105 統計學 期末考 參考答案

$\boxed{\text{I}}$

(1) $$\Lambda(X)=\frac{\prod_{j=1}^{n} f(x_j\mid\lambda_1)}{\prod_{j=1}^{n} f(x_j\mid\lambda_0)}=\frac{\lambda_1^{n}\exp\left(-\lambda_1(x_1+\cdots+x_n)\right)}{\lambda_0^{n}\exp\left(-\lambda_0(x_1+x_2+\cdots+x_n)\right)}$$

$$=\left(\frac{\lambda_1}{\lambda_0}\right)^{n}\exp\Big(-\underbrace{(\lambda_1-\lambda_0)}_{>0}\underbrace{(x_1+\cdots x_n)}_{\lambda\text{ 的充份統計量}}\Big)$$

$x_1+x_2+\cdots+x_n$：增加 $\Leftrightarrow$ $\Lambda(X)$：減少

$$\therefore\ C=\{(x_1,x_2,\cdots x_n)\mid \Lambda(X)\geq k\}$$
$$=\{(x_1,x_2,\cdots x_n)\mid x_1+x_2+\cdots+x_n\leq c\}$$

根據 Neyman Pearson's lemma,

$$\varphi(X)=\begin{cases}1 & (x_1\cdots x_n)\in C\\ 0 & (x_1\cdots x_n)\notin C\end{cases}$$

為對 $H_0:\lambda=\lambda_0$ vs 對 $H_1:\lambda=\lambda_1$ 之最強力檢定

$$E[\varphi(X)\mid H_0]=P_r((x_1,\cdots x_n)\in C\mid\lambda=\lambda_0)=\alpha$$

$= Pr(X_1+X_2+\cdots+X_n \leq C \mid \lambda=\lambda_0)$

$= Pr(\underbrace{2\lambda_0(X_1+\cdots X_n)} \leq C' \mid \lambda=\lambda_0)$

$2\lambda_0(X_1+X_2+\cdots X_n) \sim \Gamma(n, \frac{1}{2}) = \chi^2_{2n}$

$\therefore C' = \chi^{2^{-1}}_{2n}(\alpha)$

$\therefore C = \frac{1}{2\lambda_0}\chi^{2^{-1}}_{2n}(\alpha)$

$\therefore$ 棄卻域 $C = \{(X_1, X_2, \cdots X_n) \mid X_1+X_2+\cdots+X_n \leq \frac{1}{2\lambda_0}\chi^{2^{-1}}_{2n}(\alpha)\}$

(2) $\beta_\varphi(\lambda_1) \overset{def}{=} E[\varphi(X) \mid \lambda=\lambda_1]$ (檢定力)

$\beta_\varphi(\lambda_1) \geq \beta_{\varphi^*}(\lambda_1)$ $(\forall \varphi^*,\ E[\varphi^*(X) \mid \lambda=\lambda_0]=\alpha)$

($\because \varphi(X)$ 為顯著水準 α 下的最強力檢定)

對於任意 $\lambda > \lambda_0$，$\beta_\varphi(\lambda) \geq \beta_{\varphi^*}(\lambda)$ 均成立，

故此，對於 $H_0: \lambda=\lambda_0$ vs $H_1: \lambda>\lambda_0$，

$\varphi(X)$ 依然為最強力檢定.
(均勻)

$\therefore$ 證明完成.

㊄ 這題亦可取 $f(x)=\frac{1}{\lambda}\exp\left(\frac{-x}{\lambda}\right)$

(1) $X_1+X_2+\dots X_n \geq \frac{1}{2\lambda_0}\chi^{2\,-1}_{2n}(1-\alpha)$

(2) 同理

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(1) n_j $(j=1\sim m)$ 應為已知參數

$$Pr(X_1=x_1, X_2=x_2, \dots, X_m=x_m)$$

$$= {}_{n_1}C_{x_1} \cdot p_1^{x_1}(1-p_1)^{n_1-x_1} \cdot {}_{n_2}C_{x_2}\, p_2^{x_2}(1-p_2)^{n_2-x_2} \cdots {}_{n_m}C_{x_m} \cdot p_m^{x_m}(1-p_m)^{n_m-x_m}$$

① $p_1=p_2=\cdots p_m=p_0$ 時

$$L_0(p_0)=\left({}_{n_1}C_{x_1}\,{}_{n_2}C_{x_2}\cdots{}_{n_m}C_{x_m}\right)\cdot p_0^{x_1+x_2+\cdots x_m}(1-p_0)^{(n_1+n_2+\cdots n_m-(x_1+\cdots x_m))}$$

$$l_0(p_0)=\ln\left({}_{n_1}C_{x_1}\cdots{}_{n_m}C_{x_m}\right)+(x_1+x_2+\cdots x_m)\ln p_0$$

$$+\left(n_1+n_2+\cdots n_m-x_1-x_2\cdots-x_m\right)\ln(1-p_0)$$

$$\frac{\partial l_0}{\partial p_0}=\frac{1}{p_0}(x_1+x_2+\cdots+x_m)-\frac{1}{1-p_0}(n_1+n_2+\cdots+n_m-x_1-x_2\cdots-x_m)=0$$

$$\frac{\partial^2 l_0}{\partial p_0^2}=\frac{-1}{p_0^2}(x_1+\cdots x_m)-\frac{1}{(1-p_0)^2}(n_1+n_2+\cdots n_m-x_1-x_2\cdots x_m)<0$$

$$\frac{\partial l_0}{\partial p_0}=0 \Rightarrow (1-p_0)(x_1+x_2+\cdots+x_m)-p_0(n_1+n_2+\cdots n_m-x_1-x_2\cdots-x_m)=0$$

$$\Leftrightarrow x_1+x_2+\cdots+x_m=p_0(n_1+n_2+\cdots+n_m)$$

$$\therefore \hat{p}_{0\,MLE}=\frac{x_1+x_2+\cdots+x_m}{n_1+n_2+\cdots+n_m}$$

② 無限制條件時

$$L(p_1, p_2, \ldots, p_m) = {}_{n_1}C_{x_1}\, p_1^{x_1}(1-p_1)^{n_1-x_1} \cdots {}_{n_m}C_{x_m}\, p_m^{x_m}(1-p_m)^{n_m-x_m}$$

$$\ell(p_1, p_2, \ldots, p_m) = \ln({}_{n_1}C_{x_1}\, {}_{n_2}C_{x_2} \cdots {}_{n_m}C_{x_m}) + \sum_{j=1}^{m} x_j \ln p_j + (n_j - x_j)\ln(1-p_j)$$

$$\frac{\partial \ell}{\partial p_j} = \frac{x_j}{p_j} - \frac{n_j - x_j}{1-p_j}$$

$$\frac{\partial^2 \ell}{\partial p_j^2} = \frac{-x_j}{p_j^2} - \frac{n_j - x_j}{(1-p_j)^2} < 0$$

$$\frac{\partial \ell}{\partial p_j} = 0 \Rightarrow \hat{p}_j = \frac{x_j}{n_j}$$

對 $H_0: p_1 = p_2 = \cdots = p_m$ vs $H_1:$ not $p_1 = p_2 = \cdots = p_m$

之概似比檢定……

$$\Lambda(X) = \frac{\sup_{p_1 = p_2 = \cdots = p_m} L(p_1, p_2, \ldots, p_m)}{\sup_{0 \le p_1 \sim p_m \le 1} L(p_1, p_2, \ldots, p_m)} = \frac{\sup_{p_0} L_0(p_0)}{\sup_{p_1 \sim p_m} L(p_1, \ldots, p_m)}$$

$$= \frac{L_0(\hat{p}_0)}{L(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m)} = \frac{\hat{p}_0^{\,x_1+\cdots+x_m}(1-\hat{p}_0)^{n_1+n_2+\cdots+n_m-(x_1+\cdots+x_m)}}{\hat{p}_1^{x_1}(1-\hat{p}_1)^{n_1-x_1} \cdots \hat{p}_m^{x_m}(1-\hat{p}_m)^{n_m-x_m}}$$

$$C = \{(X_1, X_2, \ldots, X_m) \mid \Lambda(X) \le k\} \quad (\text{棄卻域})$$

(2) 、H_0下參數的自由度=1

、參數的個數=m

∴ 差=m-1

$-2\ln\Lambda(X) \xrightarrow{d} \chi^2_{m-1}$ （大樣本時）

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(1) $nF_n(x) = \sum_{j=1}^{n} \underbrace{I(X_j \leq x)}_{\text{bernoulli}(F(x))} \sim \text{Bin}(n, F(x))$

- $E[nF_n(x)] = nF(x) \quad \therefore \quad E[F_n(x)] = F(x)$
- $V[nF_n(x)] = n^2 V[F_n(x)] = nF(x)(1-F(x))$

$\therefore V[F_n(x)] = \frac{1}{n} F(x)(1-F(x))$

(2) $\mathrm{cov}[\overset{n}{F_n(u)}, \overset{n}{F_n(v)}]$

$= \mathrm{cov}\left[\sum_{i=1}^{n} I(X_i \leq u), \sum_{j=1}^{n} I(X_j \leq v)\right]$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{cov}[I(X_i \leq u), I(X_j \leq v)]$

$= \sum_{i=j}^{n} \mathrm{cov}[I(X_i \leq u), I(X_j \leq v)]$

$+ \sum\sum_{i \neq j} \mathrm{cov}[I(X_i \leq u), I(X_j \leq v)]$

$= \sum_{i=1}^{n} \mathrm{cov}[I(X_i \leq u), I(X_i \leq v)]$

$= n\,\mathrm{cov}[I(X_1 \leq u), I(X_1 \leq v)]$

$$n\Big(E[I(X_1 \leq u) I(X_1 \leq v)]$$

$$- E[I(X_1 \leq u)]\, E[I(X_1 \leq v)] \Big)$$

$$= n \Pr(X_1 \leq u \cap X_1 \leq v) - n \Pr(X_1 \leq u) \Pr(X_1 \leq v)$$

$$= n\big(F(\min\{u, v\}) - F(u)F(v) \big)$$

$$\therefore \operatorname{cov}[nF_n(u), nF_n(v)] = n\big(F(\min\{u, v\}) - F(u)F(v) \big)$$

$$\therefore \frac{1}{n}\big(F(\min\{u, v\}) - F(u)\, F(v) \big)$$

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(1) 全確率 $= \int_0^\infty f_X(x)\,dx = 1$

$Y \overset{\text{def}}{=} aX \quad (a>0) \qquad \frac{dy}{dx} = a$

$1 = \int_0^\infty f_X(x)\,dx = \int_0^\infty \underbrace{f_X\left(\frac{y}{a}\right)\cdot\frac{1}{a}}_{aX\text{の pdf}}\,dy$

$\Pr(aX \leq y) = \Pr\left(X \leq \frac{y}{a}\right) = \underbrace{F_X\left(\frac{y}{a}\right)}_{aX\text{の cdf}}$

$\therefore \dfrac{\frac{1}{a} f_X\left(\frac{y}{a}\right)}{1 - F_X\left(\frac{y}{a}\right)} \cdots$ aX の hazard 函数

X の hazard 函数 $= h(t) = \dfrac{f_X(t)}{1 - F_X(t)}$

$\therefore$ aX の hazard 函数 $= \frac{1}{a} h\left(\frac{t}{a}\right)$

(2) $X \sim U(0,1)$

$$F(x) = \begin{cases} 0 & (x<0) \\ x & (0 \leqq x \leqq 1) \\ 1 & (x>1) \end{cases}$$

$$f(x) = \begin{cases} 0 & (x<0 \text{ or } x>1) \\ 1 & (0 \leqq x \leqq 1) \end{cases}$$

- $x<0 \cdots \quad \dfrac{f(x)}{1-F(x)} = 0$
- $0 \leqq x < 1 \cdots \quad \dfrac{f(x)}{1-F(x)} = \dfrac{1}{1-x}$
- $x \geqq 1 \cdots$ 無法定義

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(3) $\dfrac{f(t)}{1-F(t)} = \dfrac{1}{1+t}$

$$\frac{d}{dt}\ln(1-F(t)) = \frac{-f(t)}{1-F(t)} = \frac{-1}{1+t}$$

$$\ln(1-F(t)) = -\ln(1+t) + C = \ln\left(\frac{k}{1+t}\right)$$

$$\therefore\ 1-F(t) = \frac{k}{1+t}$$

$$\therefore\ F(t) = 1-\frac{k}{1+t}$$

$$\therefore\ f(t) = \frac{k}{(1+t)^2}$$

$$\int_0^\infty f(x)\,dt = \left[\frac{-k}{1+t}\right]_0^\infty = k = 1$$

$$\therefore\ f(x) = \frac{1}{(1+x)^2}\ (x>0)$$

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(1) rank sum $= T = \frac{n(n+1)}{2} + \sum_{i=1}^{n}\sum_{j=1}^{n} I(X_i > Y_j)$

$$E[T] = E\left[\frac{n(n+1)}{2} + \sum_{i=1}^{n}\sum_{j=1}^{n} I(X_i > Y_j)\right]$$

$$= \frac{n(n+1)}{2} + n^2 \Pr(X_1 > Y_1)$$

$$\Pr(X_1 > Y_1) = \int_{y=0}^{y=\infty}\int_{x>y} \lambda_1 \exp(-\lambda_1 x)\, \lambda_2 \exp(-\lambda_2 y)\, dx\, dy$$

$$= \int_{y=0}^{y=\infty} \left[-\exp(-\lambda_1 x)\right]_{x=y}^{x=\infty} \lambda_2 \exp(-\lambda_2 y)\, dy$$

$$= \int_{y=0}^{y=\infty} \lambda_2 \exp(-(\lambda_1 + \lambda_2) y)\, dy$$

$$= \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

$$\therefore\ E[T] = \frac{n(n+1)}{2} + n^2 \cdot \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

(2) $U_i \stackrel{def}{=} \sum_{j=1}^{n} I(X_i > Y_j)$

$$T = \frac{1}{2} n(n+1) + \sum_{i=1}^{n} U_i$$

$$\therefore\ V[T] = V\left[\frac{1}{2} n(n+1) + \sum_{i=1}^{n} U_i\right]$$

$$= V\left[\sum_{i=1}^{n} U_i\right]$$

Chryu culture

$$= \sum_{i=1}^{n} V[U_i] + \sum\sum_{i \neq j} \mathrm{cov}[U_i, U_j]$$

$$= \underbrace{n\, V[U_1]}_{②} + n(n-1) \underbrace{\mathrm{cov}[U_1, U_2]}_{①} \quad \lrcorner$$

① $$\mathrm{cov}[U_1, U_2] = \mathrm{cov}\left[\sum_{i=1}^{n} I(X_1 > Y_i), \sum_{j=1}^{n} I(X_2 > Y_j)\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{cov}[I(X_1 > Y_i), I(X_2 > Y_j)]$$

($i \neq j \Rightarrow$ 独立)

$$= \sum_{j=1}^{n} \mathrm{cov}[I(X_1 > Y_j), I(X_2 > Y_j)]$$

$$= n\left(\underbrace{Pr(X_1, X_2 > Y_1)}_{\left(\frac{\lambda_2}{2\lambda_1 + \lambda_2}\right)} - \underbrace{Pr(X_1 > Y_1) \cdot Pr(X_2 > Y_1)}_{\left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^2} \right)$$

注

$$Pr(X_1, X_2 > Y_1) = \int_{y=0}^{y=\infty} \left[-\exp(-\lambda_1 x_1)\right]_{x_1=y}^{\infty} \left[-\exp(-\lambda_1 x_2)\right]_{x_2=y}^{\infty} \cdot \lambda_2 \exp(-\lambda_2 y)\, dy$$

$$= \int_{y=0}^{y=\infty} \lambda_2 \exp(-(2\lambda_1 + \lambda_2) y)\, dy$$

$$= \frac{\lambda_2}{2\lambda_1 + \lambda_2}$$

② $V[U_1] = V[\sum_{j=1}^{n} I(X_1 > Y_j)]$

$$= n\underbrace{V[I(X_1>Y_1)]}_{} + n(n-1)\underbrace{cov[I(X_1>Y_1), I(X_1>Y_2)]}_{}$$

bernoulli $\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)$

$\downarrow$

$$\frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2}$$

$\Pr(X_1>Y_1, X_1>Y_2) - \Pr(X_1>Y_1)\Pr(X_1>Y_2)$

$$\Pr(X_1>Y_1,Y_2) = \int_{x=0}^{x=\infty}\left(\int_0^x \lambda_2 \exp(-\lambda_2 y)\,dy\right)^2 \cdot \lambda_1 \exp(-\lambda_1 x)\,dx$$

$$= \int_0^\infty (1-\exp(-\lambda_2 x))^2 \cdot \lambda_1 \exp(-\lambda_1 x)\,dx$$

$$= \int_0^\infty \lambda_1 \exp(-\lambda_1 x)(1-2\exp(-\lambda_2 x)+\exp(-2\lambda_2 x))\,dx$$

$$= \lambda_1 \int_0^\infty (\exp(-\lambda_1 x) - 2\exp(-(\lambda_1+\lambda_2)x) + \exp(-(\lambda_1+2\lambda_2)x))\,dx$$

$$= \left(1 - \frac{2\lambda_1}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_1+2\lambda_2}\right)$$

$$\therefore V[U_1] = \frac{n\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2} + n(n-1)\left\{\left(1-\frac{2\lambda_1}{\lambda_1+\lambda_2}+\frac{\lambda_1}{\lambda_1+2\lambda_2}\right) - \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2\right\}$$

$\therefore$ ①+②

$$= \frac{n^2\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2} + n^2(n-1)\left\{\left(1-\frac{2\lambda_1}{\lambda_1+\lambda_2}+\frac{\lambda_1}{\lambda_1+2\lambda_2}\right) - \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2\right\}$$

$$+ n^2(n-1)\left\{\frac{\lambda_2}{2\lambda_1+\lambda_2} - \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2\right\}$$

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(1) $\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma_X}\right)^2 \sim \chi^2_{n-1}$

$\sum_{j=1}^{m}\left(\frac{Y_j-\bar{Y}}{\sigma_Y}\right)^2 \sim \chi^2_{m-1}$

$$\frac{\frac{1}{m-1}\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma_X}\right)^2}{\frac{1}{n-1}\sum_{j=1}^{m}\left(\frac{Y_j-\bar{Y}}{\sigma_Y}\right)^2} \sim F_{n-1,m-1}.$$

$$\frac{\sigma_Y^2}{\sigma_X^2}\cdot\frac{S_X^2}{S_Y^2} \sim F_{n-1,m-1}$$

(2) 令 ψ 為 $F_{n-1,m-1}$ 分佈的 cdf.

$$\Pr\left(\psi^{-1}\left(\frac{\alpha}{2}\right) \leq \frac{\sigma_Y^2}{\sigma_X^2}\cdot\frac{S_X^2}{S_Y^2} \leq \psi^{-1}\left(1-\frac{\alpha}{2}\right)\right) = 1-\alpha$$

$$\therefore \Pr\left(\frac{S_X^2}{S_Y^2}\cdot\frac{1}{\psi^{-1}(1-\frac{\alpha}{2})} \leq \frac{\sigma_X^2}{\sigma_Y^2} \leq \frac{S_X^2}{S_Y^2}\cdot\frac{1}{\psi^{-1}(\frac{\alpha}{2})}\right) = 1-\alpha$$

$$\therefore \left[\frac{S_X^2}{S_Y^2}\cdot\frac{1}{\psi^{-1}(1-\frac{\alpha}{2})},\ \frac{S_X^2}{S_Y^2}\cdot\frac{1}{\psi^{-1}(\frac{\alpha}{2})}\right]$$

(3) 利用概似比檢定：

$$\Lambda = \frac{\sup_{\substack{\sigma_X=\sigma_Y \\ \mu_X,\mu_Y}} L(\sigma_X,\sigma_Y,\mu_X,\mu_Y)}{\sup_{\substack{\sigma_X,\sigma_Y \\ \mu_X,\mu_Y}} L(\sigma_X,\sigma_Y,\mu_X,\mu_Y)} \le k$$

$\Leftrightarrow \quad \frac{S_X^2}{S_Y^2} \le k_1$ or $\frac{S_X^2}{S_Y^2} \ge k_2 \quad (k_1 < k_2)$

H_0 為真時 $\frac{S_X^2}{S_Y^2} \sim F_{n-1,m-1}$. $\therefore$ 取 $k_1 = \Psi^{-1}(\frac{\alpha}{2})$, $k_2 = \Psi^{-1}(1-\frac{\alpha}{2})$

棄卻域 $C = \left[\Psi^{-1}(\frac{\alpha}{2}), \Psi^{-1}(1-\frac{\alpha}{2})\right]^c$

$\left(= \left(-\infty, \Psi^{-1}(\frac{\alpha}{2})\right) \cup \left(\Psi^{-1}(1-\frac{\alpha}{2}), \infty\right)\right)$

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(1) Y_{ij} 的 機率密度函數

$$f_{ij}(y_{ij}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y_{ij}-\mu-\alpha_i)^2 \right)$$

$\{Y_{ij}\}$ $i=1\sim I,\ j=1\sim J$ 的 聯合機率密度函數。

$$\prod_{i=1}^{I}\prod_{j=1}^{J} f_{ij}(y_{ij})$$

利用 Lagrange 未定乘數法。

$$l(\mu, \alpha_1\sim\alpha_I, \sigma^2) = \left\{ \sum_{i=1}^{I}\sum_{j=1}^{J} -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(y_{ij}-\mu-\alpha_i)^2 \right\}$$

$$- \lambda(\alpha_1+\alpha_2+\cdots\alpha_I)$$

$$\frac{\partial l}{\partial \mu} = 0 \iff \sum_i\sum_j (y_{ij}-\mu-\alpha_i) = 0$$

$$\iff \left(\sum_i\sum_j y_{ij}\right) - \mu(IJ) = 0 \quad (\because \alpha_1+\cdots\alpha_I = 0)$$

$$\therefore \hat{\mu} = \frac{1}{IJ}\sum_i\sum_j y_{ij} = \bar{y}_{..}$$

$$\frac{\partial l}{\partial \sigma^2} = 0 \iff \sum_{i=1}^{I}\sum_{j=1}^{J} -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}(y_{ij}-\mu-\alpha_i)^2 = 0$$

$$\therefore \hat{\sigma}^2 = \frac{1}{IJ}\sum_{i}^{I}\sum_{j}^{J}(y_{ij}-\bar{y}_{..}-\hat{\alpha}_i)^2$$

$$\frac{\partial}{\partial \alpha_i} l = 0 \iff \sum_{j=1}^{J} \frac{1}{\sigma^2}(y_{ij} - \mu - \alpha_i) - \lambda = 0$$

$$\iff \sum_{j=1}^{J} (y_{ij} - \mu - \alpha_i) = \lambda \sigma^2$$

$$\iff \overline{y_{i\cdot}} - \mu - \alpha_i = \frac{1}{J} \lambda \sigma^2$$

$$(i = 1 \sim I)$$

$$\Sigma \downarrow \quad \therefore \overline{y_{i\cdot}} - \hat{\mu} - \hat{\alpha}_i = \frac{1}{J} \lambda \sigma^2$$

$$\underbrace{\sum_{i=1}^{I} (\overline{y_{i\cdot}} - \hat{\mu} - \hat{\alpha}_i)}_{=0} = \frac{I}{J} \lambda \sigma^2$$

$$(\hat{\mu} = \overline{y_{\cdot\cdot}})$$

$$\therefore \lambda = 0$$

$$\hat{\alpha}_i = \overline{y_{i\cdot}} - \hat{\mu} = \underline{\overline{y_{i\cdot}} - \overline{y_{\cdot\cdot}}}$$

$$\therefore \begin{cases} \hat{\mu}_{MLE} = \overline{y_{\cdot\cdot}} \\ \hat{\alpha}_i = \overline{y_{i\cdot}} - \overline{y_{\cdot\cdot}} \\ \hat{\sigma}^2 = \dfrac{1}{IJ} \displaystyle\sum_{i=1}^{I} \sum_{j=1}^{J} (y_{ij} - \overline{y_{i\cdot}})^2 \end{cases}$$

(2) $SS_W = \sum_{i}^{I}\sum_{j}^{J}(y_{ij}-\bar{y}_{i\cdot})^2$

注意 : $\sum_{j}^{J}\left(\frac{y_{ij}-\bar{y}_{i\cdot}}{\sigma}\right)^2 \sim \chi^2_{J-1}$

$\therefore \sum_{i=1}^{I}\sum_{j=1}^{J}\left(\frac{y_{ij}-\bar{y}_{i\cdot}}{\sigma}\right)^2 \sim \chi^2_{IJ-I}$

$\therefore E\left[\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\frac{y_{ij}-\bar{y}_{i\cdot}}{\sigma}\right)^2\right] = I(J-1)$

$\therefore E[SS_W] = \sigma^2 I(J-1)$

$SS_B = \sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{i\cdot}-\bar{y}_{\cdot\cdot})^2$

$= J\sum_{i=1}^{I}(\bar{y}_{i\cdot}-\bar{y}_{\cdot\cdot})^2 \qquad \bar{y}_{i\cdot} \sim N(\mu+\alpha_i, \frac{\sigma^2}{J})$

$= J\sum_{i=1}^{I}(\bar{w}_{i\cdot}+\alpha_i-\bar{w}_{\cdot\cdot} - \bar{\alpha}_{\cdot})^2$

$$\begin{cases} \bar{w}_{i\cdot} \overset{def}{=} \bar{y}_{i\cdot} - \alpha_i \sim N(\mu, \frac{\sigma^2}{J}) \\ \bar{w}_{\cdot\cdot} \overset{def}{=} \frac{\bar{w}_{1\cdot}+\cdots+\bar{w}_{I\cdot}}{n} \\ \quad = \bar{y}_{\cdot\cdot} - \bar{\alpha}_{\cdot} \sim N(\mu, \frac{\sigma^2}{IJ}) \end{cases}$$

$= J\sum_{i=1}^{I}\left\{(\bar{w}_{i\cdot}-\bar{w}_{\cdot\cdot})^2 + 2(\bar{w}_{i\cdot}-\bar{w}_{\cdot\cdot})(\alpha_i-\bar{\alpha}_{\cdot}) + (\alpha_i-\bar{\alpha}_{\cdot})^2\right\}$

$\therefore E[SS_B] = E\left[\sum_{i=1}^{I}\left\{\frac{\sqrt{J}(\bar{w}_{i\cdot}-\bar{w}_{\cdot\cdot})}{\sigma}\right\}^2 \cdot \sigma^2\right] + \quad \chi^2_{I-1}$

$J\sum_{i=1}^{I}E[\underbrace{2(\bar{w}_{i\cdot}-\bar{w}_{\cdot\cdot})}_{\to 0}(\alpha_i-\bar{\alpha}_{\cdot})] +$

$J\sum_{i=1}^{I}(\alpha_i-\bar{\alpha}_{\cdot})^2 = \sigma^2(I-1) + \sum_{i=1}^{I}(\alpha_i-\bar{\alpha}_{\cdot})^2$

$\bar{\alpha}_{\cdot}=0$ under H_0 $(\sum \alpha_i = 0)$

$\therefore \sigma^2(I-1) + J\sum_{i=1}^{I}\alpha_i^2$

(3)

	SS	df
Between	$\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{i\cdot}-\bar{y}_{\cdot\cdot})^2$	$I-1$
Within	$\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\bar{y}_{i\cdot})^2$	$I(J-1)$
Total	$\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\bar{y}_{\cdot\cdot})^2$	$IJ-1$

	MS	F
Between	$SS_B/I-1$	$MS_B/MS_W \quad (\sim F_{I-1,I(J-1)})$
Within.	$SS_W/I(J-1)$	*

(4) 棄却域 $C=\left\{\{y_{ij}\}_{i,j} \;\middle|\; \frac{MS_B}{MS_W} \geq \psi^{-1}(1-\alpha)\right\}$

ψ… $F_{I-1,I(J-1)}$ 分佈的 cdf