1}{10} \cdot \frac{-1}{x^2}\,dx$$

$$= \int_{\frac{1}{20}}^{\frac{1}{10}} \frac{1}{10x^2}\,dx$$

$$f_X(x) = \frac{1}{10x^2} \quad \left(\frac{1}{20} \le x \le \frac{1}{10}\right)$$

注 $\int_a^b f(x)\,dx$

$= -\int_b^a f(x)\,dx$

1

作業1 hint

可能有些人已經忘記跟機率有關的概念.
所以在此整理一下「作業1」會用到的知識.

① X: 隨機變數, $f(x)$… X之機率密度函數 (probability density function) (pdf)

$\Rightarrow Pr(X \in A) = \int_A f(x)dx$, $\int_{-\infty}^{\infty} f(x)dx = 1$ (全機率)

例 $X \sim U(0,1)$ (均勻分布) $\left(f(x) = \begin{cases} 1 & (0 \leq x \leq 1) \\ 0 & (\text{else}) \end{cases}\right)$

$A: [0.5, 0.75]$

$Pr(X \in A) = Pr(0.5 \leq X \leq 0.75)$

$= \int_{0.5}^{0.75} 1 \cdot dx = [x]_{0.5}^{0.75} = 0.25$

② X的期望值 $= E[X] = \int_{-\infty}^{\infty} x f(x) dx$

例 $X \sim U(0,1)$

$E[X] = \int_{-\infty}^{\infty} x \cdot f(x) dx$

$= \int_0^1 x \cdot 1\, dx + \underbrace{\int_{x \notin [0,1]} x \cdot 0\, dx}_{0}$ $\left(\because x \notin [0,1] \Rightarrow f(x) = 0\right)$

$= \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$

③ $E[X^2]=\int_{-\infty}^{\infty}\underline{\underline{x^2}}\,f(x)\,dx$ ⊛ (不可以寫成 $f(x^2)$ 喔!)

$E[X^3]=\int_{-\infty}^{\infty}\underline{\underline{x^3}}\,f(x)\,dx$

$E[X^4]=\int_{-\infty}^{\infty}\underline{\underline{x^4}}\,f(x)\,dx$ 　　㊟ $E[X^2]-E[X]^2$

$=V[X]$

變異數

$E[\boxed{g(x)}]=\int_{-\infty}^{\infty}\underline{g(x)}\,f(x)\,dx$

(動差母函數)

④ X的 Moment Generating Function

$M_X(t)=E[e^{tX}]=\int_{-\infty}^{\infty}e^{tx}f(x)\,dx$

變成t的函數

若 X, Y 的 Moment Generating Function 一致

⇒ X與Y的分佈相同.