

電磁學（一）

（100學年度第一學期）

授課教師：賀培銘（辦公室 R812; 電話：3366-5192; Email: pmho@phys.ntu.edu.tw）

上課時間：星期一 第8節；星期四 第678節

上課地點：新物833

Office Hours: 星期四 1:00PM – 2:00PM（也可以email或當面另外約時間。）

助教：盧奕銓 <allen82217@gmail.com> ； 林鼎恩 <r99222068@ntu.edu.tw>

（可以用 email 與助教約時間討論。）

課程網頁：<https://ceiba.ntu.edu.tw/1001EM>

課本：Griffiths: Introduction to Electrodynamics (3rd ed.)

學期成績：作業、期中考、期末考 各佔約三分之一。

附註：

網頁提供之講義僅供作為課本之補充教材，而非替代品。

課本之完整內容均需熟讀。

所有課本及講義之練習題均應自行習作，並非限於指派作業之習題。

課程大綱：

週次	日期	單元主題
第1週	9/15	Chap. 0 Introduction/Orientation；Chap. 1 Vector Analysis
第2週	9/19, 9/22	Chap. 1 Vector Analysis（9/24 網路加選課程截止；9/25 網路退選課程截止）
第3週	9/26, 9/29	Chap. 2 Electrostatics
第4週	10/03, 10/06	Chap. 2 Electrostatics
第5週	10/13	Chap. 3 Special Techniques (10/10 國慶日放假.)
第6週	10/17, 10/20	Chap. 3 Special Techniques
第7週	10/24, 10/27	Chap. 3 Special Techniques；Chap. 4 Electric Fields in Matter
第8週	10/31, 11/03	Chap. 4 Electric Fields in Matter
第9週	11/07, 11/10	11/07 複習；11/10 期中考
第10週	11/14, 11/17	11/15 校慶停課；Chap. 4 Electric Fields in Matter
第11週	11/21, 11/24	Chap. 5 Magnetostatics
第12週	11/28, 12/01	Chap. 5 Magnetostatics
第13週	12/05, 12/8	Chap. 5 Magnetostatics（12/09 停修申請截止）
第14週	12/12, 12/15	Chap. 6 Magnetic Fields in Matter
第15週	12/19, 12/22	Chap. 6 Magnetic Fields in Matter
第16週	12/27, 12/30	Chap. 7 Electrodynamics
第17週	1/02, 1/05	Chap. 7 Electrodynamics
第18週	1/09, 1/12	1/09 複習；1/12 期末考

- 關於課程：

Go through Syllabus:
 課程網站的使用：公佈欄、討論區、作業、資源分享
 3+1的安排（其他國家的情形）
 作業遲交規定
 1/10 被當；沒有通融（二一沒有不好）
- 關於上課：

如何問問題。沒有笨問題。
 "My responses are limited... You must ask the right question!"
 "That, detective, is the right question"! [iRobot]
- 關於讀書：

唸書時間、方式。
 如何確定自己已經了解？
 你可以試著自己想一個例題，自己出題自己解。
 Morpheus: There is a difference between knowing the path and walking the path. [Matrix]
 Feynman: What I cannot create, I do not understand.
 Feynman: What does it mean to understand... I don't know.
 解題之前要先猜答案。
 解完題之後要看看自己猜的對不對，想一想是不是應該可以猜得更準？解題過程有沒有可能更精簡？有哪幾個步驟是最重要的？這一題的解法還可以解哪些問題？還有哪些解法可以解這個問題？
 楊振寧：了解一件事到像自己的手一樣的程度。
 （讓你的身心靈與物理之神合一。）
- 關於物理：

物理是有深度的，需要平時的潛移默化。
 學物理就像了解一個人，要用心體會。
 學物理像練功、會有 phase transition。
 Einstein: I think and think for months and years. Ninety-nine times, the conclusion is false. The hundredth time I am right.
 物理是抽象的，也是自然的。
 物理是一種態度，態度對了，處處皆是物理。
 物理是要理解的，不能靠死背。
 Agent Smith: Never send a human to do a machine's job. [Matrix]
 物理要研究的是事情和事情之間的關係，而且要把本質用最精簡的方式理解。
 Einstein: Everything should be made as simple as possible ... but not simpler.

要盡量練習從不同的角度看問題，才會對本質有好的理解。

- Einstein's favorite.
- E&M is everywhere.
- The role of E&M in theoretical physics:
 Classical vs Quantum,
 Classical vs Relativistic,
 Particle vs Field, (Einstein vs. Feynman)
 [Wheeler, Feynman, "Classical Electrodynamics in Terms of Direct Interparticle Action", Rev. in Mod. Phys. 21 (1949) 425.]
 Unification of Electricity, Magnetism and Optics,
 Gauge symmetry.
- E&M and weak interactions are unified by the electroweak theory.
 Standard Model includes (but does not unify) electroweak and strong interactions.
 Electroweak and strong interactions are unified in Grand Unified Theories (GUT).
 Electroweak, strong and gravitational interactions are unified in string theory.
 All the above are quantum field theories.

CM	QM
SR	QFT
GR	ST

Chapter 1

Vector Analysis

1.1 Vector Algebra

1.1.1 Basic Definitions

Einstein summation convention: repeated indices are summed over.

A vector space (or linear space) is a set \mathcal{V} of elements that is closed under superpositions, that is,

$$\text{if } \mathbf{A}, \mathbf{B} \in \mathcal{V}, \text{ then } a\mathbf{A} + b\mathbf{B} \in \mathcal{V} \quad \forall a, b \in \mathbb{C}. \quad (1.1)$$

In this book, unless otherwise specified, the term *vector* is reserved for vectors in the Euclidean 3 dimensional space of the physical world, and its tangent space. (See Sec. 1.1.3 below.)

A vector $\mathbf{A} \in \mathcal{V}$ can be specified in many ways:

$$\mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3 = A_i\hat{\mathbf{e}}_i, \quad (1.2)$$

$$\mathbf{A} = (A_1, A_2, A_3), \quad (\mathbf{A})_i = A_i, \quad (1.3)$$

where $\hat{\mathbf{e}}_i$ is a complete basis of the vector space \mathcal{V} . Roughly speaking, \mathbf{A} can be considered as the information of a magnitude $|\mathbf{A}|$ and a direction $\hat{\mathbf{A}}$.

1.1.2 Basic Operations

In an orthonormal basis, the basic operations are

$$(f\mathbf{A})_i = fA_i, \quad (1.4)$$

$$(\mathbf{A} \cdot \mathbf{B}) = A_iB_i, \quad (1.5)$$

$$\mathbf{A} \times \mathbf{B} = \epsilon_{ijk}\hat{\mathbf{x}}_iA_jB_k, \quad (\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk}A_jB_k, \quad (1.6)$$

where ϵ_{ijk} is the totally antisymmetrized tensor with $\epsilon_{123} = 1$.

Q: Let $\mathbf{A} = \hat{\mathbf{x}} + 2\hat{\mathbf{y}}$, $\mathbf{B} = \hat{\mathbf{y}} - 3\hat{\mathbf{z}}$. What are $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{A} \times \mathbf{B}$?

For example, $A_iB_{ij}C_{jk} = D_k$ is a short-hand for $\sum_i \sum_j A_iB_{ij}C_{jk} = D_k$.

This section is related to Sec. 1.1.1, Sec. 1.1.2 in Griffiths.

Electric field is a vector, magnetic field is a pseudo-vector, electric potential is a scalar, and magnetic potential is a vector.

In this course, we will mostly consider orthonormal basis for which $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$.

This section is related to Sec. 1.1.3 in Griffiths.

$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z,$
 $\mathbf{A} \times \mathbf{B} = \hat{\mathbf{x}}(A_yB_z - A_zB_y) + \hat{\mathbf{y}}(A_zB_x - A_xB_z) + \hat{\mathbf{z}}(A_xB_y - A_yB_x).$

Q: What are the geometrical meanings of these operations?

The magnitude or norm of a vector \mathbf{A} is

$$|\mathbf{A}| \equiv \sqrt{\mathbf{A} \cdot \mathbf{A}}. \quad (1.7)$$

We will use the notation that a hat implies a unit vector

$$\hat{\mathbf{A}} \equiv |\mathbf{A}|^{-1} \mathbf{A}. \quad (1.8)$$

Combinations of basic operations:

The volume of the parallelepiped spanned by $\mathbf{A}, \mathbf{B}, \mathbf{C}$:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \epsilon_{ijk} A_i B_j C_k \quad (1.9)$$

is totally antisymmetrized w.r.t. $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

BAC-CAB:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}). \quad (1.10)$$

Using the above, it is easy to prove that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0. \quad (1.11)$$

1.1.3 Scalar, Vector and Tensor Fields

The notion about tensor fields (including scalar and vector fields) is a combination of the notion about tensor and the notion about fields.

Rotations and Translations

The flat 3-dimensional space \mathbb{R}^3 is invariant under rotation, translation, and inversion.¹ They are transformations of the form

$$x_i \rightarrow x'_i = R_{ij} x_j + a_i, \quad (1.12)$$

where R_{ij} is a 3×3 matrix satisfying

$$R_{ik} R_{jk} = \delta_{ij}, \quad \text{or equivalently,} \quad R^T R = I = R R^T. \quad (1.13)$$

From the relation above, one deduces that

$$\det(R) = \pm 1. \quad (1.14)$$

¹ By this we mean that geometric notions such as distance, inner product of tangent vectors, etc. are invariant.

The norm has two nice properties:

(1) $|f\mathbf{A}| = |f||\mathbf{A}|$,

(2) $|\mathbf{A}'| = |\mathbf{A}|$,

where \mathbf{A}' is any rotation of \mathbf{A} .

The RHS of this identity is fixed up to an overall constant from the requirements: (1) tri-linearity, (2) rotation symmetry, (3) skew-symmetry of B, C .

This section is related to Sec. 1.1.4 and Sec. 1.1.5.

Eq.(1.13) is the condition that the transformation (1.12) preserves the lengths of all position vectors $|x|^2$. Equivalently, the inner products of any two vectors $x_i y_i$ is invariant. (Griffiths Prob. 1.8 (b))

A matrix R satisfies (1.13) and $\det(R) = 1$ if and only if R corresponds to a 3 dimensional rotation. A matrix R satisfies (1.13) and $\det(R) = -1$ if and only if R corresponds to a 3 dimensional rotation plus an inversion $(x, y, z) \rightarrow (-x, y, z)$.

Q: Show that both transformations $(x, y, z) \rightarrow (-x, y, z)$ and $(x, y, z) \rightarrow (-x, -y, -z)$ have $\det(R) = -1$ and they differ by a 3D rotation.

Tensors

Tensors are objects that transform covariantly under rotations.

$$V \rightarrow V' = V, \tag{1.15}$$

$$A_i \rightarrow A'_i = R_{ij}A_j, \tag{1.16}$$

$$T_{ij} \rightarrow T'_{ij} = R_{ik}R_{jl}T_{kl}, \tag{1.17}$$

$$S_{ijk} \rightarrow S'_{ijk} = R_{il}R_{jm}R_{kn}S_{lmn}, \tag{1.18}$$

$$\vdots \quad \vdots \quad \vdots \tag{1.19}$$

Tensors transform *linearly* under rotations.

In general, for a rank- n tensor $T_{i_1 \dots i_n}$,

$$T_{i_1 \dots i_n} \rightarrow T'_{i_1 \dots i_n} = R_{i_1 j_1} \dots R_{i_n j_n} T_{j_1 \dots j_n}. \tag{1.20}$$

Scalars and vectors are rank-0 and rank-1 tensors.

Under an *active* rotation, $\hat{\mathbf{x}}_i$'s are invariant, so that $\mathbf{A} = \hat{\mathbf{x}}_i A_i$ is changed upon a rotation to $\mathbf{A}' = \hat{\mathbf{x}}_i A'_i$. Under a *passive* rotation, $\hat{\mathbf{x}}_i$ is transformed like a vector, so that \mathbf{A} is invariant. That is $\mathbf{A} = \hat{\mathbf{x}}_i A_i = \hat{\mathbf{x}}'_i A'_i$.

Q: Show that \mathbf{A} is invariant under a passive rotation. (What is the transformation law for \mathbf{x}_i ?)

Here are some examples.

Position vector in physical space:

$$\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z = \hat{\mathbf{x}}_i x_i. \tag{1.21}$$

Separation vector in physical space:

$$\mathbf{z} = \mathbf{r} - \mathbf{r}', \tag{1.23}$$

Tangent vector in physical space:

$$\mathbf{v} = \hat{\mathbf{x}}v_x + \hat{\mathbf{y}}v_y + \hat{\mathbf{z}}v_z = \hat{\mathbf{x}}_i v_i. \tag{1.24}$$

A pseudo-scalar and a pseudo-vector transform under (1.12) as

$$W \rightarrow W' = \det(R)W, \tag{1.25}$$

$$B_i \rightarrow B'_i = \det(R)R_{ij}B_j. \tag{1.26}$$

A column of 3 components is not necessarily a vector.

Whether a physical quantity is a scalar, or a component of a vector or a tensor, is a physical question to be answered by experiments/observations.

An active transformation changes the physical state, while a passive transformation is merely a change of coordinates.

We will also use the notation

$$z = |\mathbf{z}|, \quad \hat{\mathbf{z}} = \frac{\mathbf{z}}{z}. \tag{1.22}$$

Q: Check that $\mathbf{A} \cdot \mathbf{B}$ is a scalar and $\mathbf{A} \times \mathbf{B}$ is a pseudo-vector if \mathbf{A} and \mathbf{B} are both vectors.

Q: Design an experiment to test whether the magnetic field is a pseudo-vector.

Invariant Tensors

A tensor $T_{i_1 \dots i_n}$ is invariant under a transformation R if

$$T'_{i_1 \dots i_n} \equiv R_{i_1 j_1} \cdots R_{i_n j_n} T_{j_1 \dots j_n} = T_{i_1 \dots i_n}. \quad (1.27)$$

There are two constant tensors invariant under rotations

$$\delta_{ij}, \quad \epsilon_{ijk},$$

δ_{ij} is a tensor but ϵ_{ijk} is a pseudo-tensor.

which are defined by

$$\delta_{ij} = \begin{cases} 1 & (i = j), \\ 0 & (i \neq j), \end{cases} \quad (1.28)$$

$$\epsilon_{ijk} = \begin{cases} 1 & (i, j, k) = \text{cyclic perm. of } (1, 2, 3), \\ -1 & (i, j, k) = \text{cyclic perm. of } (2, 1, 3), \\ 0 & i = j \text{ or } j = k \text{ or } k = i. \end{cases} \quad (1.29)$$

δ_{ij} is symmetric and ϵ_{ijk} is totally antisymmetrized

$$\delta_{ij} = \delta_{ji}, \quad (1.30)$$

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{jik} = -\epsilon_{kji} = -\epsilon_{ikj}. \quad (1.31)$$

Q: Show that $\delta_{ij} A_j = A_i$, and $\epsilon_{ijk} A_j A_k = 0$ for arbitrary A_i .

δ_{ij} , ϵ_{ijk} and their combinations are the only *constant* tensors.

There is an important relation that relates them ²

$$\epsilon_{ijm} \epsilon_{klm} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}, \quad (1.33)$$

which leads to the BAC-CAB formula.

Q: What do you get by contracting the two indices of δ_{ij} with two vectors?

What do you get by contracting two of the three indices of ϵ_{ijk} with two vectors, or by contracting all indices of ϵ_{ijk} with three vectors?

Q: Compute

$$\delta_{ij} \delta_{jk}, \quad \delta_{ij} \epsilon_{jkl}, \quad \delta_{ij} \epsilon_{ijk}, \quad \epsilon_{ijk} \epsilon_{jkl}. \quad (1.34)$$

² The only other nontrivial relation that the epsilon tensor satisfies is the Plucker relation:

$$\epsilon_{i_1 i_2 j_1} \epsilon_{j_2 j_3 j_4} - \epsilon_{i_1 i_2 j_2} \epsilon_{j_1 j_3 j_4} + \epsilon_{i_1 i_2 j_3} \epsilon_{j_1 j_2 j_4} - \epsilon_{i_1 i_2 j_4} \epsilon_{j_1 j_2 j_3} = 0. \quad (1.32)$$

Q: If A_{ij} is symmetric and B_{ij} is anti-symmetric, prove that $A_{ij}B_{ij} = 0$.

Q: Show that the determinant of a 3×3 matrix M can be expressed as

$$\det(M)\epsilon_{ijk} = M_{il}M_{jm}M_{kn}\epsilon_{lmn}. \quad (1.35)$$

(Hint: First show that $\det(M) = M_{1l}M_{2m}M_{3n}\epsilon_{lmn}$.)

Q: Use the relation above to check that

$$M_{il}M_{jm}\epsilon_{lmn} = \det(M)M_{nk}^{-1}\epsilon_{ijk}. \quad (1.36)$$

Fields

A *field* is a map that gives an element in \mathcal{T} for every point in space (or spacetime). \mathcal{T} is a set, e.g. \mathbb{R}^n , \mathbb{C}^n , S^2 , \mathbb{Z}_2 , etc. Often we assume that this map is smooth, in that case \mathcal{T} is normally a continuous (rather than discrete) space.

The vector space for a vector field $\mathbf{A}(\mathbf{r})$ at a point \mathbf{r} can not be identified with the 3D physical space. (In fact, unless our world is exactly flat, the physical space is not even a vector space.) To describe the configuration of a vector field \mathbf{A} , one should first associate a vector space $\mathcal{V}_{\mathbf{r}}$ to every point \mathbf{r} in space-time, and then assign a vector $\mathbf{A}(\mathbf{r})$ on each $\mathcal{V}_{\mathbf{r}}$. In general the basis $\hat{\mathbf{e}}_i(\mathbf{r})$ of $\mathcal{V}_{\mathbf{r}}$ also depends on \mathbf{r} . For flat spaces, it is natural to use the same notation (as we did with $\hat{\mathbf{x}}_i$) for position, separation and tangent vectors.

One refers to a field as a scalar, vector, or tensor field, if it transforms under the transformation (1.12) as

$$f(\mathbf{r}) \rightarrow f'(\mathbf{r}') = f(\mathbf{r}), \quad (1.37)$$

$$A_i(\mathbf{r}) \rightarrow A'_i(\mathbf{r}') = R_{ij}A_j(\mathbf{r}), \quad (1.38)$$

$$T_{ij}(\mathbf{r}) \rightarrow T'_{ij}(\mathbf{r}') = R_{ik}R_{jl}T_{kl}(\mathbf{r}). \quad (1.39)$$

A pseudo-scalar field and a pseudo-vector field transform under (1.12) as

$$W(\mathbf{r}) \rightarrow W'(\mathbf{r}') = \det(R)W(\mathbf{r}), \quad (1.40)$$

$$B_i(\mathbf{r}) \rightarrow B'_i(\mathbf{r}') = \det(R)R_{ij}B_j(\mathbf{r}). \quad (1.41)$$

Griffiths Prob. 1.10.

Q: How do pseudo-scalar fields and pseudo-vector fields transform under the transformation $(x, y, z) \rightarrow (-x, y, z)$?

Covariance of Physical Laws

A physical theory respects a symmetry if all the equations of motion are *covariant*. That is, the equations of motion before and after the symmetry transformation must be satisfied by the same set of solutions.

Q: Which of the following is (are) not covariant?

$$A_i = 1, \quad T_{ij} = 0, \quad A_i B_j = T_{ij}, \quad A_{ij} B_j = C_i, \quad A_{ij} B_j = C_j. \quad (1.42)$$

Maxwell Equations

Let

$$\nabla \equiv \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i}, \quad (1.43)$$

the Maxwell equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B}, \quad (1.44)$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \mathbf{E}. \quad (1.45)$$

Q: Check that ∂_i is a vector, and that ∇ is invariant.

1.2 Differential Calculus

For a function $f(x_1, x_2, \dots, x^n)$ of n variables, we define $\partial_i f(x)$ by

Read Sec. 1.2 and Sec. 1.3

$$df(x) = dx_1 \partial_1 f(x) + dx_2 \partial_2 f(x) + \dots + dx_n \partial_n f(x) = dx_i \partial_i f(x). \quad (1.46)$$

That is, $\partial_i f$ is the ratio of the change in f and the change in x_i when all other variables $x_{j \neq i}$ are held fixed.

1.2.1 General Properties of Derivations

The only important algebraic properties of derivations are the Leibniz rule, the chain rule and the commutativity:

$$\partial_i f(x) = f(x) \partial_i + (\partial_i f(x)), \quad (1.47)$$

$$\partial_i f(g(x)) = (\partial_i g(x)) f'(g(x)), \quad (1.48)$$

$$\partial_i \partial_j = \partial_j \partial_i. \quad (1.49)$$

where $f'(g) \equiv \frac{df}{dg}$.

When we change coordinates, the derivatives change according to the chain rule. Upon the change of coordinates by

$$x^i \rightarrow x'^i = x'^i(x), \quad (1.50)$$

their derivatives change by

In (1.47) $f(x)$ is viewed as an operator acting on functions on the right (by multiplication).

Notice that even if some of the variables are not changed, their partial derivatives may change due to the change of other variables.

$$\frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i} \frac{\partial}{\partial x^j}, \quad (1.51)$$

where $\frac{\partial x^j}{\partial x'^i}$ can be computed using the inverse function $x^j(x')$ of the transformation (1.50), or it can be computed as the inverse matrix of $\frac{\partial x'^j}{\partial x^i}$.

Q: Prove that

$$\frac{\partial x'^j}{\partial x^i} \frac{\partial x^k}{\partial x'^j} = \delta_i^k. \quad (1.52)$$

Q: For

$$x' = f(x, y), \quad y' = y, \quad (1.53)$$

show that the partial derivatives with respect to y and y' are different:

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} - \left(\frac{\partial f}{\partial x}\right)^{-1} \left(\frac{\partial f}{\partial y}\right) \frac{\partial}{\partial x}. \quad (1.54)$$

In this section we will discuss how partial derivatives ∂_i can act on scalar and vector fields. We will focus on the following operations:

Gradient, divergence and curl:

$$\nabla f = \hat{\mathbf{x}}_i (\partial_i f), \quad (1.55)$$

$$\nabla \cdot \mathbf{A} = (\partial_i A_i), \quad (1.56)$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{x}}_i \epsilon_{ijk} \partial_j A_k. \quad (1.57)$$

These operations are special because they are the first derivatives whose action on scalars/vectors always produce tensors.

1.2.2 Gradient

Recall Taylor expansion of a function with multi-variables

$$f(\mathbf{r} + d\mathbf{l}) = f(\mathbf{r}) + dx_i \partial_i f(\mathbf{r}) + \dots, \quad (1.58)$$

where $d\mathbf{l} = \hat{\mathbf{x}}_i dx_i$.

Thus

$$f(\mathbf{r} + d\mathbf{l}) - f(\mathbf{r}) = \nabla f(\mathbf{r}) \cdot d\mathbf{l}. \quad (1.59)$$

This is the change in the value of f over an infinitesimal displacement $d\mathbf{l}$ at \mathbf{r} .

Patching infinitesimal displacements $d\mathbf{l}$ together to form a path \mathcal{P} from \mathbf{r}_1 to \mathbf{r}_2 , we obtain

$$\int_{\mathcal{P}} d\mathbf{l} \cdot (\nabla f) = f(\mathbf{r}_2) - f(\mathbf{r}_1). \quad (1.60)$$

Q: Argue that $\nabla r = \hat{\mathbf{r}}$ using geometric notions. Then check it by explicit computation.

As a result of (1.60),

Q: Show that if f is a scalar, ∇f is a vector.

$$\oint d\mathbf{l} \cdot (\nabla f) = 0.$$

Q: Find the most general solution of $f(x, y, z)$ for

$$\nabla f(x, y, z) = a \hat{\mathbf{r}}r \quad (1.61)$$

for a given constant a .

1.2.3 Curl

$$\sum \mathbf{A}(\mathbf{r}) \cdot d\mathbf{l} = (\nabla \times \mathbf{A}) \cdot d\mathbf{a}. \quad (1.62)$$

This gives how much the vector field \mathbf{A} curls around an infinitesimal area element $d\mathbf{a}$ at \mathbf{r} .

For the area element spanned by two vectors $d\mathbf{l}_1, d\mathbf{l}_2$, we have

$$d\mathbf{a} = d\mathbf{l}_1 \times d\mathbf{l}_2, \quad (1.63)$$

and it is bounded by 2 pairs of linear elements $\pm d\mathbf{l}_1, \pm d\mathbf{l}_2$. The sum $\mathbf{A} \cdot d\mathbf{l}$ over the first pair $\pm d\mathbf{l}_1$ is

$$\mathbf{A}(\mathbf{r} + d\mathbf{l}_2) \cdot (-d\mathbf{l}_1) + \mathbf{A}(\mathbf{r}) \cdot d\mathbf{l}_1 = -\partial_j A_i(\mathbf{r})(d\mathbf{l}_1)_i(d\mathbf{l}_2)_j. \quad (1.64)$$

Similarly, the sum $\mathbf{A} \cdot d\mathbf{l}$ over the 2nd pair $\pm d\mathbf{l}_2$ is

$$\mathbf{A}(\mathbf{r} + d\mathbf{l}_1) \cdot d\mathbf{l}_2 + \mathbf{A}(\mathbf{r}) \cdot (-d\mathbf{l}_2) = \partial_i A_j(\mathbf{r})(d\mathbf{l}_1)_i(d\mathbf{l}_2)_j. \quad (1.65)$$

The total sum is therefore

$$\sum \mathbf{A}(\mathbf{r}) \cdot d\mathbf{l} = (\partial_i A_j - \partial_j A_i)(d\mathbf{l}_1)_i(d\mathbf{l}_2)_j = \epsilon_{ijk} \partial_i A_j \epsilon_{kmn} (d\mathbf{l}_1)_m (d\mathbf{l}_2)_n = (\nabla \times \mathbf{A}) \cdot d\mathbf{a}. \quad (1.66)$$

Patching infinitesimal area elements together to form a surface \mathcal{S} bounded by a closed curve \mathcal{C} , we have

$$\int_{\mathcal{S}} d\mathbf{a} \cdot (\nabla \times \mathbf{A}) = \oint_{\mathcal{C}} d\mathbf{l} \cdot \mathbf{A}. \quad (1.67)$$

If \mathbf{A} represents the flow of a fluid, i.e., $\hat{\mathbf{A}}$ is the direction of the flow and $|\mathbf{A}|$ the velocity, then $\nabla \times \mathbf{A}$ is an attempt to account for the flow by superposing infinitely many infinitesimal vortices together with a certain distribution. The direction of $\nabla \times \mathbf{A}$ is determined by the right hand rule, and $|\nabla \times \mathbf{A}|$ is proportional to the density of the vortices (imagining that each vortex are of the same magnitude).

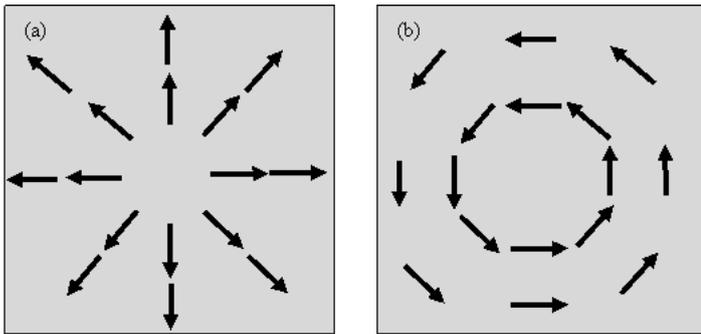


Figure 1.1: (a) a vector field with positive divergence at the center. (b) a vector field with positive curl at the center.

1.2.4 Divergence

$$\sum \mathbf{A}(\mathbf{r}) \cdot d\mathbf{a} = \nabla \cdot \mathbf{A}(\mathbf{r}) d\tau. \quad (1.68)$$

This is the amount of “flux” generated by sources within an infinitesimal volume element $d\tau$ nearby the point \mathbf{r} .

For the parallelogram spanned by $\hat{\mathbf{x}}dx$, $\hat{\mathbf{y}}dy$, $\hat{\mathbf{z}}dz$, we have the infinitesimal volume element $d\tau = dxdydz$ bounded by 3 pairs of rectangular infinitesimal area elements $(\pm\hat{\mathbf{x}}dydz)$, $(\pm\hat{\mathbf{y}}dzdx)$, $(\pm\hat{\mathbf{z}}dxdy)$. The sum of $\mathbf{A} \cdot d\mathbf{a}$ over the pair of area elements $\hat{\mathbf{x}}dydz$ is

$$\mathbf{A}(\mathbf{r} + \hat{\mathbf{x}}dx) \cdot (\hat{\mathbf{x}}dydz) + \mathbf{A}(\mathbf{r}) \cdot (-\hat{\mathbf{x}}dydz) = \partial_x A_x(\mathbf{r}) dxdydz. \quad (1.69)$$

Apparently, the sum over all 3 pairs of area elements is

$$(\partial_x A_x + \partial_y A_y + \partial_z A_z) dxdydz = (\nabla \cdot \mathbf{A}) d\tau. \quad (1.70)$$

Q: Generalize the derivation above to a parallelepiped spanned by 3 vectors $d\mathbf{l}_1$, $d\mathbf{l}_2$, $d\mathbf{l}_3$.

Q: Compute $\nabla \cdot \mathbf{A}$ and $\nabla \times \mathbf{A}$ for (1) $\mathbf{A} = \hat{\mathbf{r}}f(r)$, and (2) $\mathbf{A} = \hat{\phi}f(r)$.

Q: Compute $\nabla V(\mathbf{r})$ and $\nabla^2 V(\mathbf{r})$ for (1) $V(\mathbf{r}) = f(r)$, and (2) $V(\mathbf{r}) = f(\cos \phi_0 x + \sin \phi_0 y)$.

Patching infinitesimal volume elements together to form a volume \mathcal{V} with a boundary \mathcal{S} , we get

$$\int_{\mathcal{V}} d\tau (\nabla \cdot \mathbf{A}) = \oint_{\mathcal{S}} d\mathbf{a} \cdot \mathbf{A}. \quad (1.71)$$

Q: Check the following identities:

$$\nabla \times (\nabla V) = 0, \quad (1.72)$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0, \quad (1.73)$$

$$\nabla(fg) = (\nabla f)g + f(\nabla g), \quad (1.74)$$

$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + (\nabla f) \cdot \mathbf{A}, \quad (1.75)$$

$$\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} + (\nabla f) \times \mathbf{A}. \quad (1.76)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}). \quad (1.77)$$

Q: Expand $\nabla \times (\mathbf{A} \times \mathbf{B})$ so that cross product is avoided.

Again, if \mathbf{A} represents a flow, then $\nabla \cdot \mathbf{A}$ is an attempt to account for the flow by infinitely many infinitesimal openings (water sources) where fluid goes out (+) or in (-). $\nabla \cdot \mathbf{A}$ gives the density of such infinitesimal sources, assuming that each source is of the same magnitude.

These identities can be quickly derived whenever you need them by noting that ∇ is at the same time a vector and an operator. Inner product or cross product with ∇ refers to its vectorial nature, and it acts on the right as an operator via Leibniz rule.

1.3 Curvilinear Coordinates

The only curvilinear coordinates we will use in this course are spherical coordinates and cylindrical coordinates.

In **spherical coordinates**,

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z), \quad \phi = \tan^{-1}(y/x),$$

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta,$$

Read Sec. 1.4 and Appendix A.

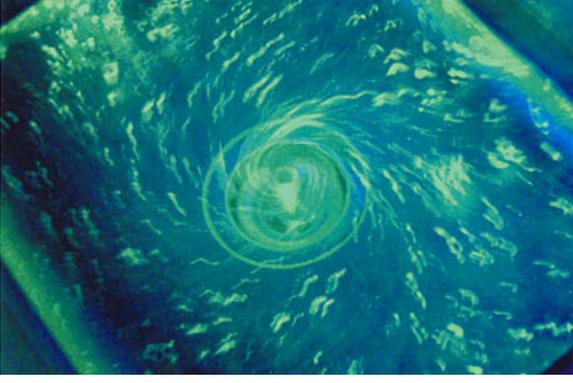


Figure 1.2: The velocity field corresponding to a vortex in a sink has negative divergence at the center (sink) and positive or negative curl (depending on the orientation of the flow) at the center. The curl off the center can be positive or negative (depending on whether the vortex is forced) The divergence off the center is zero, assuming that the fluid is incompressible.

$$\begin{aligned}
 \nabla V &= \frac{\partial V}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\phi}, \\
 \nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} A_\phi, \\
 \nabla \times \mathbf{A} &= \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta A_\phi) - \frac{\partial}{\partial \phi} A_\theta \right] \hat{\mathbf{r}} + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \phi} A_r - \frac{\partial}{\partial r} (r A_\phi) \right] \hat{\theta} \\
 &\quad + \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial}{\partial \theta} A_r \right] \hat{\phi}, \\
 \nabla^2 V &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}.
 \end{aligned}$$

In cylindrical coordinates,

$$\begin{aligned}
 s &= \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}(y/x), \quad z = z, \\
 x &= s \cos \theta, \quad y = s \sin \theta, \quad z = z.
 \end{aligned} \tag{1.78}$$

One should have tried to avoid using the same symbol z in both Cartesian and cylindrical coordinates.

$$\begin{aligned}
 \nabla V &= \frac{\partial V}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial V}{\partial \phi} \hat{\phi} + \frac{\partial V}{\partial z} \hat{\mathbf{z}}, \\
 \nabla \cdot \mathbf{A} &= \frac{1}{s} \frac{\partial}{\partial s} (s A_s) + \frac{1}{s} \frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z} A_z, \\
 \nabla \times \mathbf{A} &= \left[\frac{1}{s} \frac{\partial}{\partial \phi} A_z - \frac{\partial}{\partial z} A_\phi \right] \hat{\mathbf{s}} + \left[\frac{\partial}{\partial z} A_s - \frac{\partial}{\partial s} A_z \right] \hat{\phi} \\
 &\quad + \frac{1}{s} \left[\frac{\partial}{\partial s} (s A_\phi) - \frac{\partial}{\partial \phi} A_s \right] \hat{\mathbf{z}}, \\
 \nabla^2 V &= \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial V}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}.
 \end{aligned}$$

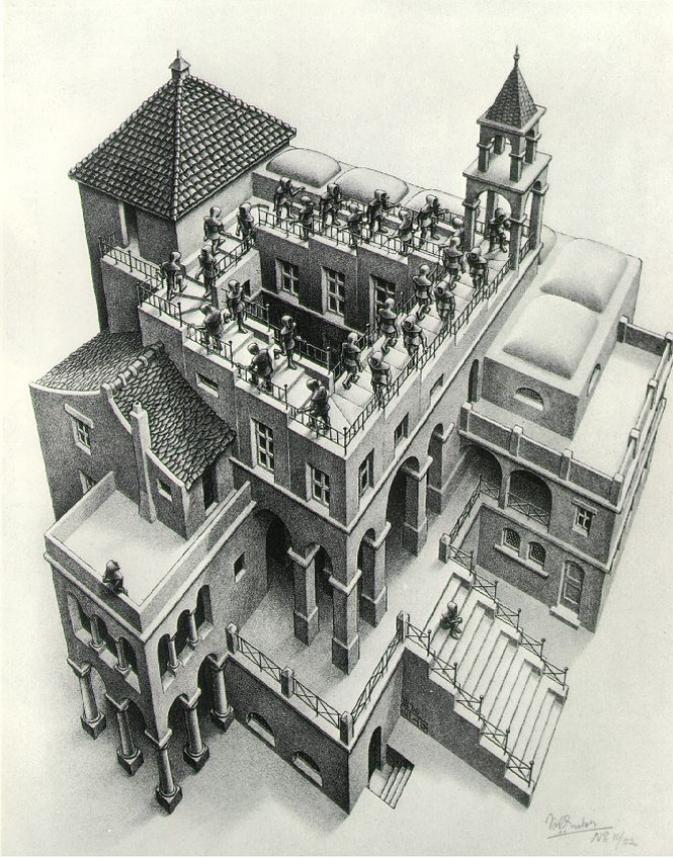


Figure 1.3: The relation $\nabla \times \nabla V = 0$ is violated in the imaginary world of Escher.

1.4 Dirac Delta Function

A *distribution* $D(x)$ is to be defined by

$$\int_{-\infty}^{\infty} dx f(x)D(x) \in \mathbb{C}, \quad (1.79)$$

that is, it defines a linear map from a well defined smooth function $f(x)$ to a number in \mathbb{C} .

Change of variables and integration by parts are always assumed to be legitimate manipulations.

Distributions are not necessarily well defined functions.

Products of distributions may not be well defined.

Two distributions D_1, D_2 are considered the same distribution if

$$\int dx f(x)D_1 = \int dx f(x)D_2 \quad (1.80)$$

for all well defined functions $f(x)$.

The Dirac Delta function is a distribution defined by

Read Sec. 1.5.

The function $F(x)$ that equals 1 at $x = 0$ and vanishes everywhere else is considered the same distribution as the constant 0.

$$\int_{-\infty}^{\infty} f(x)\delta(x) = f(0). \quad (1.81)$$

Q: The Gaussian function is defined as

$$G_a(x) = \frac{1}{\sqrt{2\pi a}} e^{-x^2/2a}. \quad (1.82)$$

For $a > 0$, show that $\int_{-\infty}^{\infty} dx G_a(x) = 1$. Argue that

$$\delta(x) = \lim_{a \rightarrow 0} G_a(x) \quad (1.83)$$

by evaluating $\int dx f(x)G_a(x)$ for an analytic function

$$f(x) = f(0) + xf'(0) + \frac{1}{2}x^2f''(0) + \dots. \quad (1.84)$$

Q: Write an expression for the electric charge density $\rho(\mathbf{r})$ of a point charge q with the trajectory $\mathbf{R}(t)$. What is the current density?

Q: For a positive integer n ,

$$\int_{-\infty}^{\infty} dx f(x) \frac{d^n}{dx^n} \delta(x - x_0) = ? \quad (1.85)$$

Q: For a function $f(x)$ with n zeros at $x = x_1, \dots, x_n$, find A_n 's in the expression

$$\delta(f(x)) = \sum_{i=1}^n A_n \delta(x - x_n). \quad (1.86)$$

Q: Suppose two functions $f(x, y)$ and $g(x, y)$ are both zero only when $(x, y) = (x_0, y_0)$, find A in the expression

$$\delta^{(2)}(f(x, y), g(x, y)) = A\delta^{(2)}(x - x_0, y - y_0). \quad (1.87)$$

Dirac delta functions can be defined in higher dimensions

$$\delta^{(3)}(\mathbf{r}) \equiv \delta(x)\delta(y)\delta(z). \quad (1.88)$$

The following identities will become important in the next chapter

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi\delta^{(3)}(\mathbf{r}), \quad (1.89)$$

$$\nabla^2 \left(\frac{1}{r} \right) = -4\pi\delta^{(3)}(\mathbf{r}). \quad (1.90)$$

To prove (1.89), first show that $\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = 0$ everywhere $r > 0$. Next show that for the inside \mathcal{V} of a sphere of radius R ,

$$\int_{\mathcal{V}} d\tau f(r) \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi f(0) \quad (1.91)$$

independent of R for any well defined function $f(r)$. Since $\nabla \cdot \left(\frac{\mathbf{r}}{r^2}\right)$ is invariant under rotations, one can argue that (1.89) is correct.

Eq.(1.90) is straightforward to prove.

Due to (1.90), there is (almost) always a solution V to the equation

$$\nabla^2 V = -\rho \quad (1.92)$$

for a given function ρ . First note that after a translation by a vector \mathbf{r}' , eq.(1.90) becomes

$$\nabla^2 \left(\frac{1}{z}\right) = -4\pi\delta^{(3)}(\mathbf{z}), \quad (1.93)$$

where

$$\mathbf{z} = \mathbf{r} - \mathbf{r}', \quad z = |\mathbf{r} - \mathbf{r}'|. \quad (1.94)$$

This allows us to write the solution to the equation above as

$$V(\mathbf{r}) = \frac{1}{4\pi} \int d^3\tau' \frac{\rho(\mathbf{r}')}{z}. \quad (1.95)$$

You can check that this is indeed a solution by plugging it back into the differential equation for V . Thus we can always solve that differential equation at least whenever the integral (1.95) is well defined (does not diverge).

This is one of the useful applications of delta functions.

1.5 The Theory of Vector Fields

Helmholtz theorem:

For any (sufficiently smooth) vector field \mathbf{F} ,

$$\mathbf{F} = -\nabla U + \nabla \times \mathbf{W}, \quad (1.96)$$

for some U and \mathbf{W} (not unique).

Special case:

Let

$$\mathbf{C} \equiv \nabla \times \mathbf{F}, \quad D \equiv \nabla \cdot \mathbf{F}. \quad (1.97)$$

If \mathbf{F} goes to zero, and \mathbf{C} , D go to zero faster than $1/r^2$ as $r \rightarrow \infty$, F is uniquely determined by \mathbf{C} and D via

$$U(\mathbf{r}) \equiv \frac{1}{4\pi} \int \frac{D(\mathbf{r}')}{z} d\tau', \quad (1.98)$$

$$\mathbf{W}(\mathbf{r}) \equiv \frac{1}{4\pi} \int \frac{\mathbf{C}(\mathbf{r}')}{z} d\tau'. \quad (1.99)$$

Theorem 1:

$\nabla \times \mathbf{A} = 0$ iff there exists a function V such that $\mathbf{A} = -\nabla V$.

The \Leftarrow part of the statement is trivial to prove.

Read Sec. 1.6 and Appendix B.

In the analogy with the fluid dynamics, in general the flow \mathbf{A} can not be completely described by infinitesimal vortices alone, or by infinitesimal sources alone. Rather it is a superposition, but no more than that.

The \Rightarrow part can be proved by computation as follows. Let $V(\mathbf{r}) = V_0 - \int_{\mathcal{P}(\mathbf{r})} d\mathbf{l}' \cdot \mathbf{A}(\mathbf{r}')$, where V_0 is an arbitrary constant, and $\mathcal{P}(\mathbf{r})$ is an arbitrary path from the origin to \mathbf{r} . Since $\nabla \times \mathbf{A} = 0$, the integral of the 2nd term is independent of the choice of the path, as long as it starts at the origin and ends on \mathbf{r} . Thus $V(\mathbf{r})$ is well defined, and one can check that $\nabla V = -\mathbf{A}$.

Theorem 2:

$\nabla \cdot \mathbf{A} = 0$ iff there exists \mathbf{B} such that $\mathbf{A} = \nabla \times \mathbf{B}$.

The \Leftarrow part of the statement is trivial to prove.

If $\nabla \times \mathbf{A}$ goes to zero at infinities sufficiently fast, the \Rightarrow part of the statement is included in the special case of the Helmholtz theorem. We will not give the proof for the general case here.

If $\nabla \cdot \mathbf{A}$ goes to 0 at infinities sufficiently fast, the special case of the Helmholtz theorem is already a proof.

Given a flow \mathbf{B} , $\mathbf{A} = \nabla \times \mathbf{B}$ corresponds to the direction and density of infinitesimal vortices. The fact that $\nabla \cdot \mathbf{A} = 0$ implies that the center of the vortices form continuous lines and there is no "source for the vortices".

1.6 Exercises

- (tensor) Let A_i, B_j, C_k be vectors, which of the following is a tensor? (1) $A_1 B_i + A_2 B_j + A_3 B_k$; (2) $(A_1 + A_2 + A_3) B_i C_j$; (3) $\partial_1 B_2 + \partial_2 B_3 + \partial_3 B_1$; (4) $(A_1 B_1 + A_2 B_2 + A_3 B_3) A_i B_j C_k$.
- (rotation) Upon a 3D rotation along the axis $(\hat{\mathbf{y}} + \hat{\mathbf{z}})$ by an angle θ , all vectors \mathbf{A} are transformed as

$$A_i \rightarrow A'_i = R_{ij} A_j. \tag{1.100}$$

Find R_{ij} .

- (δ -fx) Find $A(s_0, \phi_0)$ defined by

$$\delta(x - s_0 \cos \phi_0) \delta(y - s_0 \sin \phi_0) = A(s_0, \phi_0) \delta(s - s_0) \delta(\phi - \phi_0), \tag{1.101}$$

where x, y are 2D Cartesian coordinates, and s, ϕ are 2D polar coordinates. ($s_0 > 0, 0 \leq \phi_0 < 2\pi$).

- (δ -fx) For given $a, b, c \in \mathbb{R}$, find A, B, C in the expression

$$\delta((x - a)(x - b)(x - c)) = A\delta(x - a) + B\delta(x - b) + C\delta(x - c). \tag{1.102}$$

- (δ -fx) Suppose D functions $f_i(x_1, \dots, x_D)$ vanish only when $x_i = y_i$ ($i = 1, 2, \dots, D$) for given real numbers y_i . Find A in the expression

$$\delta^{(D)}(\mathbf{f}) = A\delta^{(D)}(\mathbf{x} - \mathbf{y}). \tag{1.103}$$

- (line integral) Let a curve be defined by $\mathcal{P} = \{(x(t), y(t)) | t \in [0, 2\pi]\}$ with $x(t) = \cos t, y(t) = \sin t$.

(a) Let $\mathbf{A} = \hat{\mathbf{x}}y - \hat{\mathbf{y}}x$.

i. $\oint_{\mathcal{P}} d\mathbf{l} \cdot \mathbf{A}(\mathbf{r}) = ?$

ii. $\int d\mathbf{a} \cdot (\nabla \times \mathbf{A}) = ?$

Check that the answer agrees with the previous question.

(b) Let $\mathbf{A} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$.

i. $\oint_{\mathcal{P}} d\mathbf{l} \cdot \mathbf{A}(\mathbf{r}) = ?$

ii. Find V such that $\mathbf{A} = -\nabla V$.

How is this fact related to the question above?

7. (surface integral) Let a surface be defined by

$$\mathcal{S} = \{(x(\alpha, \beta), y(\alpha, \beta), z(\alpha, \beta)) | \alpha \in [0, 1], \beta \in [0, 2\pi)\} \quad (1.104)$$

with

$$x(\alpha, \beta) = \alpha \cos \beta, \quad y(\alpha, \beta) = \alpha \sin \beta, \quad z(\alpha, \beta) = \frac{1}{2}c\alpha^2, \quad (1.105)$$

for a constant $c \in \mathbb{R}$. Evaluate the surface integral

$$\int_{\mathcal{S}} d\mathbf{a} \cdot \hat{\mathbf{z}} z e^{-(x^2+y^2)/2}. \quad (1.106)$$

(Choose the area element $d\mathbf{a}$ to have a positive projection on $\hat{\mathbf{z}}$.)

8. (differential calculus) For a region V including the origin, evaluate

$$\int_V d\tau \frac{\hat{\mathbf{r}}}{r^2} \cdot \nabla f(\mathbf{r}), \quad (1.107)$$

assuming that $f(\mathbf{r})$ vanishes on the boundary of V .

9. (theory of vector fields) If a vector field \mathbf{A} satisfies

$$\nabla \cdot \mathbf{A} = 0, \quad \text{and} \quad \nabla \times \mathbf{A} = 0, \quad (1.108)$$

prove that \mathbf{A} can be solved in terms of a scalar V which satisfies the Laplace equation

$$\nabla^2 V = 0. \quad (1.109)$$

10. By finding infinitely many explicit solutions to the equations

$$\nabla \cdot \mathbf{A} = 0, \quad \nabla \times \mathbf{A} = 0, \quad (1.110)$$

Hint: Use the result of Prob. 12.

prove that there are infinitely many solutions to (1.97) for given \mathbf{C} and D if \mathbf{F} is not assumed to vanish at infinity.

Appendix: More about the Helmholtz Theorem

The Helmholtz theorem states that (almost) any vector field \mathbf{F} can be expressed as

$$\mathbf{F} = -\nabla U + \nabla \times \mathbf{W} \quad (1.111)$$

for some U and \mathbf{W} .

Instead of aiming at the highest generality, we prove this theorem via direct computation, using the solution (1.95) to the Poisson equation (1.92).

A crucial ingredient in proving the theorem is to notice that a vector field \mathbf{V} can be shifted

$$\mathbf{V} \rightarrow \mathbf{V}' = \mathbf{V} + \nabla f \quad (1.112)$$

for an arbitrary function f without affecting $\nabla \times \mathbf{V}$. This degeneracy allows us to choose \mathbf{V} to be divergenceless if we only care about $\nabla \times \mathbf{V}$ because if \mathbf{V} is not divergenceless, we can choose f to satisfy

$$\nabla^2 f = -\nabla \cdot \mathbf{V} \quad (1.113)$$

so that the new vector field \mathbf{V}' is divergenceless.

To prove that we can always find U and \mathbf{W} for given \mathbf{F} , we note that if U is already known, we should choose \mathbf{W} such that

$$\nabla \times \mathbf{W} = \mathbf{F} + \nabla U. \quad (1.114)$$

Let

$$\mathbf{W} = \nabla \times \mathbf{V}, \quad (1.115)$$

the equation above becomes

$$\nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V} = \mathbf{F} + \nabla U. \quad (1.116)$$

As mentioned above, we can always choose \mathbf{V} to be divergenceless, and so we want \mathbf{V} to satisfy

$$\nabla^2 \mathbf{V} = -(\mathbf{F} + \nabla U). \quad (1.117)$$

This equation can be solved because for each component of \mathbf{V} it is of the same form as (1.92). But this is consistent only if the right hand side is also divergenceless

$$\nabla \cdot \mathbf{F} + \nabla^2 U = 0. \quad (1.118)$$

This condition can be achieved by choosing U to solve this equation, which is also of the form (1.92). This completes the proof.

Q: Prove (1.98) and (1.99).

f can (almost) always be solved because the equation for f is of the same form as (1.92).

Chapter 2

Electrostatics

2.1 Charges and Fields

Charges:

1. There are two kinds of charges.
2. Charges are *locally* conserved.
3. Charges are quantized and additive.

The electric field at a point \mathbf{r}_0 is often defined as the electric force on a point charge q in the limit

$$\mathbf{E}(\mathbf{r}_0) = \lim_{q \rightarrow 0} \mathbf{F}/q. \quad (2.1)$$

The purpose of taking the limit is to ensure that the addition of the test charge does not affect the charge distribution ρ in space. However, since charges are quantized, strictly speaking the limit $q \rightarrow 0$ does not make sense, although practically it does for macroscopic phenomena.

Let us try the following definition of \mathbf{E} to avoid taking the limit. If the initial charge distribution is $\rho_0(\mathbf{r})$, after introducing a finite charge q at \mathbf{r}_0 , the charge distribution becomes $\rho(\mathbf{r}) = \rho_0(\mathbf{r}) + q\delta(\mathbf{r} - \mathbf{r}_0)$, assuming that the charge distribution ρ_0 is fixed by other forces. We define the electric field \mathbf{E} generated by ρ_0 at \mathbf{r}_0 to be

$$\mathbf{E}(\mathbf{r}_0) = \mathbf{F}/q. \quad (2.2)$$

Note that $\mathbf{E}(\mathbf{r}_0)$ is defined by excluding the field generated by q . Including the field due to a point charge q at \mathbf{r}_0 , the electric field at \mathbf{r}_0 always diverges.

We imagine that \mathbf{E} is fully determined by the charge distribution ρ , and we only care about the final charge distribution ρ , i.e., we believe that \mathbf{E} is independent of ρ_0 and also independent of the mechanism which determines

Gravity has a single kind of charge (energy) parametrized by a non-negative real number. Electromagnetism has two kinds of charges parametrized by a real number. Weak interactions and strong interactions have charges that have to be described by more than one parameters.

The phrase “test charge” refers to a small charge that does not affect the original charge distribution.

how charges distribute. We can imagine that ρ is held fixed by mechanical forces so that it is independent of q and \mathbf{r}_0 . However, **this definition of \mathbf{E} is consistent only if \mathbf{F} is proportional to q .**

The nature of the definition of \mathbf{E} above seems to suggest that it is fictitious, because the distinction between the test charge q and the rest of the charges ρ is artificial. Feynman and Wheeler wrote a paper claiming that the electric field is not a real physical entity in the context of classical electrodynamics. They constructed a formulation in which there are only charges, and interactions between them can be described without referring to electric or magnetic fields. An alternative formulation is to use the fields to describe everything. The point charges are described as holes in space where the fields diverge, and the field equations determine how the holes move. When the electric field is viewed as physical entity by itself, the interaction between \mathbf{E} and a charge q in principle can be more general than the linear relation $\mathbf{F} = q\mathbf{E}$. Perhaps this formula has to be modified when the electric field is very strong.

Both viewpoints make sense. One can say that we never detect \mathbf{E} directly without using a test charge. But we can say equally well that we never see a charge directly, but we only see the effect of the field created by the charge (e.g. we see the light, which is a wave of the field, emitted from the charge).

The quantum field theory of electromagnetism (quantum electrodynamics – abbreviated as QED) is currently the most precise theory for electromagnetism (if we avoid talking about the unification of EM and weak interaction into the electroweak theory). In this theory both \mathbf{E} and the charges appear as fields. (One can also choose to “integrate out” one of them and use only the other to describe physics.)

2.2 \mathbf{E}, V, ρ and Their Relations

Everything about electrostatics, including the formulas for boundary conditions, work and energy to be introduced below, can be derived from one of the many equivalent descriptions of electrostatics given in this section.

The laws of electrostatics as differential equations of \mathbf{E} :

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \tag{2.3}$$

$$\nabla \times \mathbf{E} = 0. \tag{2.4}$$

The laws of electrostatics as integral equations of \mathbf{E} :

The physical meaning of \mathbf{E} can be different for different physicists. There are at least 3 possibilities: (1) \mathbf{E} is not a real physical entity. It is defined only as an intermediate step in the middle of a calculation. It can be completely avoided if we compute differently.

(2) \mathbf{E} corresponds to certain mechanical properties such as stress and strains in an invisible jellylike stuff called “ether”. This viewpoint is already abandoned by physicists after the advent of special relativity.

(3) \mathbf{E} has its fundamental physical meaning and electromagnetism should be described as a field theory.

Often physicists do not care as long as they know how to compute to get the correct answer.

$$\epsilon_0 \simeq 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}.$$

$$\oint_S d\mathbf{a} \cdot \mathbf{E} = Q/\epsilon_0, \quad (2.5)$$

Gauss law.

$$\oint_C d\mathbf{l} \cdot \mathbf{E} = 0. \quad (2.6)$$

Eqs. (2.4) and (2.6) will be modified when we consider time-dependent magnetic field.

The laws of electrostatics as Coulomb's law plus superposition principle in terms of \mathbf{E} :

In the following, we will use the notation

$\hat{\mathbf{z}} \equiv \mathbf{r} - \mathbf{r}'$, and $z \equiv |\hat{\mathbf{z}}|$.

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q\hat{\mathbf{z}}}{z^2}, \quad + \quad \text{superposition principle} \quad (2.7)$$

or equivalently,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d\tau' \frac{\rho(\mathbf{r}')\hat{\mathbf{z}}}{z^2}. \quad (2.8)$$

For point charges q_i at \mathbf{r}_i ,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}. \quad (2.9)$$

Formulas for surface or line charge densities can be easily written down by noticing that the notion of a small quantity of charge dq can be expressed differently for different situations

$$dq \sim d\tau\rho \sim da\sigma \sim dl\lambda. \quad (2.10)$$

Q: What is the charge density $\rho(\mathbf{r})$ for a collection of point charges q_i at \mathbf{r}_i ? Derive (2.9) from (2.8).

The notion of electric potential is not only convenient for many calculations, it will play a more fundamental role when we talk about gauge symmetry.

The definition of V requires the choice of a reference point.

The relations between \mathbf{E} and V :

$$\mathbf{E} = -\nabla V, \quad (2.11)$$

$$V = -\int d\mathbf{l} \cdot \mathbf{E}. \quad (2.12)$$

The law of electrostatics as a differential equation of V :

$$\nabla^2 V = -\rho/\epsilon_0. \quad (2.13)$$

The law of electrostatics as Coulomb's law plus superposition principle in terms of V :

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{z}, \quad + \quad \textit{superposition principle} \quad (2.14)$$

or equivalently,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d\tau' \frac{\rho(\mathbf{r}')}{z}. \quad (2.15)$$

Finally, the law of electrostatics can also be derived from Gauss' law, 3D rotation and translation symmetry, together with superposition principle.

Q: Find \mathbf{E} a distance z above the midpoint between two equal charges q , a distance d apart. Check that the result is consistent with what you would expect when $z \gg d$.

Griffiths Prob.2.2

The equivalence among different descriptions of the laws of electrostatics is not rigorous. As an example, consider the configuration with a constant charge distribution $\rho(\mathbf{r}) = c$ constant. The electric field $\mathbf{E} = cx\hat{x}$ satisfies both differential equations (2.3) and (2.4), but the integral (2.8) is not well defined.

It is often assumed in a physical problem that the charge distribution decays to zero sufficiently fast as $r \rightarrow \infty$ so that the integral (2.8) is well defined. Throughout this course we will make a lot of physical assumptions unless otherwise states.

2.3 Field Lines

The invisible electric field can be visualized through the concept of **field lines**.

Interpretation of field lines:

1. The direction of the tangent vector of the field lines at a point \mathbf{r} is $\hat{\mathbf{E}}(\mathbf{r})$.
2. The density of the field lines at a point \mathbf{r} (the number of field lines crossing a unit area element with normal vector parallel to $\mathbf{E}(\mathbf{r})$) is $|\mathbf{E}(\mathbf{r})|$.

Following the 2nd point, the total number of field lines out of a surface \mathcal{S} (more precisely, the number of field lines coming out minus the number of field lines going in) is given by

$$\# \text{ of field lines} = \int_{\mathcal{S}} d\mathbf{a} \cdot \mathbf{E} \equiv \text{electric flux}. \quad (2.16)$$

Rules about field lines:

1. Field lines do not intersect and they repel one another.

2. There is always a charge at an endpoint of a field line.
3. The number of field lines generated by a charge q is q/ϵ_0 .

The notion of field lines (Gauss law) together with the requirement of certain symmetries are sufficient to determine the electric field in some cases.

1. spherical symmetry (invariant under 3D rotations)
2. cylindrical symmetry (invariant under 2D rotation of the $x - y$ plane and translation along the z -axis)
3. 2D translational symmetry (translation along the $x - y$ plane)

Note that there is a symmetry of 2 degrees of freedom in each case, so that the electric field depends on a single variable in 3D.

Q: Compute the electric field due to a spherical shell of uniform charge distribution in two approaches: (1) Notion of field lines and symmetry. (2) Superposition of Coulomb's law.

Q: Find the electric field in space due to two parallel infinite plane of uniform charge distribution σ and $\pm\sigma$.

2.4 Boundary Conditions

The discontinuity is due to the field generated by the surface charge σ .

Across a surface charge density σ , there is a discontinuity in the electric field

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}}, \quad (2.17)$$

where $\hat{\mathbf{n}}$ is the normal vector of the surface.

On the other hand, the electric potential is continuous unless the electric field diverges on the interface

$$V_{\text{above}} - V_{\text{below}} = 0. \quad (2.18)$$

Eq. (2.17) gives the discontinuity of the first derivative of V .

On the boundary, consider a point p and a small patch surrounding the point. The electric field at the point (slightly above or below the boundary) is generated by charges on the patch and charges outside the patch. The field due to charges outside the patch is always continuous at p . Let us denote this field by \mathbf{E}_o . The field due to charges on the patch is discontinuous at p . At a scale much smaller than the size of the small patch, the charges near p can be approximated by uniformly distributed charges over a large plate. Hence

The patch is "small" in the sense that (1) the surface charge density does not change significantly over the patch, and that (2) the total charge on the small patch is negligible compared with the rest of the charges, so that \mathbf{E}_o does not change significantly when the size of the patch is further reduced.

the field due to the small patch at p immediately above/below the boundary is $\pm \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}}$. Thus we have

$$\mathbf{E}_{\text{above}} = \mathbf{E}_o + \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}}, \quad \mathbf{E}_{\text{below}} = \mathbf{E}_o - \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}}. \quad (2.19)$$

This implies (2.17). Furthermore, we get

$$\mathbf{E}_o = \frac{1}{2}(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}}), \quad (2.20)$$

which is the electric field used to compute the force on the boundary charges. The pressure (force per unit area) on the boundary due to electrostatic force is thus

$$\text{Press.} = \sigma \mathbf{E}_o. \quad (2.21)$$

2.5 Work and Energy in Electrostatics

The work needed to move a charge q across a potential difference ΔV is $W = q\Delta V$. This is path-independent, and so the electrostatic force is conservative.

To compute the potential energy stored in the configuration with a given charge distribution $\rho(\mathbf{r})$, we can consider an arbitrary process that brings charges from infinity to specified locations until the charge distribution is $\rho(\mathbf{r})$. The work done will be independent of the process.

The result of the computation is that, for a continuous distribution of charges, the electrostatic energy W is

$$W = \frac{1}{2} \int d\tau \rho V = \frac{\epsilon_0}{2} \int d\tau E^2. \quad (2.22)$$

This is the energy it takes to collect charges from spatial infinity and relocate them to construct the given charge distribution ρ .

For point charges, the expressions above diverge, because the energy it takes to concentrate a finite amount of charge on a single point is infinity. We should not count the energy used to create an electron as part of the work we need to do, so one has to take out the divergent contribution due to self interactions. It is

$$W = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}. \quad (2.23)$$

Q: Compute the potential energy for a charge q uniformly distributed over a spherical surface of radius R .

Griffiths Ex. 2.8

Q: What formula will you use for W if the system consists of both point charges and continuous charge distributions?

Q: Is the energy to be associated with the charge or the field or both?

Q: Check that (2.22) is different from (2.23) when $\rho(\mathbf{r}) = \sum_i q_i \delta(\mathbf{r} - \mathbf{r}_i)$. Why?

2.6 Conductor

Properties of a conductor in static electric field (after everything reaches a static state):

1. $\mathbf{E} = 0$ in the bulk.
2. $\rho = 0$ in the bulk.
3. $V = \text{constant}$ on the boundary.
4. $\mathbf{E} = \sigma \hat{\mathbf{n}}/\epsilon_0$ on the boundary.

Q: A charge q is placed inside a conducting cavity. The outer surface of the conductor is a sphere of radius R . Find the electric field outside the sphere.

Griffiths Ex. 2.9, screening effect

Q: What is the pressure on a conducting surface when the surface charge density is σ and the electric field on (slightly outside) the surface is \mathbf{E} ?

2.7 Capacitor

The capacitance of a capacitor is defined as

$$C \equiv \frac{Q}{V}. \quad (2.24)$$

Farad = Coulomb/Volt.

Its electrostatic energy is

$$W = \frac{1}{2} CV^2 = \frac{1}{2} \frac{Q^2}{C}. \quad (2.25)$$

Since W can be computed in other ways, this expression allows us to find C .

This is the work to be done to charge the capacitor.

Often the capacitor is composed of two disconnected pieces of conductor, so that the total charge on the capacitor is 0. But sometimes we also regard a single piece of conductor as a capacitor by imagining that the other piece of conductor is at spatial infinity.

Q: For a parallel plate capacitor with area A and separation d , compute (1) the capacitance C , (2) the change in the potential energy when the separation is changed to d' , and the work needed, and check the conservation of energy. (Consider separately the cases of (i) fixed charges and (ii) fixed electric potential difference on the plates.)

$$C = \epsilon_0 \frac{A}{d}.$$

Exercises

1. Assume that the electron is a spherical shell of radius R with uniform charge density. According to the theory of special relativity, the electrostatic energy $E(R)$ contributes to the rest mass of an electron. That is, $m = m_0 + E(R)/c^2$, where m is the observed mass of an electron, and m_0 is the “real mass” of the electron. It is known from experiments that the mass and charge of an electron is approximately

$$m \simeq 9 \times 10^{-31} \text{ kg}, \quad q_e \simeq -1.6 \times 10^{-19} \text{ C}. \quad (2.26)$$

If $m_0 > 0$, what is the lower bound of R ? What is the energy of a photon if its wave length equals the lower bound of R ?

For a photon, the energy is given by $E = hf$, where f is the frequency and $h \simeq 6.6 \times 10^{-34} \text{ kg m}^2/\text{s}$.

2. (a) Find the electric field at $\mathbf{r} = \hat{\mathbf{z}}z$ due to a charged circular disk of radius R and surface charge density $\rho(r) = \alpha r^2 + \beta r$ for given constants α, β . The disk is centered at the origin on the $x - y$ plane.

(b) Consider the limit $R \rightarrow 0$ with the total charge on the disk fixed (so we need $\alpha \rightarrow \infty$). Check whether the result agrees with that of a point charge.
3. Find the charge density corresponding to the electric field $\mathbf{E} = ax\hat{\mathbf{x}}$ (in Cartesian coordinates), and that for $\mathbf{E} = ar\hat{\mathbf{r}}$ (in spherical coordinates). Compare and comment.
4. Two charged spherical conductors of radii a and b separated at a large distance are connected through a very thin conducting wire. Ignoring the charges on the wire, what is the ratio of the electric field strengths near the surface of the two spheres?
5. A hollow spherical shell carries charge density $\rho = kr$ (for a given constant k) in the region $a \leq r \leq b$. Find the electric field and the electric potential in the three regions (i) $r < a$, (ii) $a < r < b$, (iii) $b < r$.
6. Two spheres, each of radius R and carrying uniform charge density $+\rho$ and $-\rho$, respectively, are placed so that they partially overlap. Call the vector from the positive center to the negative center \mathbf{d} . Show that the field in the region of overlap is constant and find its value.
7. For a flat d dimensional space ($d > 1$) it is natural to generalize Gauss’ law to d dimensional space as

Griffiths Prob. 2.18

$$\partial_i E_i = \rho/\epsilon_0, \quad (2.27)$$

and replace $\nabla \times \mathbf{E} = 0$ by

$$\partial_i E_j - \partial_j E_i = 0. \quad (2.28)$$

The divergence theorem can be generalized to higher dimensions as

$$\int_V d\tau \partial_i E_i = \oint_S E_i da_i, \quad (2.29)$$

where V is a d -dimensional volume and S is its $(d - 1)$ dimensional boundary. Similarly, (2.28) implies that locally there exists a function V such that

$$E_i = -\partial_i V. \quad (2.30)$$

- (a) Find the electric field \mathbf{E} generated by a point charge q at the origin.
- (b) Define the electric potential V by (2.30) and find the electric potential of a point charge q at the origin.
- (c) Check that the $d = 2$ result agrees with that of an infinite straight line of linear charge density q .

In your answer, you can use $A(S^n)$ to denote the area of a unit n dimensional sphere.

In d dimensions with Cartesian coordinates x_1, \dots, x_d , let $r = \sqrt{x_1^2 + \dots + x_d^2}$.

See the appendix if you are interested in computing the area of a unit n -sphere.

8. Assume that the law of electrostatics is changed to

$$\nabla \cdot \mathbf{E} + \lambda \mathbf{E} \cdot \mathbf{E} = \rho/\epsilon_0, \quad (2.31)$$

$$\nabla \times \mathbf{E} = 0, \quad (2.32)$$

where λ is a constant.

- (a) Which of the following statement(s) is (are) correct?
 - i. One can still define the electric potential V as $\mathbf{E} = -\nabla V$.
 - ii. The superposition principle is still valid.
 - iii. These two equations are inconsistent.
 - (b) In order to measure the value of λ , you should do an experiment with ... (1) a very strong electric field, (2) a very weak electric field, (3) a very high potential, (4) a fast changing electric field.
9. A spherical insulator of radius R is charged with a charge density $\rho(r) = qr/(\pi R^4)$. A spherical conducting shell with inner radius a and outer radius b surrounds the insulator with coincident centers at the origin. ($R < a < b$.) The total charge on the conducting shell is Q .
- (a) Find the total charges on the spherical surface at (1) $r = a$ and (2) $r = b$.

- (b) Find the electric field \mathbf{E} and the electric potential V for (1) $b < r$, (2) $a < r < b$, (3) $R < r < a$ and (4) $0 < r < R$.
- (c) Find the total electrostatic energy of the system.
- (d) After we connect the outer surface of the conducting shell to the ground (so that its electric potential becomes 0), (i) find the total charges at (1) $r = a$ and (2) $r = b$, and (ii) find the electric potential $V(r)$ for $0 < r < a$.
10. Two infinite plates extending in the $y - z$ directions are located at $x = 0$ and $x = a$. The surface charge density at $x = 0$ is σ_1 and that at $x = a$ is σ_2 .
- (a) Find the electric field $\mathbf{E}(x)$ and electric potential $V(x)$ for (1) $x < 0$, (2) $0 < x < a$, and (3) $a < x$.
- (b) Find the surface charge densities on both sides of the surface for each plate. That is, find the surface charge densities at (1) $x = 0^-$, (2) $x = 0^+$, (3) $x = a^-$, (4) $x = a^+$.
- (c) For $\sigma_1 = -\sigma_2$, find the capacitance per unit area.
11. Consider two coaxial cylindrical conducting shells of radii a and b ($a < b$), respectively. (Their thickness is negligible.) The potential difference between the shells is V_0 .
- (a) Find the charges per unit length on each shell.
- (b) Find the capacitance per unit length.
- (c) Find the electrostatic energy per unit length.
- (d) Find the pressure at $r = a$ and that at $r = b$ by (1) computing the Coulomb force (charge density times electric field), (2) computing the changes in electrostatic energy due to changes in a or b .
- (e) If the shells are connected to a battery of constant potential V_0 , will the pressure at $r = a$ or $r = b$ be different from the answer to the previous question?
12. Given a material of constant charge density ρ of volume V . What shape should it take in order to create the maximal possible electric field at a given point?
13. Consider a system of conductors as in the figure below. The concentric conducting shells have inner radii a_1, a_2 and outer radii b_1, b_2 . The sphere of radius a has charge Q . There is no net charge on the inner shell, and

the outer shell has total charge $-Q$. Find (a) the potential $V(r)$ in the region $a < r < a_1$. (b) the surface charge density σ at $r = b_1$. (c) the pressure on the surface at $r = a_2$. (d) the capacitance C of the system. (e) the total electrostatic energy.

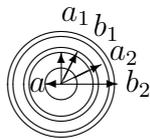


Figure for Prob. 13

14. For the charge distribution $\rho(\mathbf{r}) = q\delta^{(3)}(\mathbf{r}) + k\delta(r - R)$ for given constants q, k , find the total electrostatic energy.
15. Griffiths Prob's. 2.32, 2.37, 2.38, 2.40, 2.46, 2.47, 2.49.

Appendix: Area of S^n

A unit n -dimensional sphere (a unit n -sphere, or S^n) can be defined as an n -dimensional subspace in an $(n+1)$ -dimensional Euclidean space with Cartesian coordinates x_1, \dots, x_n, x_{n+1} by

$$S^n = \{(x_1, \dots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 = 1\}. \quad (2.33)$$

To derive the area of S^n , we start with the computation of

$$A_0^2 \equiv \int dx dy e^{-(x^2+y^2)/2} = \int_0^{2\pi} d\theta \int_0^\infty dr r e^{-r^2/2} = 2\pi. \quad (2.34)$$

Now consider

$$A_0^{n+1} = A(S^n) \int_0^\infty dr r^n e^{-r^2/2}, \quad (2.35)$$

where $r = \sqrt{x_1^2 + \dots + x_{n+1}^2}$ is the radial coordinate in $(n+1)$ -dimensional space, and $A(S^n)$ is the area of S^n . The area of S^n is thus

$$A(S^n) = \frac{A_0^{n+1}}{B_n}, \quad (2.36)$$

where

$$B_n \equiv \int_0^\infty dr r^n e^{-r^2/2}. \quad (2.37)$$

Using integration by part, we find the recursion relation for B_n

$$B_n = (n-1)B_{n-2} \quad (2.38)$$

for $n > 1$. For $n = 1$, it is easy to compute directly

$$B_1 = 1. \quad (2.39)$$

For $n = 0$, we get

$$B_0 = \frac{1}{2}A_0 = \sqrt{\frac{\pi}{2}}. \quad (2.40)$$

Thus for $n > 1$,

$$B_n = \begin{cases} (n-1)(n-3)\cdots 3 \cdot 1 \cdot \sqrt{\frac{\pi}{2}}, & n = \text{even}, \\ (n-1)(n-3)\cdots 2 \cdot 1, & n = \text{odd}. \end{cases} \quad (2.41)$$

We list a few examples of the final result:

$$A(S^1) = 2\pi, \quad A(S^2) = 4\pi, \quad A(S^3) = 2\pi^2, \quad A(S^4) = \frac{8}{3}\pi^2. \quad (2.42)$$

Chapter 3

Special Techniques

3.1 Solutions of Laplace Equation

A solution V to the Laplace equation

$$\nabla^2 V = 0 \quad (3.1)$$

in a region \mathcal{V} can be thought of as the electric potential generated by a charge distribution ρ outside the region \mathcal{V} .

Any solution V of the Laplace equation in \mathcal{V} has the following two properties:

1. The average value of V over a spherical surface (of any radius) always equals its value at the center,

$$V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{S^2} V(\mathbf{r}') da' \quad (3.2)$$

2. V has no local maxima or minima. Maxima and minima of V in \mathcal{V} can only occur on the boundary of \mathcal{V} .

The second point is an immediate consequence of the first point.

Q: For any solution of Laplace equation in 1 dimension,

$$\frac{d^2}{dx^2} V(x) = 0, \quad (3.3)$$

prove that

$$V(x) = \frac{1}{2}(V(x+d) + V(x-d)) \quad (3.4)$$

for any d .

Q: Consider the Laplace equation in 2 dimensions

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) V(x, y) = 0. \quad (3.5)$$

Solutions of Laplace equation are called harmonic functions.

The proof can be found in Griffiths. It is based on two simple ideas: (1) The potential at \mathbf{r} for a charge q at \mathbf{r}' is the same as the potential at \mathbf{r}' due to a charge q at \mathbf{r} . (2) The potential for uniformly distributed charges over a sphere equals the potential for the charge at the center.

Expand V near the origin as a Taylor expansion up to 2nd order terms

$$V(x, y) = V_0 + \alpha x + \beta y + \frac{a}{2}x^2 + bxy + \frac{c}{2}y^2 + \dots \quad (3.6)$$

Assuming that the origin is a local extremum, we set $\alpha = \beta = 0$.

1. Show that by a rotation $(x, y) \rightarrow (x', y')$, V can always be rewritten as

$$V = V_0 + \frac{a'}{2}x'^2 + \frac{b'}{2}y'^2 + \dots \quad (3.7)$$

2. Show that $a' + b' = 0$ in order for V to satisfy the Laplace equation.
3. Show that the average value of V around an infinitesimal circle centered at the origin is V_0 .

Q: Justify *Earnshaw's Theorem*: A charged particle cannot be held in a stable equilibrium by electrostatic forces alone.

Griffiths Prob. 3.2

3.2 Uniqueness Theorems

Uniqueness theorems tell us how to specify the conditions of a physical experiment such that the set-up is not ambiguous, and the result will be unique.

This idea applies to all branches of physics.

Here we give two examples of uniqueness theorems in electrostatics. Their proofs are very similar. Schematically, the basic idea is the following. To prove that there is a unique solution, we start by assuming that there are two solutions S_1, S_2 . The difference of the two solutions $\Delta S = S_1 - S_2$ has to satisfy some conditions as a result of the fact that the two solutions satisfy some conditions. Use these properties of ΔS to prove that $\int (\Delta S)^2 = 0$, which implies that $\Delta S = 0$. You can easily generalize the proof to other uniqueness theorems.

Read Griffiths p.120 on Purcell's experiment.

Theorem 1:

V is uniquely determined by its values on the boundaries (and ρ).

Theorem 2:

V is uniquely determined by the total charges on each conductors (and ρ), up to a shift by constant.

A uniqueness theorem is a warrant for imagination. If you can come up with a solution by imagination, it is the correct solution as long as it satisfies the Laplace equation and boundary conditions.

3.3 Symmetries of Laplace Equation

The Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial x_i^2} \quad (3.8)$$

is invariant under spatial translation

$$x_i \rightarrow x_i + a_i, \quad a_i \in \mathbb{R}, \quad (3.9)$$

and spatial rotation including inversion

$$x_i \rightarrow \Lambda_{ij}x_j, \quad \Lambda^T \Lambda = I. \quad (3.10)$$

These symmetries of ∇^2 imply that:
if $V(x_i)$ is a solution of the Laplace equation,

$$\nabla^2 V = 0, \quad (3.11)$$

then $V(\Lambda_{ij}x_j + a_i)$ is also a solution. These are the symmetries of the Laplace equation which do not involve a change of the function V . A simple example of a symmetry involving V is

$$V(\mathbf{r}) \rightarrow \lambda V(\mathbf{r}), \quad \lambda \in \mathbb{R}. \quad (3.12)$$

There is another symmetry of the Laplace equation. Given a solution $V(r, \theta, \phi)$ (in spherical coordinates) of the Laplace equation, we always have another solution defined by

$$\hat{V}(r, \theta, \phi) = \frac{R}{r} V(R^2/r, \theta, \phi). \quad (3.13)$$

To check that (3.13) is a solution of the Laplace equation whenever V is, we only need to check how the first term in ∇^2

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \quad (3.14)$$

acts on \hat{V} . Let

$$\hat{r} \equiv \frac{R^2}{r}, \quad (3.15)$$

then

$$\hat{V}(r, \theta, \phi) = \frac{\hat{r}}{R} V(\hat{r}, \theta, \phi), \quad (3.16)$$

and we find

$$\begin{aligned} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \hat{V}(r, \theta, \phi) &= \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \frac{\hat{r}}{R} V(\hat{r}, \theta, \phi) \\ &= \frac{\hat{r}}{R} \left(\frac{\partial}{\partial \hat{r}} \hat{r}^2 \frac{\partial}{\partial \hat{r}} V(\hat{r}, \theta, \phi) \right), \end{aligned} \quad (3.17)$$

This section has to do with the method of images

The method of images applied to a grounded $x - y$ plane works because of the inversion symmetry $z \rightarrow -z$.

Due to the symmetry (3.12), the overall factor of R in \hat{V} can be omitted. We insert it there just to preserve the dimension of V .

so that, for an arbitrary function V ,

$$\nabla^2 \hat{V}(r, \theta, \phi) = \left(\frac{\hat{r}}{R}\right)^5 \hat{\nabla}^2 V(\hat{r}, \theta, \phi). \quad (3.18)$$

In the above we defined $\hat{\nabla}^2$ as the Laplace operator with the replacement $r \rightarrow \hat{r}$, i.e.

$$\hat{\nabla}^2 = \frac{1}{\hat{r}^2} \frac{\partial}{\partial \hat{r}} \hat{r}^2 \frac{\partial}{\partial \hat{r}} + \frac{1}{\hat{r}^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\hat{r}^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \quad (3.19)$$

Obviously, if $\nabla^2 V(r, \theta, \phi) = 0$, then $\hat{\nabla}^2 V(\hat{r}, \theta, \phi) = 0$ as it can be viewed as a mere change of notation from r to \hat{r} . Thus we can make sure that \hat{V} is a solution to the Laplace equation if V is a solution.

A symmetry always means a way to generate new solutions from old solutions.

Q: If V and ρ satisfy the Poisson equation $\nabla^2 V = -\rho$, how would you define $\hat{\rho}$ such that \hat{V} and $\hat{\rho}$ also satisfy the Poisson equation?

Q: Explain example 3.2 in Griffiths using this symmetry transformation.

A similar story can be told for other dimensions. In two dimensions, the Laplace operator is

$$\nabla^2 = \frac{1}{s} \frac{\partial}{\partial s} s \frac{\partial}{\partial s} + \frac{1}{s^2} \frac{\partial^2}{\partial \phi^2}. \quad (3.20)$$

This can also be interpreted as the 3D Laplace operator in cylindrical coordinates when there is no z -dependence.

It is straightforward to check that, for

$$\hat{s} \equiv \frac{R^2}{s}, \quad (3.21)$$

we have the identity

$$\nabla^2 V(\hat{s}, \phi) = \left(\frac{\hat{s}}{R}\right)^4 \hat{\nabla}^2 V(\hat{s}, \phi) \quad (3.22)$$

for an arbitrary function V . Again, $\hat{\nabla}^2$ is obtained from replacing s by \hat{s} in (3.20). Therefore, if $V(s, \phi)$ is a solution of the 2D Laplace equation, then $V(\hat{s}, \phi)$ is also a solution.

Q: What about other dimensions?

3.4 General Solutions of Laplace Equation

Via separation of variables, the most general solution of 3D Laplace equation

For this course, one can ignore how separation of variables leads to the most general solution of the Laplace equation, but should know how to use them to solve problems.

can be obtained in Cartesian coordinates, spherical coordinates and cylindrical coordinates, for various boundary conditions. It is important to realize that the boundary condition determines which general solution one should start with.

The general solutions considered in Griffiths book include:

- Cartesian coordinates

$$V(x, y, z) = \sum_{k_1, k_2} a(k_1, k_2) \left\{ \begin{array}{c} \sin(k_1 x) \\ \cos(k_1 x) \end{array} \right\} \left\{ \begin{array}{c} \sin(k_2 y) \\ \cos(k_2 y) \end{array} \right\} \left\{ \begin{array}{c} \sinh(k_3 z) \\ \cosh(k_3 z) \end{array} \right\}, \quad (3.23)$$

where k_3 is determined by k_1 and k_2 as

$$k_3 = \pm \sqrt{k_1^2 + k_2^2}. \quad (3.24)$$

Note that the roles of x, y, z may be swapped; one of the coordinates may be skipped (e.g. by setting k_1 to zero). If the region is bounded in the x -direction, k_1 can only take discrete values.

Note also that both $\sinh(k_3 z)$ and $\cosh(k_3 z)$ diverge as $k_3 z$ goes to $\pm\infty$.

- Spherical coordinates:

Assuming that the solution is ϕ -independent:

$$V(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}} \right) P_{\ell}(\cos \theta). \quad (3.25)$$

Note that r^{ℓ} diverges at $r \rightarrow \infty$, and $1/r^{\ell+1}$ diverges at $r = 0$.

The Legendre polynomials $P_n(x)$ constitute an orthogonal basis for the dependence of V on θ ,

$$\int_{-1}^1 dx P_m(x) P_n(x) = \int_0^{\pi} d\theta \sin \theta P_m(\cos \theta) P_n(\cos \theta) = \frac{2}{2m+1} \delta_{mn}. \quad (3.27)$$

$P_n(x)$ for arbitrary n is give by the Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!} \left(\frac{d}{dx} \right)^n (x^2 - 1)^n. \quad (3.28)$$

- Cylindrical coordiantes:

Assuming that the solution is z -independent:

$$V(s, \phi, z) = a_0 + b_0 \log(s) + \sum_{n=1}^{\infty} [s^n (a_n \cos(n\phi) + b_n \sin(n\phi)) + s^{-n} (c_n \cos(n\phi) + d_n \sin(n\phi))]. \quad (3.29)$$

Note that s^n ($n \geq 1$) diverges at $s \rightarrow \infty$ and s^{-n} ($n \geq 1$) diverges at $s = 0$, and $\log(s)$ diverges at both $s = 0, \infty$.

The braces in (3.23) mean that you should consider a superposition of the two functions for each brace. Usually the superposition is dictated by the boundary condition.

The solution (3.23) still obeys Laplace equations if we replace \sin, \cos by \sinh, \cosh for the y -dependence, if we also change the relation among k_i to $k_1 = \pm \sqrt{k_2^2 + k_3^2}$. Or we can interchange the roles of y and z , etc.

The normalization of the Legendre polynomial is given by

$$P_n(1) = 1. \quad (3.26)$$

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= (3x^2 - 1)/2 \\ P_3(x) &= (5x^3 - 3x)/2 \end{aligned}$$

Q: Go through Sec. 3.3 of Griffiths and see how each example falls into a certain class of solutions listed above.

3.4.1 Boundary Conditions

There are two types of boundary conditions: *homogeneous* and *inhomogeneous* boundary conditions. For linear differential equations, homogeneous boundary conditions allows us to find new solutions by superposition of known solutions.

Homogeneous Boundary Conditions

A homogeneous boundary condition for ϕ is often of the form

$$a\phi(0) + b\phi'(0) = 0. \quad (3.30)$$

Here we suppress the coordinates parallel to the surface of boundary.

If $b = 0$, this is called Dirichlet boundary condition. If $a = 0$, it is Neumann boundary condition.

If V satisfies a homogeneous boundary condition for the boundaries of a certain coordinate, it is often convenient to use the general solution of V which automatically satisfies this boundary condition.

In general, we first choose a coordinate system $(\lambda_1, \lambda_2, \lambda_3)$, and then a general solution of V in the form

$$V(\lambda_1, \lambda_2, \lambda_3) \sim \text{superposition of } \Phi_{s_1}(\lambda_1)\Psi_{s_2}(\lambda_2)\Xi_{s_3}(\lambda_3). \quad (3.31)$$

If a homogeneous boundary condition is imposed on the boundary of λ_1 , we should find the most general form of Φ_{s_1} to express V .

Q: If the boundary conditions on x are $\partial_x V(0, y, z) = 0$ and $\partial_x V(L, y, z) = 0$, what are the functions of x that we can use to expand V ?

Q: What if the boundary conditions are $V(0, y, z) = 0$ and $\partial_x V(L, y, z) = 0$?

For example, if V satisfies Dirichlet boundary condition at $x = 0$ and $x = L$, we use (3.23) with the x -dependence $\sin(n\pi x/L)$.

Inhomogeneous Boundary Conditions

An inhomogeneous boundary condition is typically of the form

$$\left(a + b\frac{\partial}{\partial\lambda_1}\right)V(0, \lambda_2, \lambda_3) = f(\lambda_2, \lambda_3) \quad (3.32)$$

for a certain given function f . There are (at least) two ways to satisfy an inhomogeneous boundary condition.

The first approach is to find a special solution that satisfies the inhomogeneous boundary condition. Then the solution you want is a superposition of this special solution and another solution which satisfies a homogeneous

boundary condition. The problem of inhomogeneous boundary condition is thus turned into a problem of homogeneous boundary condition, which is discussed above.

The second approach is as follows. We choose the general solution of V (3.31) such that $\Psi_{s_2}(\lambda_2)\Xi_{s_3}(\lambda_3)$ form a complete basis of functions on the space of λ_2, λ_3 , so that we can use the *Fourier's trick*.

Fourier's trick is a trick to find the coefficients A_s that allows us to expand a given function f in terms of a complete basis ϕ_s

$$\sum_s A_s \phi_s(\lambda) = f(\lambda) \tag{3.33}$$

for a given function f . This trick works best when the basis ϕ_s is an orthogonal basis, that is,

$$\int d\lambda \mu(\lambda) \phi_s(\lambda) \phi_t(\lambda) = c_s \delta_{st} \tag{3.34}$$

for some constants c_s . (The index s is not summed over on the right hand side.) Here $\mu(\lambda)$ is called *measure* and it comes with the definition of the complete basis. Multiplying both sides of (3.33) by ϕ_t and integrate over λ (with the measure), we find

$$A_t = \frac{1}{c_t} \int d\lambda \mu(\lambda) f(\lambda) \phi_t(\lambda). \tag{3.35}$$

Q: Extend Fourier's trick to two dimensions.

Q: Use Fourier's trick to find the coefficients A_n in

$$\sum_{n=1}^{\infty} A_n \sin(n\pi x/L) = \sin^2(\pi x/L). \tag{3.36}$$

Implicit Boundary Conditions

All the coordinates are defined with a range with boundaries. The Cartesian coordinates are defined with boundaries at $\pm\infty$. One can impose boundary conditions at infinities by specifying the asymptotic behavior of a solution at infinities. For example, if we need $E_x \rightarrow$ a constant C as $x \rightarrow \infty$, then we must have $V = V_0 - Cx +$ constant for some function V_0 that goes to 0 at $x = \infty$.

Similarly, the radial coordinate r is defined with boundaries at $r = 0$ and $r = \infty$. If the solution is to be defined at $r = 0$ or $r = \infty$, we have to check that it does not diverge at $r = 0$, and that it has the desired asymptotic behavior at $r = \infty$.

The angular coordinate ϕ has boundaries at $\phi = 0$ and $\phi = 2\pi$. A function $V(\phi)$ must be periodic in ϕ if it is defined over all possible angles. If, however, the function is defined only over a range of ϕ , $[\phi_0, \phi_1]$, then it does not have to be periodic.

Often we have to choose the dependence of V in other directions to satisfy certain homogeneous boundary conditions at the same time.

See Griffiths Ex. 3.3, 3.4, 3.5 as examples.

Often the measure is just 1. But actually it depends on the choice of variables. For example, the Legendre polynomials have $\mu(x) = 1$ and $\mu(\theta) = \sin\theta$. The difference is just the Jacobian for the change of coordinates. The functions $\sin(n\pi x/L)$ is a complete basis of functions for functions of x on the domain $[0, L]$.

3.5 Multiple Expansion

When viewed from very far away, any localized charge distribution with total charge Q looks like a point charge Q (or it is not yet far enough). Its potential is approximately $V \sim Q/(4\pi\epsilon_0 r)$ for very large r . More precisely, the precise potential V can be expanded in powers of $1/r$, and the expression above gives the leading order term. Generically the next to leading order term will be of order $1/r^2$, and so on. The multipole expansion corresponds to this $1/r$ expansion when we look at the electrostatic field of a localized charge distribution.

Using the identity

$$\frac{1}{z} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos \theta'), \tag{3.37}$$

where θ' is the angle between \mathbf{r} and \mathbf{r}' , we find

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int d\tau' \frac{\rho(\mathbf{r}')}{z} \tag{3.38} \\ &= \frac{1}{4\pi\epsilon_0} \frac{Q}{r} + \frac{*}{r^2} + \frac{*}{r^3} + \dots \\ &= V_{monopole}(\mathbf{r}) + V_{dipole}(\mathbf{r}) + V_{quadrupole}(\mathbf{r}) + \dots \tag{3.39} \end{aligned}$$

To derive this expansion, note that $1/z$ is a solution to Laplace equation at large r , and thus it can be expressed as the general solution (3.25) with $A_\ell = 0$ (with \mathbf{r} on the z -axis so that $\theta = \theta'$). To fix the values of B_ℓ , one can consider the special case when $\theta' = 0$, $1/(r-r') = \sum_n r'^n / r^{n+1}$.

The explicit expressions for the dipole, quadrupole and higher order terms can be derived by plugging the explicit expressions of the Legendre polynomials into (3.38).

The electric field \mathbf{E} can be expanded in powers of $1/r$ similarly. The expressions are more complicated because it is a vector.

The dipole potential and field are

$$V_{dipole}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}, \tag{3.40}$$

$$\mathbf{E}_{dipole}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}], \tag{3.41}$$

where \mathbf{p} is the *dipole moment*

$$\mathbf{p} \equiv \int d\tau' \mathbf{r}' \rho(\mathbf{r}'). \tag{3.42}$$

An ideal electric dipole can be defined as the limit $d \rightarrow 0$ for two charges $\pm q$ separated by a vector \mathbf{d} , with the dipole moment $\mathbf{p} = q\mathbf{d}$ fixed. (The vector \mathbf{d} points from the charge $-q$ to q .)

Exercises

1. Find the average potential over a spherical surface of radius R . Show that in general

$$V_{ave} = V_{center} + \frac{Q_{enc}}{4\pi\epsilon_0 R}, \quad (3.43)$$

where V_{center} is the potential at the center due to all the external charges, and Q_{enc} is the total enclosed charge.

Uniqueness Theorem
Griffiths Prob. 3.1

2. In Griffiths, a proof of the first uniqueness theorem was given based on the property that a solution of the Laplace equation does not have local maximum or minimum. There is another proof that formally resembles the proof of the 2nd uniqueness theorem.

Consider $\int_{\mathcal{V}} d\tau V \nabla^2 V$ and prove that if V is a solution of the Laplace equation in the region \mathcal{V} and V vanishes on the boundary, then $V = 0$ identically everywhere in \mathcal{V} .

Uniqueness Theorem

3. Griffiths Prob. 3.4, 3.41, 3.43.

Uniqueness Theorem

4. Two parallel infinite conducting planes at $z = a, -a$ are grounded (at potential 0). A point charge q is placed at the origin. Find all the auxiliary charges and their locations.

Method of Images

5. A long, straight conducting pipe of radius R is placed parallel to, and at a distance d from an infinite conducting plane at $z = 0$. The axis of the pipe is located at $z = d, y = 0$. The pipe is at potential V_0 and the plane is at potential 0. Find the potential for $z > 0$.

Method of Images

6. Griffiths Probs. 3.8, 3.9, 3.10, 3.11.

Method of Images

7. Let $V(x)$ be a step function over the region $[0, \pi]$ defined by

$$V(x) = \begin{cases} 0 & (0 < x < \pi/2), \\ 1 & (\pi/2 < x < \pi). \end{cases} \quad (3.44)$$

Assuming that $V(x)$ can be expanded in terms of $\sin(nx)$, find the coefficients a_n in the expansion

$$V(x) = \sum_{n=1}^{\infty} a_n \sin(nx). \quad (3.45)$$

To make sure that (3.45) is correct, check that

$$\int_0^{\pi} dx \left[V(x) - \sum_{n=1}^{\infty} a_n \sin(nx) \right]^2 = 0. \quad (3.46)$$

Despite the condition (3.46), the "identity" (3.45) does not hold everywhere. At the point of discontinuity, the RHS of (3.45) differs from the LHS by a finite number. But the RHS and the LHS are the same as distributions.

Hint: You can use the identity

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \quad (3.47)$$

Separation of Variables

8. Two infinitely long grounded metal plates at $y = 0$ and $y = a$ are connected at $x = b$ and $x = -b$ by metal strips at constant potentials V_0 and $-V_0$, respectively. ($V = V_0$ at $x = b$ and $V = -V_0$ at $x = -b$.) (A thin layer of insulation at each corner prevents them from shorting out.) Find the potential inside the rectangular pipe.

Separation of Variables

9. The potential at the surface of a sphere of radius R is given by

$$V(\theta) = k \cos(2\theta), \quad (3.48)$$

where k is a constant. Find the potential inside and outside the sphere, as well as the surface charge density $\sigma(\theta)$ on the sphere.

Separation of Variables

10. Someone has glued a surface charge

$$\sigma(\theta) = k \cos \theta \quad (3.49)$$

over a thin spherical shell of radius a . This charge is surrounded by a concentric thin conducting spherical shell of radius b ($b > a$). The total charge on the conducting shell is zero.

- (a) Find the electric potential for (i) $r < a$, (ii) $a < r < b$ and (iii) $b < r$, respectively.
- (b) Find the induced surface charge density σ on the inner surface of the conducting shell.
- (c) Find the surface charge density σ on the outer surface of the conducting shell.

Separation of Variables

Griffiths Prob. 3.23

11. Solve Laplace's equation by separation of variables in cylindrical coordinates, assuming there is no dependence on z (cylindrical symmetry). [Make sure you find *all* solutions to the radial equation; in particular, your result must accommodate the case of an infinite line charge, for which (of course) we already know the answer.]

Separation of Variables

12. Griffiths Probs. 3.14, 3.15, 3.19, 3.20, 3.22, 3.24, 3.25, 3.39, 3.47, 3.48.

Multipole Expansion

13. Show that the quadrupole term in the multipole expansion can be written

Griffiths Prob. 3.45

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{2r^3} \sum_{i,j=1}^3 \hat{r}_i \hat{r}_j Q_{ij}, \quad (3.50)$$

where

$$Q_{ij} = \int d\tau' [3r'_i r'_j - r'^2 \delta_{ij}] \rho(\mathbf{r}'). \quad (3.51)$$

Q_{ij} is the quadrupole moment of the charge distribution.

Multipole Expansion

14. Griffiths Probs. 3.27, 3.28,

Appendix: Complete Basis of Functions

There is a close analogy between the pairs (*differential operators, functions*) and (*matrices, vectors*). Just like any vector in a vector space can be expanded as a superposition of vectors in a complete basis, any function in a certain class of functions can be expanded as a superposition of functions in a complete basis.

As an example, there is theorem saying that any periodic function

$$f(x + 2\pi) = f(x) \quad (3.52)$$

can be expanded by sin and cos which are periodic over the same range

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)). \quad (3.53)$$

This is called *Fourier series*. The class of periodic functions with period 2π has the complete basis $\{\cos(nx) (n \geq 0); \sin(nx) (n > 0)\}$. All the functions in the basis are *linearly independent*. That is, none of the functions in the basis can be expressed as a superposition of the rest. As a result, the coefficients a_0, a_n, b_n of the Fourier series are uniquely determined by $f(x)$.

The linear independence of a set of functions $f_n(x)$ means that the condition

$$a_1 f_1(x) + a_2 f_2(x) + \dots + a_n f_n(x) = 0$$

can be satisfied for all values of x only if all coefficients a_n vanish. In our discussions we always assume that the complete basis is linearly independent. Otherwise Fourier's trick will not work.

Q: Find the complete basis of functions for periodic functions with period L .

This kind of situation is not rare. There are mathematical theorems (“Sturm-Liouville theorems”) which help us to find complete bases. Roughly speaking, given a differential operator \mathcal{D} (analogous to a matrix), you can find its eigenfunctions $f_n(x)$ (analogous to eigenvectors) satisfying

$$\mathcal{D}f_n(x) = \lambda_n f_n(x), \quad (3.54)$$

where $\lambda_n \in \mathbb{R}$ is the eigenvalue, and the complete set of eigenfunctions $\{f_n(x)\}$ will be a complete basis for a certain class of functions, assuming some conditions (analogous to the Hermiticity of the matrix) are met.

We can construct more examples from the example above without using Sturm-Liouville theorems. The set of even periodic functions with period 2π

$$f(x + 2\pi) = f(x) = f(-x) \quad (3.55)$$

is the same as the set of functions defined on $[0, \pi]$ with Neumann boundary conditions

$$f'(0) = f'(\pi) = 0. \quad (3.56)$$

The theorem about Fourier series then implies that $\{\cos(nx) \ (n \geq 0)\}$ is a complete basis for functions defined on the interval $[0, \pi]$ with Neumann boundary conditions.

Q: Find a complete basis for functions over the interval $[0, L]$ with Neumann boundary conditions.

Similarly, consideration about odd periodic functions leads to a complete basis for functions with Dirichlet boundary conditions.

Q: Find a complete basis for functions over the interval $[0, L]$ with Dirichlet boundary conditions.

Chapter 4

Electric Fields in Matter

4.1 Dielectrics

- **Dielectrics** are insulators that can be polarized.
- **Polarization** is the density of dipole moments inside a dielectric.

In general the polarization of a dielectric depends on the electric field

$$\mathbf{P} = \hat{\mathbf{x}}_i \left(C_i + M_{ij} E_j + \frac{1}{2} R_{ijk} E_j E_k + \dots \right), \quad (4.1)$$

where the constants $C_i, M_{ij}, R_{ijk}, \dots$ are parameters characterizing the dielectric.

For simplicity, we often consider linear dielectrics satisfying the relation

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \quad (4.2)$$

for a constant χ_e (*electric susceptibility*). We also often consider permanent polarizations \mathbf{P}_0 independent of \mathbf{E} .

A polarized dielectric (regardless of how it is polarized, whether it is permanently polarized or polarized by external electric fields) with polarization \mathbf{P} is *equivalent* to the charge distribution

$$\rho_b = -\nabla \cdot \mathbf{P}, \quad (4.3)$$

$$\sigma_b = \hat{\mathbf{n}} \cdot \mathbf{P}. \quad (4.4)$$

After all, dipoles are just a particular kind of charge distribution.

Q: Check that the total charge due to polarization is always zero. That is,

$$\int_V \rho_b d\tau + \oint_{\partial V} \sigma_b da = 0. \quad (4.5)$$

Note that here \mathbf{E} is the electric field at the same point where \mathbf{P} is evaluated. The value of \mathbf{E} may have contributions from the dipole moments represented by \mathbf{P} , but it does not concern us.

Electrostatics is only concerned with the relation among quantities $(\mathbf{E}, \mathbf{P}, \rho, \sigma, V, \dots)$ after everything is settled down.

Q: Compare $\nabla \cdot \mathbf{P} = -\rho_b$ with $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. Can we conclude that $\mathbf{P} = -\epsilon_0 \mathbf{E}$ if $\rho = \rho_b$ (that is, there is no free charge)?

Q: A measurement of \mathbf{E} in practice involves some sort of averaging (with a probe of finite size) over a finite region. Similarly any measurement of ρ or \mathbf{P} also involves some averaging. If the physical laws relating \mathbf{E} to ρ and \mathbf{P} are exact, how do we know that the relation between the averaged quantities $\mathbf{E}_{ave}, \rho_{ave}, \mathbf{P}_{ave}$ also satisfy exactly the same relations?

See Griffiths Sec. 4.2.3.

At the boundary, the polarization \mathbf{P} that determines the surface bound charge density $\sigma_b = \hat{\mathbf{n}} \cdot \mathbf{P}$ is related to the electric field \mathbf{E} at the boundary via $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$. Which value of \mathbf{E} should we use here? There are three possibly natural choices: (1) \mathbf{E}_{above} , (2) \mathbf{E}_{below} , (3) $(\mathbf{E}_{above} + \mathbf{E}_{below})/2$.

The electric field \mathbf{E} that we should use to determine \mathbf{P} at the boundary is the value of the electric field *inside* the linear medium. After all, polarization (and the parameter χ_e) is only defined inside the medium.

Another way to see this is to view the “boundary” of a medium and vacuum as an approximate description of a smooth change of $\chi_e(\mathbf{r})$ from a given value χ_e well inside the medium to 0 (vacuum). Let the boundary be located at $z = 0$, and the region $z < 0$ is filled with a linear medium. The bound charge density is

$$\rho_b(z) = -\nabla \cdot \mathbf{P} = -\epsilon_0 \nabla \cdot [\chi_e(z) \mathbf{E}(z)]. \quad (4.6)$$

If the value of $\chi_e(z)$ changes from χ_e to 0 over a short region $z \in (-\epsilon, \epsilon)$ with small ϵ , we can approximate the charges ρ_b in this region by a surface charge density

$$\begin{aligned} \sigma_b &= \int_{-\epsilon}^{\epsilon} dz \rho_b(z) \\ &= -\epsilon_0 \int_{-\epsilon}^{\epsilon} dz \nabla \cdot [\chi_e(z) \mathbf{E}(z)] \\ &\simeq -\epsilon_0 \int_{-\epsilon}^{\epsilon} dz \partial_z [\chi_e(z) E_z(z)] \\ &= -\epsilon_0 (\chi_e(\epsilon) E_z(\epsilon) - \chi_e(-\epsilon) E_z(-\epsilon)) \\ &= \epsilon_0 \chi_e(-\epsilon) E_z(-\epsilon), \end{aligned} \quad (4.7)$$

where $E_z(-\epsilon)$ is the electric field inside the linear medium. This proves our claim above that it is the electric field inside the medium that determines the surface bound charge density σ_b .

4.2 Electric Displacement

There are two kinds of charge density

$$\rho = \rho_f + \rho_b, \quad (4.8)$$

the *free charge density* and the *bound charge density*. The bound charge density is the charge density due to the polarization of a dielectric. See (4.3) and (4.4). All the rest of the charges are called free charges.

Using (4.3), we find that the electric displacement

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P} \quad (4.9)$$

satisfies

$$\nabla \cdot \mathbf{D} = \rho_f \quad \Leftrightarrow \quad \oint \mathbf{D} \cdot d\mathbf{a} = Q_f. \quad (4.10)$$

In general we do not know what $\nabla \times \mathbf{D}$ is, so we can not determine \mathbf{D} from ρ_f directly using Coulomb's law, like how we can determine \mathbf{E} directly from ρ . But when there is sufficient symmetry, it is possible to use (4.10) as a modified Gauss law to determine \mathbf{D} .

Q: For a given free charge density ρ_f , in the presence of a dielectric, how do we determine \mathbf{E} in electrostatics?

For a given dielectric with a known relation between \mathbf{P} and \mathbf{E} , we can express \mathbf{D} as a function of \mathbf{E}

$$\mathbf{D} = \mathbf{D}(\mathbf{E}), \quad (4.11)$$

and then we can use

$$\nabla \cdot \mathbf{D} = \rho_f, \quad (4.12)$$

$$\nabla \times \mathbf{E} = 0 \quad (4.13)$$

to determine \mathbf{E} from a given ρ_f .

For simplicity, we often consider the following two special cases.

1. Linear Dielectric

For linear dielectrics, \mathbf{P} is proportional to \mathbf{E} , and so according to (4.9), \mathbf{D} is also proportional to \mathbf{E} ,

$$\mathbf{D} = \epsilon \mathbf{E}, \quad (4.14)$$

where

$$\epsilon = \epsilon_0(1 + \chi_e) \quad (4.15)$$

The ratio ϵ/ϵ_0 is called "dielectric constant".

is the *permittivity* of the material. The laws of electrostatics in a linear dielectric are thus simply the results of replacing ϵ_0 by ϵ and ρ by ρ_f in the laws of electrostatics for vacuum:

$$\nabla \cdot \mathbf{E} = \rho_f/\epsilon, \quad (4.16)$$

$$\nabla \times \mathbf{E} = 0. \quad (4.17)$$

2. Permanent Polarization

If the polarization \mathbf{P} is a given function, the bound charge density ρ_b and σ_b are determined. We can simply solve the electrostatic equations in vacuum with total charge $\rho = \rho_f + \rho_b$. (Of course, depending on the problem, there may be other even simpler ways to solve the problem.)

Regardless of any properties of the materials, the boundary conditions on an interface between two materials are

$$D_{above}^\perp - D_{below}^\perp = \sigma_f, \quad (4.18)$$

$$\mathbf{E}_{above}^\parallel - \mathbf{E}_{below}^\parallel = 0. \quad (4.19)$$

4.3 Energy in Dielectric Systems

There are two formulas for the work done to change an electrostatic state:

$$\Delta W = \int (\Delta\rho)V d\tau = \int \epsilon_0(\Delta\mathbf{E}) \cdot \mathbf{E} d\tau, \quad (4.20)$$

$$\Delta W_f = \int (\Delta\rho_f)V d\tau = \int (\Delta\mathbf{D}) \cdot \mathbf{E} d\tau. \quad (4.21)$$

- The first equation is the work done on all charges, including both ρ_f and ρ_b . The 2nd equation is the work done on ρ_f alone. When ρ_f changes, ρ_b also changes according to how dipoles react to changes in external fields.
- The difference of the two equations is

$$\Delta W_b = \Delta W - \Delta W_f = - \int (\Delta\mathbf{P}) \cdot \mathbf{E} d\tau. \quad (4.22)$$

Imagine that within a small volume $d\tau$ the polarization is represented by a dipole $\mathbf{p} = q\mathbf{d}$ with two charges $\pm q$ separated by a displacement vector \mathbf{d} . Then $-(\Delta\mathbf{P}) \cdot \mathbf{E}d\tau = (\Delta\mathbf{d})(-q \cdot \mathbf{E})$ is the work done on the bound charges by an external force $\mathbf{F} = -q\mathbf{E}$ exerted against the electrostatic force.

- The external force on bound charges comes from the “spring”, so the work done on the bound charges (by the “spring”) is the same as the negative of the work done (by the bound charges) on the “spring”

$$W_b = -W_{spring}. \quad (4.23)$$

- As a result,

$$W_f = W_f + W_b + W_{spring} = U_{electrostatic} + U_{spring}. \quad (4.24)$$

That is, the sum of the electrostatic energy and the potential energy of the spring equals W_f , which is the work done on the free charges alone.

Recall that the potential energy of an electric dipole \mathbf{p} in an electric field \mathbf{E} is $U = -\mathbf{p} \cdot \mathbf{E}$.

We can not apply force directly to the bound charges.

The “spring” represents anything that keeps the charges of a dipole in a bound state at the atomic scale. Although it is in fact the (quantum) electrostatic force between the electron and the nucleus, the field is averaged out at a macroscopic scale. That is, the microscopic electrostatic force cannot be described as a macroscopic electrostatic force, and we replace it by imaginary springs.

- The first equation (4.20) gives the change in the total electrostatic energy, which is $\epsilon_0 E^2/2$ per unit volume. The 2nd equation (4.21) gives the total change in the sum of the total electrostatic energy ($\epsilon_0 E^2/2$ per unit volume) plus the potential energy of the “springs” for each dipole moment in the dielectric that are stretched to react to the change in ρ_f .
- The two equations can be integrated to

$$W = \frac{\epsilon_0}{2} \int E^2 d\tau, \quad (4.25)$$

$$W_f = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} d\tau \quad (4.26)$$

but the 2nd equation only applies to linear dielectrics.

Q: Which formula do we use to compute the work we need to do in order to assemble a system of free charges and dielectrics.

Exercises

1. A sphere of radius R is polarized with the polarization density

$$\mathbf{P} = \alpha \frac{\hat{\mathbf{r}}}{r}, \quad r < R. \quad (4.27)$$

Find ρ_b , σ_b and \mathbf{E} .

permanent polarization
See also Griffiths Probs. 4.11, 4.12, 4.13, 4.15.

2. A spherical shell of inner radius a and outer radius b is uniformly polarized with the polarization vector \mathbf{P}_0 . Find \mathbf{E} in the three regions (i) $r < a$, (ii) $a < r < b$, (iii) $b < r$.

permanent polarization

3. Suppose the field inside a large piece of dielectric is \mathbf{E}_0 , so that the electric displacement is $\mathbf{D}_0 = \epsilon_0 \mathbf{E}_0 + \mathbf{P}$.

boundary conditions
Griffiths Prob. 4.16

- (a) Now a small spherical cavity is hollowed out of the material. Find the field at the center of the cavity in terms of \mathbf{E}_0 and \mathbf{P} . Also find the displacement at the center of the cavity in terms of \mathbf{D}_0 and \mathbf{P} .
- (b) Do the same for a long needle-shaped cavity running parallel to \mathbf{P} .
- (c) Do the same for a thin wafer-shaped perpendicular to \mathbf{P} .

Assume the cavities are small enough that \mathbf{P} , \mathbf{E}_0 and \mathbf{D}_0 are essentially uniform. *Hint:* Carving out a cavity is the same as superimposing an object of the same shape but opposite polarization.

4. A conducting sphere of radius a with charge Q is surrounded by a linear dielectric with permittivity ϵ_1 from $r = a$ to $r = b$. Another layer of dielectric with permittivity ϵ_2 extends from $r = b$ to $r = c$. Find $\mathbf{D}, \mathbf{E}, V(r)$ for all regions of r (i) $r < a$, (ii) $a < r < b$, (iii) $b < r < c$ (iv) $c < r$. Find σ_b at $r = a, b$ and c .
linear dielectric with symmetry
See also Griffiths Probs. 4.18, 4.19, 4.20, 4.21, 4.32.
5. An infinitely long, grounded conducting cylinder of radius a is covered by a layer of dielectric material with permittivity ϵ out to radius b . Find the electric potential of all space ($s < a$, $a < s < b$, $b < s$) if it is placed in an otherwise uniform electric field \mathbf{E}_0 perpendicular to the axis of the cylinder.
linear dielectric + separation of variables
See also Griffiths Probs. 4.22, 4.23, 4.24, 4.34.
6. A uniformly polarized sphere of radius a with $\mathbf{P} = P_0\hat{\mathbf{z}}$ is surrounded by a spherical shell of linear dielectric $\epsilon = 2\epsilon_0$ from radius a to $b = 2a$. Find (a) the electric potential V in the regions $r > b$ and (b) the surface bound charge density σ_b at $r = b$.
linear dielectric + permanent polarization + separation of variables
7. A spherical shell of inner radius a and outer radius b ($b > a$) is made of linear dielectric with permittivity $\epsilon = 2\epsilon_0$. It is placed in an otherwise uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. Find the electric field in the cavity ($0 < r < a$).
linear dielectric + separation of variables
8. A long cylindrical capacitor of length L with inner radius a and outer radius c is filled with a linear dielectric with permittivity ϵ from $s = a$ to $s = b$ ($a < b < c$). The charge on the inner and outer capacitors are Q and $-Q$, respectively. When the dielectric shell is pulled out by a distance x ($x < L$), find (a) the total electrostatic energy of the system, (b) the direction and magnitude of the force on the linear dielectric, and (c) the answer to (b) if the capacitor is maintained at a constant potential V (rather than a constant charge).
energy
See also Griffiths Probs. 4.26, 4.27, 4.28.
9. Suppose the region above the plane $z = 0$ is filled with linear dielectric of permittivity ϵ_1 , and the region below with linear dielectric of permittivity ϵ_2 . Find the force on a point charge q at a distance d above the origin.
method of images
Related to Ex. 4.8 and Prob. 4.25.
10. Consider a crystal for which $D_i = \epsilon_i E_i$ ($i = 1, 2, 3$), where ϵ_i are constants. Derive the general expression for the energy of the material for given electric field \mathbf{E} analogous to (4.58).
anisotropic material
11. Suppose the region $z > 0$ is filled with a material with $\mathbf{D} = (a+b|\mathbf{E}|)\mathbf{E}$ for some constants $a, b > 0$; and the region $z < 0$ filled with a linear dielectric with permittivity ϵ . If the electric field at $z = 0_-$ (immediately below the interface) is $E_0\hat{\mathbf{z}}$ with $E_0 > 0$, and the surface free charge density is 0, find the total surface bound charge density at $z = 0$.
nonlinear media

12. Griffiths Probs. 4.29, 4.35, 4.36, 4.37, 4.38.

miscellaneous topics

Chapter 5

Magnetostatics

5.1 Electrostatics vs. Magnetostatics

Electrostatics		Magnetostatics	
$\mathbf{F} = q\mathbf{E}$	$\mathbf{F} = \int d\tau \rho \mathbf{E}$	$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$	$\mathbf{F} = I \int d\mathbf{l} \times \mathbf{B}$
$\nabla \cdot \mathbf{E} = \rho/\epsilon_0$	(Gauss' law)	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$	(Ampère's law)
$\nabla \times \mathbf{E} = 0$		$\nabla \cdot \mathbf{B} = 0$	
$\oint_S \mathbf{E} \cdot d\mathbf{l} = \int_V d\tau \rho/\epsilon_0 = Q_{enc}/\epsilon_0$	(Gauss' law)	$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{a} = I_{enc}/\epsilon_0$	(Ampère's law)
$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$		$\oint_S \mathbf{B} \cdot d\mathbf{a} = 0$	
$\mathbf{E} = -\nabla V$		$\mathbf{B} = \nabla \times \mathbf{A}$	
$V(\mathbf{r}) = V(\mathbf{r}_0) - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{l}'$		$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \int d\tau' \mathbf{B}(\mathbf{r}') \times \hat{\mathbf{z}}/z^2$	(gauge: $\nabla \cdot \mathbf{A} = 0$)
V is defined up to constant		\mathbf{A} is defined up to $\nabla \lambda$	
$\nabla^2 V = -\rho/\epsilon_0$		$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$	
$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d\tau' \rho(\mathbf{r}') \hat{\mathbf{z}}/z^2$	(Coulomb's law)	$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d\tau' \mathbf{J}(\mathbf{r}') \times \hat{\mathbf{z}}/z^2$	(Biot-Savart law)
$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d\tau' \rho(\mathbf{r}')/z$	(Coulomb's law)	$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d\tau' \mathbf{J}(\mathbf{r}')/z$	(gauge: $\nabla \cdot \mathbf{A} = 0$)
$\mathbf{E}_{above} - \mathbf{E}_{below} = \hat{\mathbf{n}}\sigma/\epsilon_0$	(B.C.)	$\mathbf{B}_{above} - \mathbf{B}_{below} = \mu_0 \mathbf{K} \times \hat{\mathbf{n}}$	(B.C.)
$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\sigma/\epsilon_0$	$V_{above} = V_{below}$	$\frac{\partial \mathbf{A}_{above}}{\partial n} - \frac{\partial \mathbf{A}_{below}}{\partial n} = -\mu_0 \mathbf{K}$	$\mathbf{A}_{above} = \mathbf{A}_{below}$
electrostatic energy	$u_e \equiv \frac{\epsilon_0}{2} E^2$	magnetostatic energy	$u_m \equiv \frac{1}{2\mu_0} B^2$
electric dipole moment	$\mathbf{p} = \int d\tau \rho \mathbf{r}$	magnetic dipole moment	$\mathbf{m} = I \int d\mathbf{a}$
permittivity		permeability	
$\epsilon_0 \simeq 8.85 \times 10^{-12} C^2/(Nm^2)$		$\mu_0 \equiv 4\pi \times 10^{-7} N/A^2$	
unit for \mathbf{E} : $V/m = N/C$		unit for \mathbf{B} : $T = N/(Am)$	

Q: Can you derive all the equations in the list above from any one of the expressions (PDE, integral eq, or Biot-Savart law) of the fundamental laws of Magnetostatics?

5.2 Definition of Magnetostatics

Magnetostatics is the theory of magnetic fields when both charges and currents are time-independent, that is, $\partial \mathbf{J} / \partial t = 0$, and $\partial \rho / \partial t = 0$. In other words, we have (1) steady currents and (2) stationary charges. The continuity equation (conservation of charges) implies that

$$\nabla \cdot \mathbf{J} = 0. \quad (5.1)$$

The continuity equation is

$$\nabla \cdot \mathbf{J} + \partial \rho / \partial t = 0.$$

Q: What is the definition of *electric current*?

Q: How could there be any current if all the charges are at rest?

Q: What is the surface current density $\mathbf{K}(\theta)$ for a rotating spherical shell of radius R with surface charge density $\sigma(\theta)$ at angular velocity ω along the z -axis? What is the (volume) current density $\mathbf{J}(r, \theta)$ for the same current? Is this a configuration described by magnetostatics?

Q: Prove that

$$\int_{\mathcal{V}} \mathbf{J} d\tau = \frac{d\mathbf{p}}{dt}, \quad (5.2)$$

Griffiths Prob. 5.7

where \mathbf{p} is the dipole moment.

5.3 Lorentz Force

The Lorentz force law for a point charge q is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (5.3)$$

It implies that the magnetic force on a current is

$$\mathbf{F}_{mag} = I \int (d\mathbf{l} \times \mathbf{B}), \quad (5.4)$$

$$\mathbf{F}_{mag} = \int da (\mathbf{K} \times \mathbf{B}), \quad (5.5)$$

$$\mathbf{F}_{mag} = \int d\tau (\mathbf{J} \times \mathbf{B}). \quad (5.6)$$

Q: What is *cyclotron motion*?

Q: What is *cycloid motion*?

5.4 Magnetic Field Lines

Magnetic field lines have no endpoints.

Ampère's law states that the density of field lines around a closed loop of field line is proportional to the total current enclosed in the loop.

The right hand rule applies.

When the configuration has a large symmetry, we can determine the magnetic field by the Ampère's law without using the other PDE $\nabla \cdot \mathbf{B} = 0$.

Q: What are the examples of sufficiently large symmetries for the Ampère's law to determine \mathbf{B} ?

Griffiths Sec. 5.3.3

5.5 Vector Potential \mathbf{A}

Since $\nabla \cdot \mathbf{B} = 0$, locally there always exists vector field \mathbf{A} such that

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (5.7)$$

Q: Find \mathbf{A} that satisfies (5.7) for $\mathbf{B} = B_0 \hat{z}$ for a constant B_0 .

The equation above is equivalent to the integral equation

$$\oint d\mathbf{l} \cdot \mathbf{A} = \Phi \equiv \int d\mathbf{a} \cdot \mathbf{B}, \quad (5.8)$$

where the right hand side is the magnetic flux enclosed in the loop on which \mathbf{A} is integrated. (The right hand rule applies.) In cases when there are large symmetries, we can determine \mathbf{A} from this equation for given \mathbf{B} .

There are in fact infinitely many vector fields that satisfy the same relation with \mathbf{B} (5.7), because if \mathbf{A}_0 satisfies $\mathbf{B}_0 = \nabla \times \mathbf{A}_0$ for a given magnetic field configuration \mathbf{B}_0 , then \mathbf{A} defined by

$$\mathbf{A} = \mathbf{A}_0 + \nabla f(\mathbf{r}) \quad (5.9)$$

will also satisfy $\mathbf{B}_0 = \nabla \times \mathbf{A}$ for an arbitrary function $f(\mathbf{r})$. This is analogous to the fact that if we have a V_0 such that $\mathbf{E}_0 = -\nabla V_0$ for a given electric field \mathbf{E}_0 , then any constant shift of V_0 , i.e. $V = V_0 + c$ also satisfies $\mathbf{E}_0 = -\nabla V$ for an arbitrary constant c .

The fact that a physical state (corresponding to a given magnetic field \mathbf{B}) is not changed when \mathbf{A} is changed under the transformation

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla f \quad (5.10)$$

is call a *gauge symmetry*. The most salient feature that distinguishes a gauge symmetry from a global symmetry (such as the translation and rotation in space) is that a gauge symmetry transformation does not change the physical state, while a global symmetry transformation does.

The freedom in choosing \mathbf{A} for a given \mathbf{B} is more useful than the freedom in choosing V for a given \mathbf{E} . Any particular choice of \mathbf{A} from the infinitely many possible choices that correspond to the same magnetic field \mathbf{B} is called a choice of *gauge*. We can always choose \mathbf{A} locally such that

cf. Theorem 2 in Sec.1.5.

Note that formally the relation between \mathbf{A} and \mathbf{B} is exactly the same as the relation between \mathbf{B} and $\mu_0 \mathbf{J}$.

$$\nabla \times \mathbf{A} = \mathbf{B}, \quad (5.11)$$

$$\nabla \cdot \mathbf{A} = 0 \quad (5.12)$$

at the same time for any given \mathbf{B} . The condition $\nabla \cdot \mathbf{A} = 0$ that we use to pick a gauge is called a *gauge fixing condition*. In this gauge, the relation between \mathbf{A} and $\mu_0 \mathbf{J}$ is formally the same as the relation between V and ρ/ϵ_0 ,

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \quad (5.13)$$

so we can solve \mathbf{A} by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d\tau' \frac{\mathbf{J}(\mathbf{r}')}{z}, \quad (5.14)$$

which can of course turn into

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int da' \frac{\mathbf{K}(\mathbf{r}')}{z}, \quad (5.15)$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int dl' \frac{I(\mathbf{r}')}{z}, \quad (5.16)$$

depending on the nature of the current.

In the case when $\mathbf{J} = 0$ in a region, both the divergence and curl of \mathbf{B} vanishes; the PDE's satisfied by \mathbf{B} are exactly the same as those PDE's satisfied by \mathbf{E} when ρ vanishes. In this region we can define W by $\mathbf{B} = -\nabla W$, and W satisfies the Laplace equation $\nabla^2 W = 0$ in this region. The technique of separation of variables may be useful here.

More about gauge symmetry and gauge fixing conditions in Chap.7.

More about this in Ch.6.

5.6 Boundary Conditions

The boundary conditions for \mathbf{B} is

$$\mathbf{B}_{above}^{\parallel} - \mathbf{B}_{below}^{\parallel} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}}). \quad (5.17)$$

$\nabla \cdot \mathbf{B} = 0$ implies that B^{\perp} is continuous across the boundary. The magnitude of the discontinuity of \mathbf{B}^{\parallel} at the boundary is $\mu_0 K$, and its direction can be determined by the right hand rule.

In terms of \mathbf{A} , it is

$$\mathbf{A}_{above} - \mathbf{A}_{below} = 0, \quad (5.18)$$

$$\frac{\partial \mathbf{A}_{above}}{\partial n} - \frac{\partial \mathbf{A}_{below}}{\partial n} = -\mu_0 \mathbf{K}. \quad (5.19)$$

Q: Derive these boundary conditions from laws of magnetostatics.

5.7 Multipole Expansion of \mathbf{A}

The mathematical part of the story is very similar to that of V . The basic mathematical relation used here is

$$\frac{1}{z} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos \theta'). \quad (5.20)$$

But the physical part is different. The “monopole” term of \mathbf{A} in the multipole expansion is absent because there is no magnetic monopole in our theory of magnetostatics. As a result, the first term (most important term at large r) is the *magnetic dipole* term

$$\mathbf{A}_{dipole} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}. \quad (5.21)$$

The magnetic dipole moment is defined by

$$\mathbf{m} \equiv I \int d\mathbf{a} = I\mathbf{a}, \quad (5.22)$$

where \mathbf{a} is the “vector area” of the loop.

Q: What is the magnetic field \mathbf{B} for a magnetic dipole \mathbf{m} ?

Q: What is the magnetic quadrupole term in \mathbf{A} ?

We will introduce a theory that includes magnetic monopoles in Chap.7.

As the electric dipole term, the magnetic dipole term is proportional to $1/r^2$.

Griffiths Prob. 5.33

Exercises

- Griffiths Probs. 5.13–5.17, 5.25, 5.26, 5.33, 5.36, 5.42, 5.49, 5.52, 5.54, 5.55, 5.56, 5.57, 5.60 are recommended.
- A sphere of radius R with constant surface charge density σ rotates at constant angular velocity ω . Let the axis of rotation be the z -axis and the center of the sphere at the origin.
 - Find the surface current density $\mathbf{K}(\theta)$.
 - Find $\mathbf{B}(\mathbf{r})$ using Biot-Savart law for $r < R$.
 - For $\mathbf{B}(\mathbf{r})$ given in (2b), solve (5.59), (5.61) to find $\mathbf{A}(\mathbf{r})$.
 - For $\mathbf{B}(\mathbf{r})$ given in (2b), calculate $\mathbf{A}(\mathbf{r})$ using an integral of the same form as the Biot-Savart law but with $\mathbf{B}(\mathbf{r})$ replaced by $\mathbf{A}(\mathbf{r})$ and $\mu_0 \mathbf{J}(\mathbf{r}')$ replace by $\mathbf{B}(\mathbf{r}')$.
 - Check your answer in (2b) by verifying Ampère’s law (eq.(5.42) in Griffiths) for the amperian loop which includes the range $[-a, a]$ on the z -axis and a semicircle of radius a for $a < R$.
 - Find $\mathbf{A}(\mathbf{r})$ using (5.64) in Griffiths for $r < R$ and $r > R$.

Hint for (2b): To make the integration easier, mimic the steps of Example 5.11 in Griffiths, i.e., choose a new coordinate system in which the point of observation \mathbf{r} is on the z -axis.

(2f) is Example 5.11 in Griffiths.

- (g) Compute \mathbf{B} from the result of (2f) via $\mathbf{B} = \nabla \times \mathbf{A}$, for both regions $r < R$ and $r > R$.
- (h) Using the results of the problem above, check that the boundary conditions (5.74) in Griffiths is satisfied.
- (i) For $r < R$ or $r > R$ (any region where $\mathbf{J} = 0$), let $\mathbf{B} = -\nabla U$ for some function U . Prove that U satisfies Laplace equation $\nabla^2 U = 0$.
- (j) Following the previous question, find the boundary conditions for U on the boundary $r = R$. (Express your answer in terms of \mathbf{K} without quoting the explicit expression of \mathbf{K} from (1a) above.
- (k) Following (2j), for $\mathbf{K}(\theta)$ given in (2a), find $U(\mathbf{r})$ for both $r > R$ and $r < R$ by quoting the general solution to the Laplace equation and solving the boundary conditions for U .
- (l) Compute \mathbf{B} from the solution of U obtained above.
- (m) Find the magnetic force per unit area $\mathbf{F}(\theta)$ on the spherical surface.
- (n) Find the magnetic dipole moment \mathbf{m} of the rotating sphere.
- (o) Check that for $r > R$ the potential \mathbf{A} for the spinning sphere is that of a perfect dipole.

Hint for (2j): There should be two conditions for U .

Griffiths Prob. 5.42.

Griffiths Prob. 5.36.

Griffiths Prob. 5.36.

3. An infinite cylinder of radius R with a charge density $\rho(s) = ks$ (for some constant k) rotates at an angular velocity ω .

- (a) Find the current density $\mathbf{J}(s, \phi)$.
- (b) Find $\mathbf{B}(\mathbf{r})$ using Ampère's law for $s < R$ and $s > R$.
- (c) Find the force per unit volume $\mathbf{F}(s)$ for $s < R$.
- (d) Find $\mathbf{A}(\mathbf{r})$ using the notion of magnetic flux Φ as it was done in Example 5.12 in Griffiths.
- (e) For $r \gg R$, view the cylinder as a stack of magnetic dipole moments and calculate $\mathbf{A}(\mathbf{r})$ for the cylinder as an integral of $\mathbf{A}_{dip}(\mathbf{r})$ (see eq.(5.83) in Griffiths).

4. A pair of parallel long straight wires move at a constant velocity v . Each wire carries a constant line charge density of λ and $-\lambda$, respectively.

This problem resembles Prof. 5.16 in Griffiths.

- (a) Find the magnetic force between the wires.
- (b) At what velocity v would the magnetic force balance the electric force?

Chapter 6

Magnetic Fields in Matter

6.1 Materials

1. paramagnetism, diamagnetism, ferromagnetism, antiferromagnetism, ...
2. insulator, semiconductor, conductor, superconductor, ...
3. piezoelectricity, pyroelectricity, thermoelectricity, ...
4. piezomagnetism, ...
5. GMR: giant magnetoresistance (Nobel prize 2007)

For a macroscopic piece of material, the effects of magnetic quadrupole and higher multipoles are very small and so we can ignore them. Instead we focus our attention on magnetic dipole moments as a good approximation. The macroscopic quantity suitable for this approximation is

$$\mathbf{Magnetization} = \frac{\text{Magnetic dipole moment}}{\text{volume}}. \quad (6.1)$$

For the sake of determining the static magnetic fields, objects without magnetic “charges” or electric currents are irrelevant. (Magnetic field is “blind” to electric charges at rest, for example.) In other words, if object A and object B carry exactly the same magnetic charges and electric currents, we do not have to distinguish A from B as far as magnetic field is concerned. As an example, it is shown in Griffiths as well as in the lectures that a piece of material with magnetization \mathbf{M} is equivalent to a distribution of currents $\mathbf{J}_b = \nabla \times \mathbf{M}$ in the bulk and $\mathbf{K}_b = \mathbf{M} \times \hat{n}$ on the boundary.

Higher multipole moments would be of more interest when we consider objects at microscopic scale, e.g., a single molecule.

Here, by magnetic “charges” we mean magnetic monopoles (if they exist), magnetic dipoles, magnetic quadrupoles, etc.

6.2 Polarization vs. Magnetization

Electric field in matter	Magnetic field in matter
\mathbf{P} Polarization	\mathbf{M} Magnetization
$\rho_b = -\nabla \cdot \mathbf{P}$ bound charge density $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$ surface bound charge density	$\mathbf{J}_b = \nabla \times \mathbf{M}$ bound current density $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$ surface bound current density
$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ displacement $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$ $\mathbf{D} = \epsilon \mathbf{E}$ linear dielectric	$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}$ H -field $\mathbf{M} = \chi_m \mathbf{H}$ $\mathbf{B} = \mu \mathbf{H}$ linear media
$\nabla \cdot \mathbf{D} = \rho_f$ $\nabla \times \mathbf{E} = 0$	$\nabla \times \mathbf{H} = \mathbf{J}_f$ $\nabla \cdot \mathbf{B} = 0$
$D_{ab}^\perp - D_{bel}^\perp = \sigma_f$ B.C. $\mathbf{D}_{ab}^\parallel - \mathbf{D}_{bel}^\parallel = \mathbf{P}_{ab}^\parallel - \mathbf{P}_{bel}^\parallel$	$H_{ab}^\perp - H_{bel}^\perp = -(M_{ab}^\perp - M_{bel}^\perp)$ B.C. $\mathbf{H}_{ab}^\parallel - \mathbf{H}_{bel}^\parallel = \mathbf{K}_f \times \hat{\mathbf{n}}$
$E_{ab}^\perp - E_{bel}^\perp = \sigma / \epsilon_0$ B.C. $\mathbf{E}_{ab}^\parallel - \mathbf{E}_{bel}^\parallel = 0$	$B_{ab}^\perp - B_{bel}^\perp = 0$ B.C. $\mathbf{B}_{ab}^\parallel - \mathbf{B}_{bel}^\parallel = \mu_0 \mathbf{K}_f \times \hat{\mathbf{n}}$
$\epsilon_{ab} \frac{\partial V_{ab}}{\partial n} - \epsilon_{bel} \frac{\partial V_{bel}}{\partial n} = -\sigma_f$ B.C. $V_{ab} = V_{bel}$	$\frac{1}{\mu_{ab}} \frac{\partial \mathbf{A}_{ab}}{\partial n} - \frac{1}{\mu_{bel}} \frac{\partial \mathbf{A}_{bel}}{\partial n} = -\mathbf{K}_f$ B.C. $\mathbf{A}_{ab} = \mathbf{A}_{bel}$

Q: How to use $\nabla \times \mathbf{H} = \mathbf{J}_b$ to determine \mathbf{H} from the free current when there is sufficient symmetries?

Q: What kind of materials will be the magnetic analogues of conductors?

Q: How to use the technique of separation of variables to solve for magnetic fields in matter?

Q: Show that \mathbf{J} , \mathbf{J}_f and \mathbf{J}_b are all linearly related to each other in linear media.

6.3 Separation of Variables

The technique of separation of variables can be used to solve problems involving magnetic fields. The assumption needed is that $\mathbf{J} = 0$ or $\mathbf{J}_f = 0$, so that $\nabla \times \mathbf{B} = 0$ or $\nabla \times \mathbf{H} = 0$. They allow us to define U or W by $\mathbf{B} = -\nabla U$ or $\mathbf{H} = -\nabla W$.

When there is nontrivial magnetization (including linear media), that is, when the space is more than pure vacuum, the bound current \mathbf{J}_b may be nonzero, and $\nabla \times \mathbf{B}$ may or may not be 0. In these problems only \mathbf{J}_f is given, and \mathbf{J} is to be solved. If indeed $\mathbf{J}_f = 0$, we can define

$$\mathbf{H} = -\nabla W. \quad (6.2)$$

Obviously we can shift W by a constant without changing \mathbf{H} .

Since $\nabla \cdot \mathbf{B} = 0$, if the space is filled with linear media ($\mathbf{H} = \mathbf{B}/\mu$), we also

have $\nabla \cdot \mathbf{H} = 0$, and so W satisfies the Laplace equation

$$\nabla^2 W = 0. \quad (6.3)$$

If the space is not filled with linear media, but a material with given magnetization \mathbf{M} , W still satisfies the Laplace equation if $\nabla \cdot \mathbf{M} = 0$.

Typically the problem involves regions of space divided by boundaries. In each region W satisfies the Laplace equation. We need to solve W by imposing boundary conditions on each boundary. The boundary conditions are that (1) B^\perp is continuous on the boundary, and (2) the discontinuity in \mathbf{H}^\parallel is given by the surface free current density \mathbf{K}_f . In terms of W , the condition (1) is

$$\mu_1 \frac{\partial W_1}{\partial n} = \mu_2 \frac{\partial W_2}{\partial n}. \quad (6.4)$$

The condition (2) says that the derivative of W in tangential directions of the boundary is in general discontinuous

$$\nabla^\parallel W_1 - \nabla^\parallel W_2 = \mathbf{K}_f \times \hat{\mathbf{n}}. \quad (6.5)$$

We can solve W by imposing these boundary conditions on general solutions of Laplace equations.

Suppose we shift W_1 such that $W_1 = W_2$ at a chosen point p on the boundary. The values of W at other points on the boundary can be determined by integrating the tangential derivatives of W along a curve connected to p . Thus W must be discontinuous somewhere on the boundary unless $\mathbf{K}_f = 0$ identically. If, on the other hand, indeed we have $\mathbf{K}_f = 0$ on the boundary, the 2nd boundary condition (6.5) can be replaced by the continuity of W

$$W_1 = W_2. \quad (6.6)$$

Sometimes the space is known to have $\mathbf{J} = 0$. In these cases we can define U by

$$\mathbf{B} = -\nabla U. \quad (6.7)$$

Then $\nabla \cdot \mathbf{B} = 0$ implies that U satisfies the Laplace equation. We can then express the boundary conditions in terms of U and solve U .

For example, in linear media, since $\mathbf{B} = \mu\mathbf{H}$, we have $\mu_0\mathbf{J} = \mu\mathbf{J}_f$. So $\mathbf{J}_f = 0$ implies that $\mathbf{J} = 0$.

Exercises

- Two layers of materials of constant polarization and magnetization \mathbf{P}_i and \mathbf{M}_i ($i = 1, 2$)

$$\mathbf{P}_1 = \hat{x}P_x + \hat{y}P_y + \hat{z}P_z, \quad \mathbf{M}_1 = \hat{x}M_x + \hat{y}M_y + \hat{z}M_z \quad (0 < z < a),$$

$$\mathbf{P}_2 = \hat{x}P'_x + \hat{y}P'_y + \hat{z}P'_z, \quad \mathbf{M}_2 = \hat{x}M'_x + \hat{y}M'_y + \hat{z}M'_z \quad (a < z < b),$$

extends along the $x - y$ plane. Find $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}$ for (i) $0 < z < a$, (ii) $a < z < b$, (iii) $b < z$.

2. An infinite slab of thickness d with a uniform magnetization $\mathbf{M} = M_x \hat{x} + M_y \hat{y} + M_z \hat{z}$ (M_x, M_y, M_z are constants) extends along the $x - y$ plane. Find the magnetic field \mathbf{B} for a point at a distance L from the infinite slab.
3. A long cylinder of radius R carries a magnetization parallel to the axis $\mathbf{M} = ks^2 \hat{z}$, where k is a constant and s is the distance from the axis.
 - (a) Find the magnetic field inside and outside the cylinder.
 - (b) Find \mathbf{H} inside and outside the cylinder.
 - (c) Check that the Ampère's law (6.20) in Griffiths is satisfied.
4. A long cylinder of radius R carries a current with current density $\mathbf{J}(s) = J(s) \hat{z}$ along its axis. The cylinder is a linear material with permeability μ , and the magnetic field is found to be

$$\mathbf{B} = ks^2 \hat{\phi}. \quad (6.8)$$

Find $J(s)$.

5. A long cylinder of radius R carries a current with current density $\mathbf{J}(s) = J(s) \hat{z}$ along its axis. The cylinder is a linear material with permeability $\mu(s) = a + bs$ for constants a, b . (Let $\mu(s) > 0$ for all s .) For a given total current I , find the current distribution $J(s)$ by minimizing the energy stored in the magnetic field

$$W_m = \frac{1}{2} \int d\tau \mathbf{B} \cdot \mathbf{H}. \quad (6.9)$$

6. A long cylinder of radius R made of linear magnetic material with permeability μ lies along the x -axis in an otherwise uniform magnetic field $\mathbf{B} = B_0 \hat{z}$. Find the magnetic field inside the cylinder.
7. A magnetized long cylinder of radius R with $\mathbf{M} = M_0 \hat{x}$ for constant M_0 lies along the z -axis. Let $\mathbf{H} = -\nabla W$ for some function W . Find the equation satisfied by W inside and outside the cylinder, and the boundary conditions W needs to satisfy on the surface. Then find the magnetic field \mathbf{B} inside the magnetized cylinder ($s < R$).
8. Griffiths Prob. 6.13.
9. Griffiths Probs. 6.4, 6.7, 6.8, 6.9, 6.15, 6.16, 6.18, 6.21, 6.23, 6.24, 6.26, 6.28.

Chapter 7

Electrodynamics

In Chap. 12, Maxwell's eq's will be rewritten more compactly as

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu,$$

$$\epsilon^{\mu\nu\lambda\rho} \partial_\nu F_{\lambda\rho} = 0.$$

Maxwell's equations in vacuum

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (7.1)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (7.2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (7.3)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (7.4)$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (7.5)$$

The **flux rule** is

$$\mathcal{E} = -\frac{d\Phi}{dt}. \quad (7.6)$$

The emf \mathcal{E} for a loop \mathcal{C} is defined by

$$\mathcal{E}(\mathcal{C}) = \oint_{\mathcal{C}} \mathbf{f} \cdot d\mathbf{l}, \quad (7.7)$$

where \mathbf{f} is the force on a unit charge.

Q: What is the connection between the Lorentz force law and the flux rule?

The magnetic flux Φ over a surface \mathcal{S} is defined by

$$\Phi(\mathcal{C}) = \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a}, \quad (7.8)$$

which is uniquely determined by the boundary \mathcal{C} of \mathcal{S} . That is, all surfaces with the same boundary have the same flux.

Q: Prove the statement above.

Q: What is the relation between the continuity equation

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (7.9)$$

and Maxwell's equations?

Q: Compared with the laws of electrostatics and magnetostatics, which term is the contribution of Faraday's law, and which term is the contribution of Maxwell's displacement current?

Q: Which of the 3 experiments conducted by Faraday (see Griffiths Fig. 7.20 on p. 301) shares (share) the same theoretical basis with (1) electric motors (2) electric generators, respectively.

Maxwell's equations with magnetic charges and currents

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0}, \quad (7.10)$$

$$\nabla \cdot \mathbf{B} = \mu_0 \rho_m, \quad (7.11)$$

$$\nabla \times \mathbf{E} = -\mu_0 \mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t}, \quad (7.12)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_e + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (7.13)$$

Q: Can you derive the continuity equations for both the electric and magnetic charges from the Maxwell's equations with magnetic charges and currents?

Q: Can you define the potentials V and \mathbf{A} when there are magnetic charges and currents?

Maxwell's equations in matter

$$\nabla \cdot \mathbf{D} = \rho_f, \quad (7.14)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \quad (7.15)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (7.16)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (7.17)$$

Bound charge density $\rho_b = -\nabla \cdot \mathbf{P}$.

Bound current density $\mathbf{J}_b = \nabla \times \mathbf{M}$.

Polarization current $\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}$.

Boundary conditions

$$D_1^\perp - D_2^\perp = \sigma_f, \quad (7.18)$$

$$\mathbf{H}_1^\parallel - \mathbf{H}_2^\parallel = \mathbf{K}_f \times \hat{\mathbf{n}}, \quad (7.19)$$

$$\mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0, \quad (7.20)$$

$$B_1^\perp - B_2^\perp = 0. \quad (7.21)$$

7.1 Definitions

Ohm's law is

$$\mathbf{J} = \sigma \mathbf{E}, \quad (7.22)$$

where σ is called the **conductivity** of the medium. The **resistivity** is defined by $\rho = 1/\sigma$.

The **resistance** R of a resistor is defined by

$$V = RI. \quad (7.23)$$

The **capacitance** C of a capacitor is defined by

$$Q = CV \quad \Rightarrow \quad I = C \frac{dV}{dt}. \quad (7.24)$$

The (self) **inductance** L of an inductor is defined by

$$\Phi = LI \quad \Rightarrow \quad V = L \frac{dI}{dt}. \quad (7.26)$$

The **mutual inductance** M_{12} of two loops \mathcal{C}_1 and \mathcal{C}_2 is defined by

$$\Phi(\mathcal{C}_1) = M_{12}I_2, \quad (7.27)$$

which is the flux on \mathcal{C}_1 due to a current on \mathcal{C}_2 . Straightforward calculation shows that

$$M_{12} = M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r}. \quad (7.28)$$

Q: Find L for a solenoid of N turns with radius R and length h . (*Hint:* Note that, since V is defined by adding up the emf on each turn of the wire, Φ should be taken to be N times the flux through each turn.)

Q: For a loop of conducting wire, Φ is determined by I . We can always define a quantity by $L = \Phi(I)/I$. But why do we find L to be independent of I in all the examples?

The unit of inductance is henries

$$\text{H} \equiv \frac{\text{Volt} \cdot \text{second}}{\text{amp}} \quad (7.29)$$

7.2 Energy in Electric and Magnetic Fields

To turn on a current I_0 on an inductor, we start with $I = 0$ and increase the current to I_0 . As the inductor is reluctant to change its current, we have to apply a potential difference across it to force the current to change. This implies that it takes energy to force charges (current) through the inductor. Assuming energy conservation, the energy it takes to increase the current

In these expressions, we define V and I of a circuit element such that if V is defined as the potential difference $V(+)$ – $V(-)$ between two endpoints labelled by “+” and “–”, we have $I > 0$ when the current flows from the endpoint “+” to “–”.

There is a minus sign in eq.(7.26) on p. 313

$$\mathcal{E} = -LdI/dt \quad (7.25)$$

in Griffiths. It manifests Lenz's law. Here V is defined to be $-\mathcal{E}$ in our convention explained above.

from 0 to I_0 , if not wasted on dissipation or radiation, must contribute to the potential energy associated to the inductor.

Consider an adiabatic process through which a current is slowly turned on in a an inductor, we can compute its energy W_m as a function of the current I . Recalling the analogous story for a capacitor, the energies in capacitors and inductors are

$$W_C = \frac{1}{2}CV^2, \quad (7.30)$$

$$W_L = \frac{1}{2}LI^2. \quad (7.31)$$

The potential energy of an inductor with a fixed current can be attributed to the magnetic field. This is analogous to how we defined the energy in the electric field. Together, the energies in electric and magnetic fields are

$$W_e = \frac{1}{2} \int \rho V d\tau = \frac{\epsilon_0}{2} \int E^2 d\tau, \quad (7.32)$$

$$W_m = \frac{1}{2} \int \mathbf{J} \cdot \mathbf{A} d\tau = \frac{1}{2\mu_0} \int B^2 d\tau. \quad (7.33)$$

Strictly speaking the expressions of W_e and W_m can only be trusted in electrostatics and magnetostatics because we have only considered static configurations. We will see however that these expressions are valid even for time-dependent configurations.

Supplementary Topic:

Gauss' Law and Ampère's Law Revisited

We have learned in previous chapters that when there is sufficient symmetry in the charge distribution, one can use Gauss's law alone to determine the electric field \mathbf{E} (without using $\nabla \times \mathbf{E} = 0$). Similarly, one can use Ampère's law alone to determine the magnetic field \mathbf{B} (without using $\nabla \cdot \mathbf{B} = 0$) if the symmetry of the current distribution is large enough. However, there is actually an assumption hidden behind these techniques, as we mentioned in the lectures. The assumption is that a symmetry of the source is also a symmetry of the field, as long as the symmetry is respected by the equations of motion.

In fact, we have already mentioned a counter-example. When the charge density ρ is constant throughout the whole space, the electric field has infinitely many solutions that satisfy both laws of electrostatics ($\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, $\nabla \times \mathbf{E} = 0$), but none of these solutions respect all symmetries (translation plus rotation) of the charge distribution, even though these symmetries are also respected by all relevant physical laws.

This counter example is special in the sense that the charge density does not approach to zero at infinities like we normally would assume. Perhaps the

One can compute the inductance by first computing B due to a current I , then computing W_m from B , and finally computing L from W_m .

failure of the trick of Gauss's law plus symmetry should be attributed to the subtlety involved in how we define the charge distribution at spatial infinity.

If we focus our attention on the example of a point charge, can we claim that Gauss's law is always sufficient to determine the electric field? In other words, can we consistently imagine a universe in which the Gauss' law $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ remains the same, but the rest of the laws of electrostatics are different, such that the electric field for a point charge Q is not $\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}$?

A convenient way to explore this question is not to focus on how the other law of electrostatics $\nabla \times \mathbf{E} = 0$ is changed, but on how the expression of energy density $\frac{\epsilon_0}{2} E^2$ is changed.

The Gauss's law states that in a region free of charge, the electric flux is conserved. Consider a given amount of electric flux $\Phi_E = \int \mathbf{E} \cdot d\mathbf{a}$ passing through a region of cross section A and length L . If we uniformly spread the flux over the cross section of area A , the magnitude of the electric field is $E = \Phi_E/A$. If the energy density of the field is $\propto E^2$, the total energy of the field in the region of volume LA is $\propto L(\Phi_E)^2/A$, which is inversely proportional to the area A . In other words, spreading the flux over a larger cross-sectional area A helps reducing the energy of the system. This is why the electric flux flowing out of a point charge spreads uniformly in all directions.

On the other hand, if the energy density is given by a different formula, such that the total energy in the volume LA is an increasing function of A , the flux would then rather concentrate on a small cross-sectional area to reduce the total energy. In this case, the flux coming out of a point charge Q would pass through a thin tube connecting the point charge Q with the infinity, or with another point charge $-Q$.

This consideration of a hypothetical universe with different physical laws can actually be applied to dielectrics in the real world. In a dielectric, the flux of \mathbf{D} is conserved in regions where free charges are absent. The energy density is a function of \mathbf{D} , whose precise expression depends on the nature of the material. A priori, there can exist materials for which the technique (symmetry plus Gauss's law) used to solve Example 4.4 in Griffiths does not work.

Similarly, the trick of Example 6.2 (symmetry plus Ampère's law) may not work for all kinds of materials. To check whether the same trick will work for a particular material, you need to know how the energy density depends on \mathbf{H} .

7.3 RLC Circuits

7.3.1 Basic circuit elements

For a resistor with the resistance R ,

$$V = RI. \quad (7.34)$$



For a capacitor with the capacitance C ,

$$I = C \frac{dV}{dt} \quad (7.35)$$

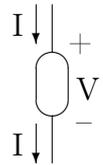


For an inductor with inductance L ,

$$V = L \frac{dI}{dt}. \quad (7.36)$$



For all the circuit elements, if we assume that the current I flows in a given direction (if it flows in the opposite direction, the value of I would be negative), the potential difference V across the circuit element is defined as the difference $V = V_{in} - V_{out}$, where V_{in} and V_{out} are the electric potential at the point where the current I flows in (by assumption) and that at the point where I comes out.



7.3.2 Kirchhoff's circuit laws

Kirchhoff's current law

At every node in a circuit, the sum over currents flowing in (or out of) the node vanishes

$$\sum_k I_k = 0. \quad (7.37)$$

Here we assume that all currents I_k are defined with respect to the directions of flowing in (or out of) the node.

charge conservation

Kirchhoff's voltage law

Around any closed loop in a circuit, the sum over potential differences vanishes

$$\sum_a V_a = 0. \quad (7.38)$$

Here we assume that the potentials V_a are all defined in the same direction along the loop.

Electric potential is a well defined function of space

Solving for V and I

Each circuit element has a definite relation between its V and I . To solve for the currents and the potentials in a circuit, you should consider independent relations among all V 's and I 's as a result of Kirchhoff's circuit laws applied to the nodes and loops of a circuit. After simplification, this would lead to a differential equation for you to solve.

See Griffiths Ex. 7.12, and Prob. 7.25 for simple examples.

Exercises

1. Find the inductance of a long solenoid of radius a and length h with N turns of wire. Find the numerical value of the inductance for $a = 5\text{mm}$, $h = 3\text{cm}$ and $N = 100$.
2. Find the inductance and capacitance per unit length for the coaxial cable of inner radius a and outer radius b (see Fig. 7.39 on p. 319 in Griffiths).
3. Find the inductance and capacitance per unit length for a pair of parallel straight wires at a distance d from each other. The currents in the two wires are in opposite directions. For simplicity, assume that the wires are thin tubes of radius a ($a \ll d$), and the current distributes uniformly over the surface of the tube.
4. Check that the product of the inductance and capacitance per unit length are the same in both problems above.
5. What would happen if a is taken to be 0 in Prob. 3? Find the numerical value of the inductance per unit length for $d = 1\text{m}$, $a = 0.1\text{mm}$, and for $d = 1\text{cm}$, $a = 0.1\text{cm}$.
6. If a monopole of magnetic charge q_m passes through a conducting loop of resistance R from infinity to infinity, the emf induced by the change in flux through the loop generates a current. Find the charge Q passing a given point on the loop.
7. Find the magnetic potential \mathbf{A} for a magnetic monopole q_m at the origin as follows.

(a) Using the magnetic version of Gauss' law, find \mathbf{B} .

(b) Use the ansatz

$$\mathbf{A} = A_\phi \hat{\phi}, \quad (7.39)$$

and solve

$$\nabla \times \mathbf{A} = \mathbf{B}. \quad (7.40)$$

(c) Since

$$\nabla \cdot \mathbf{B} = \mu_0 \rho_m, \quad (7.41)$$

when there are magnetic charges, \mathbf{A} should not exist. What is “wrong” with the solution \mathbf{A} you found?

Actually there is nothing wrong with the solution of \mathbf{A} if properly interpreted. Ref: T.T. Wu, C.N. Yang, Phys. Rev. D12, 3845 (1975).

8. In the quasistatic approximation, compute the electric field at a distance s from the axis of a long solenoid with n turns of current $I(t)$ per unit length. Assume that s is larger than the radius of the solenoid. Why does the electric field at a distance from the solenoid react to a change in the current instantaneously, instead of taking some time for the change to propagate (at the speed of light) to the point s ?
9. A long cylinder of radius R with charge density $\rho(s) = a/s$ rotates around its axis, the z -axis, with angular velocity ω . (a is a constant.) The permeability of the cylinder is μ .
 - (a) Find the vector potential \mathbf{A} inside and outside the cylinder, assuming $\nabla \cdot \mathbf{A} = 0$.
 - (b) A circular wire of radius L on the $x - y$ plane, surrounding the cylinder ($L > R$), has a constant line charge density λ . The circle is initially at rest, and then the angular velocity of the cylinder is dropped from ω to 0. Find the angular momentum of the circle after the cylinder stops rotating, assuming that the circle rotates without friction.
10. Which of the following statement(s) is (are) wrong?
 - (a) Lorentz force law can be derived from the Maxwell's equations.
 - (b) If the magnetic monopole exists, it will be possible to violate the energy conservation law.
 - (c) The electric field \mathbf{E} exerts a force on a magnetic charge in the direction of its motion.
 - (d) The fact that generators generate electric currents is an evidence of Faraday's law.
 - (e) The flux rule $\mathcal{E} = -\frac{d}{dt}\Phi$ is invalid when the magnetic field is changing and the loop is moving at the same time.
11. Griffiths' Prob's 7.1 – 7.4, 7.6 – 7.12, 7.15 – 7.20, 7.23, 7.25, 7.28, 7.29, 7.40, 7.42, 7.53, 7.54, 7.58.

電磁學（二）（100學年度第二學期）

授課教師：賀培銘（辦公室 R812; 電話：3366-5192; Email: pmho@phys.ntu.edu.tw）

上課時間：星期一 第8節；星期四 第678節

上課地點：新物 112

Office Hours: 星期四 1PM – 2PM（也可以email或當面另外約時間。）

助教：盧奕銓 <allen82217@gmail.com> ； 林鼎恩 <r99222068@ntu.edu.tw>
（可以用 email 與助教約時間討論。）

課程網頁：<https://ceiba.ntu.edu.tw/982electromagnetism/>
請隨時注意課程網頁之各項宣佈事項。

課本：Griffiths: Introduction to Electrodynamics (3rd ed.)

學期成績：作業、期中考、期末考 各佔大約三分之一。

附註：

網頁提供之講義僅供作為課本之補充教材，而非替代品。

指派作業將公佈於課程網頁之作業區。課本之完整內容均需熟讀。

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課程大綱：

週次	日期	單元主題
第1週	2/20,2/23	Review + Chap. 7
第2週	2/27,3/01	Chap. 8 Conservation Laws（2/27 調整放假）
第3週	3/05,3/8	（3/3 網路加選課程截止；3/4 網路退選課程截止）
第4週	3/12,3/15	Chap. 9 Electromagnetic Waves
第5週	3/19,3/22	
第6週	3/26,3/29	
第7週	4/02,4/05	（4/03,04,05,06 春假）
第8週	4/09,4/12	Chap. 10 Potentials and Fields
第9週	4/16,4/19	4/16 複習；4/19 期中考
第10週	4/23,4/26	
第11週	4/30,5/03	
第12週	5/07,5/10	Chap. 11 Radiation
第13週	5/14,5/17	（5/18 停修申請截止）
第14週	5/21,5/24	
第15週	5/28,5/31	Chap. 12 Electrodynamics and Relativity
第16週	6/04,6/07	
第17週	6/11,6/14	
第18週	6/21	期末考

Chapter 8

Conservation Laws

The prototype of conservation laws is the continuity equation. In differential form it is

$$\nabla \cdot \mathbf{J} + \frac{\partial}{\partial t} \rho = 0. \quad (8.1)$$

In integral form it is

$$\oint \mathbf{J} \cdot d\mathbf{a} + \frac{d}{dt} \int \rho d\tau = 0. \quad (8.2)$$

In this chapter we talk about 3 kinds of conservation laws.

1. Conservation of Energy

$$\oint \mathbf{S} \cdot d\mathbf{a} + \frac{d}{dt} [\int u_{em} d\tau + U_{mech}] = 0, \quad (8.3)$$

$$u_{em} = \frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right), \quad \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}. \quad (8.4)$$

Noether's theorem states that for each symmetry there is a conserved quantity. Translation symmetry in space-time leads to the conservation of momentum and energy, and rotation symmetry to the conservation of angular momentum. The gauge symmetry of electromagnetism leads to charge conservation.

2. Conservation of Momentum

$$\oint (-\overleftrightarrow{\mathbf{T}}) \cdot d\mathbf{a} + \frac{d}{dt} [\int \mathcal{P}_{em} d\tau + \mathbf{p}_{mech}] = 0, \quad (8.5)$$

$$T_{ij} = \epsilon_0 (E_i E_j - \frac{1}{2} \delta_{ij} E^2) + \frac{1}{\mu_0} (B_i B_j - \frac{1}{2} \delta_{ij} B^2), \quad (8.6)$$

$$\mathcal{P}_{em} = \mu_0 \epsilon_0 \mathbf{S} = \epsilon_0 \mathbf{E} \times \mathbf{B}. \quad (8.7)$$

\mathbf{S} is called the Poynting vector.

3. Conservation of Angular Momentum

$$\ell_{em} = \mathbf{r} \times \mathcal{P}_{em}. \quad (8.8)$$

The Maxwell equations considered in this course assume a flat (Minkowski) spacetime. We have translation symmetry and Lorentz symmetry (together

We will discuss Lorentz symmetry in more details in Chap. 12.

they are called Poincare symmetry). Noether's theorem implies conserved charges for all these symmetries, but why have we not talked about the conserved charges associated with the boosts? The reason is that the conserved charges corresponding to boosts are usually not very interesting. It is easier to see this for the example of a free particle in Minkowski space. In this case, the conserved charges associated with boosts are $(x - \dot{x}t)$, which is the position of the particle at $t = 0$ (the initial condition).

Lorentz symmetry includes rotations and boosts. A boost is a change of reference frame by moving at constant velocity.

8.1 Comments

8.1.1 Comments on the Conservation of Energy

The conservation laws of energy, momentum and angular momentum are slightly different from the continuity equation because they involve the "source terms". Take the conservation of energy for example. Only the sum of all kinds of energies is conserved. The energy in the EM field does not have to be conserved. Through Lorentz force, the energy in the EM field and the energy in the charges can be exchanged. The work done on the charge through Lorentz force per unit time,

$$\frac{dW}{dt} = \int d\tau \mathbf{E} \cdot \mathbf{J}, \quad (8.9)$$

is how much energy the EM field loses per unit time, or equivalently how much energy the charges gain per unit time, $\frac{dU_{mech}}{dt}$. Thus in the conservation law of energy (8.3), we should replace the term $\frac{dU_{mech}}{dt}$ by (8.9).

Q: Derive the Conservation of Energy by yourself in two different ways:

(i) Start with the energy density of the EM field and derive the expression of the Poynting vector by demanding that the changes in the energy of the EM field correspond to the effect due to energy flux density and work done to the charges. (ii) Start with the work done to the charges through Lorentz force and derive the expression of the Poynting vector and the energy density of the EM field by rewriting (8.9) as the sum of a time derivative and a gradient.

When we try to verify the conservation of energy, we are free to define the energy and energy flux for the electric magnetic field in any way. The conservation of energy is thus equivalent to the existence of an expression which can be used to define the energy and energy flux of EM field. When energy is conserved with a certain definition of energy, it is possible that there are other definitions of energy that are also conserved. It is then a problem of taste, convenience or convention to choose one of the definitions to be called "energy". On the other hand, when energy seems to be not conserved, perhaps it is only because we have not found the right definition of energy. However,

The energy is normally demanded to be a local quantity.

often we can prove that no local observable can be used as a definition of energy so that it is conserved.

Q: Consider the equation of motion for a simple harmonic oscillator

$$m\ddot{x} + kx = 0. \tag{8.10}$$

Multiply \dot{x} to it and rewrite the equation as

$$\frac{d}{dt}(\text{something}) = 0. \tag{8.11}$$

Check that this “something” is the energy of the SHO.

Q: Consider a deformation of the equation above

$$m\ddot{x} + kx + \gamma\dot{x} = 0. \tag{8.12}$$

1. Prove that it is impossible to define a conserved energy assuming that it is a local observable, i.e., a function of x and its finite time derivatives.
2. Show the conserved “energy” would exist if we do not assume that it is a local observable, but can be, say, an indefinite integral of a local observable.

What happens if energy is not conserved? For a system with time translation symmetry, Noether’s theorem tells us how to define energy such that it is conserved. According to the folklore, if the energy is not conserved, it means that there is something with energy that is not accounted for. Of course this folklore assumes that there is time translation symmetry, so it does not apply to the whole universe.

8.1.2 Comments on the Conservation of Momentum

The conservation law of momentum is similar to that of the energy. The former is slightly more complicated because the momentum \mathbf{P} is a vector and so it has 3 components. Each component P_i is conserved by itself. The conservation law for each P_i involves the density of momentum $(\mathcal{P}_{em})_i$ in the EM field, the momentum $(p_{mech})_i$ of the charges, and the momentum flux density \mathbf{T}_i of the EM field. Since the flux density is a vector, we have a vector \mathbf{T}_i for each fixed index i . This makes $\overset{\leftrightarrow}{\mathbf{T}}$ a tensor with 2 indices when we suppress the index i .

Note that the change rate of the momentum of the charges is given by

$$\mathbf{F} = \int_{\mathcal{V}} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \rho d\tau = \int_{\mathcal{V}} (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) d\tau. \tag{8.13}$$

We can replace $\frac{d\mathbf{p}_{mech}}{dt}$ by this expression in the conservation law of momentum (8.5) as the source term.

Another complication comes from the fact that the conservation of momentum has a different interpretation. By definition, force is

$$\mathbf{F} = \frac{d}{dt} \mathbf{p}_{mech}, \quad (8.14)$$

and thus (8.5) can be rewritten as

$$\mathbf{F} = \oint \overset{\leftrightarrow}{\mathbf{T}} \cdot d\mathbf{a} - \frac{d}{dt} \int \mathcal{P}_{em} d\tau. \quad (8.15)$$

Thus the conservation law of momentum can be interpreted as the law of force on charges.

This is also why the quantity $\overset{\leftrightarrow}{\mathbf{T}}$ is called the Maxwell's stress tensor. On a surface, a stress tensor $\overset{\leftrightarrow}{\mathbf{T}}$ gives the force $\overset{\leftrightarrow}{\mathbf{T}} \cdot d\mathbf{a}$ on an infinitesimal area element $d\mathbf{a}$. Consider a cube with 6 faces. One can exert a force in the x , y or z direction on any given face. Therefore we need a rank-2 tensor to describe the force exerted on a surface. One of the two indices of the tensor is used to specify the direction of the area, and the other index is used to specify the direction of the force. An interesting feature of the Maxwell's stress tensor is that it is symmetric in its two indices, so we do not have to remember which index refers to the direction of the force or the direction of the area element.

For time-independent or sinusoidal time-dependent fields, the (time-averaged) force is

$$\langle \mathbf{F} \rangle = \oint \langle \overset{\leftrightarrow}{\mathbf{T}} \rangle \cdot d\mathbf{a} \quad (8.16)$$

because the (time-average of the) time derivative term vanishes in these cases. In these cases the expression (8.15) has an advantage over the equivalent expression (8.13) because the latter is a volume integral while the former is merely a surface integral.

Q: Derive the conservation law of momentum in two different ways, analogous to the question for the conservation of energy in Sec. 8.1.1.

8.1.3 Comment of Griffiths

“Even perfectly static fields can harbor momentum and angular momentum.” (Griffiths p.359).

8.1.4 Meaning of the subscript “mech”

U_{mech} , \mathbf{p}_{mech} and \mathbf{L}_{mech} refer to the energy, linear momentum and angular momentum carried by the charges, including, for example, the kinetic energy of point charges and the potential energy of a spring to which a charge is attached. While u_{em} represents the energy carried by the electromagnetic field, all other

kinds of energy are included in U_{mech} , otherwise the conservation of energy will not hold. In fact, in certain applications, even part of the electromagnetic energy is included in U_{mech} . This happens when we focus our attention on a particular part of the electromagnetic field and ignore others. For example, the radiation as a way of heat conduction is strictly speaking also electromagnetic fields but we often focus on the coherent parts (or macroscopic part) of the electromagnetic field in order to save the trouble of describing the complicated incoherent (microscopic) part of the field. We have done the same thing earlier when we study the electric field in matter.

If the energy of charges U_{mech} has an integral form $U_{mech} = \int u_{mech} d\tau$, the conservation of energy can be put in the differential form. Similar comments apply to \mathbf{p}_{mech} and \mathbf{L}_{mech} .

8.1.5 Energy and momentum for photons

For a relativistic particle of mass m in free motion at velocity v , its energy and momentum are

$$E = m\gamma c^2, \quad p = m\gamma v, \quad (8.17)$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (8.18)$$

For a massless particle moving at the speed of light, neither of the relations above makes sense, but the ratio

$$\frac{cp}{E} = \frac{v}{c} \quad (8.19)$$

continues to make sense in the limit $m \rightarrow 0$ and it is

$$cp = E. \quad (8.20)$$

Suppose the energy density is u_{em} for a distribution of photons, the energy density flux (Poynting vector) is then

$$S = u_{em}c. \quad (8.21)$$

The momentum density is then

$$\mathcal{P}_{em} = u_{em}/c = S/c^2 = \epsilon_0\mu_0 S. \quad (8.22)$$

This is precisely the relation we found for EM fields. It is thus consistent to interpret EM fields as photons.

The relation between u_{em} and S can be established only for plane waves but not for generic configurations of fields. For plane waves, \mathbf{E} , \mathbf{B} and the direction of propagation are all perpendicular to each other. The magnitudes are related as $E = cB$, and so we have $u_{em} = \epsilon_0 E^2$, $S = \sqrt{\frac{\epsilon_0}{\mu_0}} E^2$ and thus (8.21) follows.

8.2 Exercises

1. For a spherical linear dielectric of radius R and permittivity ϵ in an otherwise uniform electric field $\mathbf{E}_0 = E_0 \hat{z}$, find the total force on
 - (a) the dielectric sphere,
 - (b) a spherical region of radius a ($a < R$) inside the dielectric sphere,
 - (c) the upper hemisphere of the dielectric sphere.

Compute the force in both ways: (1) using the stress tensor, (2) using the Lorentz force law.

2. Prove that the total force of a localized charge distribution on itself always vanishes. (Hint: Assume that all charges are located within a sphere of radius a , and then consider the stress tensor integrated over a sphere of radius R with $R \rightarrow \infty$.)
3. A pair of coaxial cylindrical conductors along the z -axis is placed in a uniform magnetic field $\mathbf{B} = B \hat{z}$. The radii of the inner and outer cylinders are a and b , and the charges on the inner and outer cylinders are Q and $-Q$, respectively.
 - (a) Find the total angular momentum in the electromagnetic field.
 - (b) Connect the cylinders by a straight wire along in the radial direction so that the cylindrical capacitor slowly discharges. If the pair of cylinders can freely rotate, find its angular momentum obtained from the Lorentz force on the wire.
 - (c) If we slowly reduce the uniform magnetic field to zero, and if the charges are glued to the cylinders, find the total angular momentum of the cylinders obtained from the electric field induced by the changes in the magnetic field.
 - (d) Check that the angular momentum is conserved in both cases.

4. Problem 8.11 on p.362 in Griffiths.
5. Problem 8.12 on p.362 in Griffiths.
6. Griffiths Prob's. 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.15.

Chapter 9

Electromagnetic Waves

The phenomenon of waves is everywhere. In general, waves are fluctuations around a stable configuration. The fluctuations exhibit the “wave” behavior because of (1) its tendency of returning to the stable state and (2) the inertia that leads to overshoots.

Any smooth function $f(x)$ can be decomposed into a superposition of simple harmonic waves ($\tilde{f}(k)e^{ikx}$ in complex notation)

$$f(x) = \int_{-\infty}^{\infty} dk \tilde{f}(k)e^{ikx}. \quad (9.1)$$

The map from $f(x)$ to $\tilde{f}(k)$ is called *Fourier transform*. Sometimes there is an extra factor of $(2\pi)^{-1}$ or $(2\pi)^{-1/2}$ as a convention. Due to Fourier transform, to understand generic solutions of wave equations, it is sufficient to focus on sinusoidal waves for linear wave equations.

The basis of Fourier transform is the equality

$$\int_{-\infty}^{\infty} dk e^{ikx} = 2\pi\delta(x). \quad (9.2)$$

Q: Use (9.2) to show that for a given smooth function $f(x)$, we can choose

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dx}{2\pi} f(x)e^{-ikx} \quad (9.3)$$

so that (9.1) is satisfied.

Q: Show that any smooth function $f(x_1, \dots, x_N)$ of N variables can be expressed as a superposition of plane waves

$$f(\mathbf{x}) = \int d^N x \tilde{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (9.4)$$

It is often convenient to use the complex notation

$$A = A_0 \cos(kx - \omega t + \theta) \rightarrow \tilde{A} = \tilde{A}_0 e^{i(kx - \omega t)}, \quad (9.5)$$

For example, the water in a cup is stable when its surface is flat. Slightly disturbing it produces fluctuations that are called waves. Large disturbances, such as turning the cup over, do not look like “waves”.

Usually the wavelength of a wave in a medium is fixed by the frequency, and vice versa.

where $\tilde{A}_0 = A_0 e^{i\theta}$ to represent a wave. It is convenient to use this complex notation when we superpose waves of the same wavelength/frequency.

Sometimes we want to compute the time-averaged product of A and B

$$\langle AB \rangle = \frac{1}{2} \mathcal{R}e \left[\tilde{A}_0^* \tilde{B}_0 \right] = \frac{1}{4} \left(\tilde{A}_0^* \tilde{B}_0 + \tilde{A}_0 \tilde{B}_0^* \right). \quad (9.6)$$

Q: Prove the relation above.

9.1 Waves on a Stretched String

Small fluctuations on a string satisfy the wave equation

$$\frac{\partial^2 \mathbf{f}}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \mathbf{f}}{\partial t^2}, \quad (9.7)$$

where $\mathbf{f}(z) = f_x(z)\hat{\mathbf{x}} + f_y(z)\hat{\mathbf{y}}$ is the transverse displacement vector for a string stretched along the z -axis. The wave velocity v is given by $v = \sqrt{T/\mu}$, where T is the tension of the string and μ is the mass per unit length.

Comments:

1. The wave equation above is the prototype of all wave equations. In higher dimensions, the wave equation is of the form

$$\nabla^2 \mathbf{f} = \frac{1}{v^2} \frac{\partial^2 \mathbf{f}}{\partial t^2}. \quad (9.8)$$

In general one can have nonlinear wave equations.

2. Superposition principle applies to linear wave equations. That is, if $\mathbf{f}_1, \mathbf{f}_2, \dots$ are solutions to the linear wave equation, then $\mathbf{f} \equiv \mathbf{f}_1 + \mathbf{f}_2 + \dots$ is also a solution of the wave equation.

Q: If the tension of the string increases with the length (like a spring) with a spring constant k per unit length, i.e., the tension is $T = T_0 + k\Delta\ell$, where T_0 is a constant and $\Delta\ell$ is the change in the length of the string per unit change in z , find how the wave equation should be modified, and when can we ignore this modification as a good approximation.

The boundary conditions at the knot of two strings are

1. The knot is a point connecting two strings, so the wave is continuous at the knot,

$$\mathbf{f}(0-) = \mathbf{f}(0+). \quad (9.9)$$

2. The motion of the knot obeys Newton's 2nd law

$$T \left(\frac{\partial \mathbf{f}}{\partial z} \right) \Big|_{0-}^{0+} = m \partial^2 \mathbf{f} / \partial t^2, \quad (9.10)$$

where m is the mass of the knot.

\mathbf{f} can be decomposed as the amplitude f ($f \in \mathbb{R}$) times the polarization vector $\hat{\mathbf{n}}$, which is a complex vector perpendicular to $\hat{\mathbf{z}}$.

For a spring with spring constant k , and the force (tension) on the spring is $T = k\Delta\ell$ when the length changes by $\Delta\ell$. If we view the spring as N equal small springs in series, the spring constant for each small spring is Nk and the change in length for each small spring is $\Delta\ell/N$, while the tension T remains the same.

Boundary conditions are always about *continuity* and *discontinuity*.

Q: Derive the boundary condition (9.10) from the wave equation (9.8) with a position-dependent mass density function $\mu(z)$ so that

$$\frac{1}{v^2} = \frac{\mu(z)}{T} = m\delta(z) + \theta(z), \quad (9.11)$$

where $\theta(z)$ is μ_1 (μ_2) for $z < 0$ ($z > 0$), respectively.

Q: In general (not necessarily related to wave equations), how do you know whether or not you have written down all the boundary conditions you need?

Q: Find the boundary conditions for the two extreme cases: (1) $\mu_2 = \infty$, (2) $\mu_2 = 0$.

9.2 Electromagnetic Monochromatic Plane Waves

Maxwell's equations in linear dielectric imply the wave equations for electromagnetic fields

$$\nabla^2 \mathbf{E} = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \mathbf{E}, \quad (9.12)$$

$$\nabla^2 \mathbf{B} = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \mathbf{B}, \quad (9.13)$$

where

$$\frac{1}{v^2} = \epsilon\mu. \quad (9.14)$$

To find a solution of the source-free Maxwell equations ($\rho = 0$, $\mathbf{J} = 0$), one can first write down a solution of \mathbf{E} of the wave equation (9.12) with some free parameters. Choose the parameters such that the source-free Gauss law ($\nabla \cdot \mathbf{E} = 0$) is satisfied. This guarantees that there exist a magnetic field configuration \mathbf{B} such that (\mathbf{E}, \mathbf{B}) constitutes a solution to the Maxwell's equations. To find this magnetic configuration \mathbf{B} , we solve \mathbf{B} from Faraday's law $\frac{\partial}{\partial t} \mathbf{B} = -\nabla \times \mathbf{E}$. The source-free Ampere's law $\nabla \times \mathbf{B} = \epsilon\mu \frac{\partial}{\partial t} \mathbf{E}$ and the equation $\nabla \cdot \mathbf{B} = 0$ are guaranteed up to time-independent terms. One can then redefine \mathbf{B} by subtracting a time-independent piece from it to satisfy all Maxwell equations.

In insulators, we have

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad (9.15)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{k}}{\omega} \times \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} = \frac{1}{v} \hat{\mathbf{k}} \times \mathbf{E}, \quad (9.16)$$

where \mathbf{k} and ω satisfy

$$\frac{\omega}{|\mathbf{k}|} = v = \frac{1}{\sqrt{\mu\epsilon}} \quad (9.17)$$

and

$$\mathbf{k} \cdot \mathbf{E}_0 = 0. \quad (9.18)$$

In conductors with finite conductivity σ , the formulas above can be modified so that the wave vector \mathbf{k} becomes a complex vector

$$\tilde{\mathbf{k}} = \mathbf{k} + i\boldsymbol{\kappa}. \quad (9.19)$$

The imaginary part leads to an attenuation factor of exponential decay $e^{-\boldsymbol{\kappa} \cdot \mathbf{r}}$. The amplitude of \mathbf{B} is then $\frac{\tilde{k}}{\omega} E_0$. More explicitly, we have

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\tilde{\mathbf{k}} \cdot \mathbf{r} - \omega t)}, \quad (9.20)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\tilde{\mathbf{k}}}{\omega} \times \mathbf{E}_0 e^{i(\tilde{\mathbf{k}} \cdot \mathbf{r} - \omega t)}. \quad (9.21)$$

9.3 Polarization

The polarization of light (photon) is the prototype of a quantum mechanical system. (See Dirac's "The Principles of Quantum Mechanics".) A good understanding of polarizations will help you to understand quantum mechanics.

The polarization of light has many applications, including sunglasses, 3D vision, and the detection of strain and stress in materials. Some animals (e.g. bees and certain types of octopuses) have "P-vision", that is, they see the difference between lights with different polarizations.

9.4 Reflection and Transmission

The main ideas behind solving the problems of reflection and transmission:

1. Decompose all waves into monochromatic plane waves.
2. Waves of different frequencies can be considered independently, because boundary conditions have to be matched at all times.
3. Boundary conditions always demand a matching of the projected wavelength on the interface.
4. Boundary conditions also impose constraints on polarizations.

The same ideas apply to cases with absorption and dispersion.

The boundary conditions derived from Maxwell's equations for linear media are

$$\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma_f, \quad (9.22)$$

$$B_1^\perp - B_2^\perp = 0, \quad (9.23)$$

$$\mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0, \quad (9.24)$$

$$\frac{1}{\mu_1} \mathbf{B}_1^\parallel - \frac{1}{\mu_2} \mathbf{B}_2^\parallel = \mathbf{K}_f \times \hat{\mathbf{n}}, \quad (9.25)$$

where σ_f and \mathbf{K}_f vanish for insulators.

The sum of the reflection coefficient $R = I_R/I_I$ and the transmission coefficient $T = I_T/I_I$ is

$$R + T = 1 \quad (9.26)$$

due to energy conservation, unless energy is absorbed or dissipated.

9.5 Wave Guides

Two kinds of wave guides are considered: (1) those composed of a single piece of conductor (2) those composed of two disconnected pieces of conductors.

We always assume (1) perfect conductors, (2) translation symmetry along the wave guide, (3) closure of the space in which EM waves propagate.

The problem of solving Maxwell's equations is essentially a two dimensional problem.

For those wave guides that are composed of a single piece of conductor, the waves propagating inside can be decomposed into a superposition of TE modes and TM modes. TE modes have transverse electric field ($E_z = 0$), and TM modes have transverse magnetic field ($B_z = 0$). There is no TEM modes.

For those wave guides that are composed of two disconnected pieces of conductors, there are TEM modes.

Actually conductors that do not form a closed space can also be used as wave guides, e.g. two parallel conducting wires.

9.6 Appendix: Fourier Transform

9.6.1 Motivation

Fourier transform is a rearrangement of the information contained in a function $f(x)$. It is based on the fact that any (smooth) function $f(x)$ can be obtained as a superposition of sinusoidal waves with suitable amplitudes and phases

$$f(x) = \int_{-\infty}^{\infty} dk \tilde{f}(k) e^{ikx}, \quad (9.27)$$

where

$$\tilde{f}(k) = |\tilde{f}(k)| e^{i\theta(k)} \in \mathbb{C} \quad (9.28)$$

This Appendix contains materials outside the scope of this course. You are not required to read it.

specifies the amplitude and phase of the sinusoidal wave e^{ikx} with wave number k . Fourier transform is the map from $f(x)$ to $\tilde{f}(k)$. There is no loss of information in this transformation (whenever this transformation is well-defined). We can reconstruct $f(x)$ from $\tilde{f}(k)$ via the *inverse Fourier transform*.

Fourier transform can be generalized to higher dimensions. It implies, for example, that we only need to consider plane waves when we learn about electromagnetic waves, because all electromagnetic waves can be viewed as a superposition of plane waves.

As we will see, sometimes it is very helpful to present the information encoded in a function in terms of its Fourier transform. In particular, it is useful to solve differential equations that are translationally invariant. In later chapters, we will also generalize the notion of Fourier transform for more general differential operators.

When the physical laws are linear equations, Fourier transform is particularly useful.

Our ears work as a Fourier transformer of the sound waves, although it only keeps the information of the amplitudes.

9.6.2 An Identity for Delta Function

The only purpose of this subsection is to derive the identity (9.35), which is the foundation of Fourier transform. The same relation (9.35) will also be derived later in Sec. 9.6.4 as a scaling limit of the relation

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta e^{in\theta} = \delta_0^n, \quad (9.29)$$

which is the foundation of the Fourier series.

Consider the distribution

$$A(x) \equiv \lim_{k \rightarrow \infty} e^{ikx}. \quad (9.30)$$

Firstly it is a distribution rather than a well-defined function of x . As a distribution, $A(x) = 0$, because

$$\int_{-\infty}^{\infty} dx A(x) f(x) = \int_{-\infty}^{\infty} dx \lim_{k \rightarrow \infty} e^{ikx} f(x) = 0 \quad (9.31)$$

for any given smooth function $f(x)$ which vanishes at infinities ($x \rightarrow \pm\infty$).

There are several ways to make this statement more rigorous:

- We can modify our definition of integral as

$$\int dx (\dots) \rightarrow \lim_{\epsilon \rightarrow 0^+} \int dx e^{-\epsilon x^2} (\dots), \quad (9.32)$$

so that $f(x)$ does not have to vanish at infinities. This modification is actually also physically motivated.

- We can use the residue theorem to evaluate the integral. First analytically continue $f(x)$ to the complex plane, and consider the contour encircling the upper half plane. The integration over infinities of the upper half plane has no contribution due to the factor e^{ikx} if $k > 0$, so the residue theorem tells us that

$$\int dx e^{ikx} f(x) = 2\pi i \sum_k e^{ikz_k} \text{Res}_{z_k} f(z).$$

But since we only need to consider residues of poles in the upper half plane, the imaginary part of z_k must be positive, and thus $e^{ikz_k} \rightarrow 0$ as $k \rightarrow \infty$.

Next we consider the distribution

$$B(x) = \int_{-\infty}^{\infty} dk e^{ikx} = \lim_{k \rightarrow \infty} \frac{1}{ix} (e^{ikx} - e^{-ikx}). \quad (9.33)$$

Again, it vanishes (as a distribution) for any $x \neq 0$, but it is ill-defined for $x = 0$. We may guess that it is proportional to $\delta(x)$, which is also 0 for $x \neq 0$ and ill-defined for $x = 0$. We can check this by computing

$$\int_{-a}^a dx B(x) = \int_{-a}^a dx \int_{-\infty}^{\infty} dk e^{ikx} = \int_{-\infty}^{\infty} dk \frac{e^{ika} - e^{-ika}}{ik} = 2\pi \quad (9.34)$$

for any $a > 0$. (You can use the residue theorem to evaluate the last step.)

Thus, if $B(x)$ is proportional to $\delta(x)$, we have $B(x) =$

$$\int_{-\infty}^{\infty} dk e^{ikx} = 2\pi\delta(x). \quad (9.35)$$

Let us try to be more rigorous in deriving this result (9.35). Since $B(x)$ vanishes for all $x \neq 0$, $\int dx B(x)f(x)$ should only depend on the behavior of $f(x)$ at $x = 0$. Because $\int dx B(x)f(x)$ is a linear map of $f(x)$ (to \mathbb{C}), the most general situation is

$$\int_{-\infty}^{\infty} dx B(x)f(x) = a_0 f(0) + a_1 f'(0) + a_2 f''(0) + \dots \quad (9.36)$$

for constant coefficients a_0, a_1, a_2, \dots . This means

$$B(x) = \int_{-\infty}^{\infty} dk e^{ikx} = a_0 \delta(x) - a_1 \delta'(x) + a_2 \delta''(x) - \dots \quad (9.37)$$

$$\int dx \delta^{(n)}(x)f(x) = (-1)^n f^{(n)}(0).$$

What we can learn from (9.34) is only that $a_0 = 2\pi$.

From the definition of $B(x)$ one can check that it is an even function of x . This implies that all $a_n = 0$ for all odd n 's. We can further fix all the rest of

$$\delta^{(n)}(-x) = (-1)^n \delta^{(n)}(x).$$

the coefficients a_n by considering a simple example

$$\begin{aligned} \int_{-\infty}^{\infty} dx B(x) e^{-\frac{\alpha}{2}x^2} &= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx e^{-\frac{\alpha}{2}(x-ik/\alpha)^2} e^{-\frac{1}{2\alpha}k^2} \\ &= \sqrt{\frac{2\pi}{\alpha}} \int_{-\infty}^{\infty} dk e^{-\frac{1}{2\alpha}k^2} \\ &= 2\pi. \end{aligned} \tag{9.38}$$

This should be identified with

$$\int_{-\infty}^{\infty} dx (2\pi\delta(x) + a_2\delta''(x) + a_4\delta^{(4)}(x) + \dots) e^{-\frac{\alpha}{2}x^2} = 2\pi + a_2\alpha + 3a_4\alpha^2 + \dots \tag{9.39}$$

For the equality to hold for all α , we conclude that $a_n = 0$ for all $n > 0$.

9.6.3 Review of Fourier Series

Periodic functions $f(x)$ of period 2π can be approximated by

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)). \tag{9.40}$$

Another equivalent expression is

$$f(x) = \sum_{n \in \mathbb{Z}} f_n e^{inx}. \tag{9.41}$$

If $f(x)$ is real, $a_n, b_n \in \mathbb{R}$ and $f_n = f_{-n}^* \in \mathbb{C}$.

The equal signs in (9.40) and (9.41) are not exact. There can be points of measure 0 where the equality breaks down. In particular there is the Gibbs' phenomenon: if there is a sudden change of the slope of $f(x)$, the value of the Fourier series typically overshoots.

The Fourier series for periodic functions is just a special case of the following general idea: For a given class of functions (e.g. periodic functions on $[-\pi, \pi)$), one can choose a complete basis of functions $\{\phi_n(x)\}$ so that it is always possible to express any function $f(x)$ in this class as a linear superposition of them: $f(x) = \sum_n f_n \phi_n(x)$, which holds almost everywhere on the domain of $f(x)$. The Fourier expansion to be discussed below is another example. Here we give two more examples.

If periodic functions on $[-\pi, \pi)$ can be expressed as (9.40), then even periodic functions can be expressed as

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx), \tag{9.42}$$

and the odd ones as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx). \tag{9.43}$$

These equalities hold in the same sense that $A(x) = 0$ for $A(x)$ defined in (9.30), that is, as equalities of distributions. Two distributions are equal if the integrations of their products with an arbitrary function give the same result. Hence their difference at points of measure 0 can be ignored.

The general theory about such ideas is called the "Sturm-Liouville theory".

These are functions defined on $[0, \pi]$, and their values in $[-\pi, 0]$ are determined by either $f(-x) = f(x)$ or $f(-x) = -f(x)$. Hence we conclude that any function $f(x)$ defined on $[0, \pi]$ can be expanded using either $\cos(nx)$ or $\sin(nx)$. Note that the RHS of (9.42) has the property that its first derivative vanishes at the boundaries $x = 0, \pi$, while the RHS of (9.43) vanishes at $x = 0, \pi$.

9.6.4 Fourier Transform as a Limit of Fourier Series

Any periodic function $f(x)$ of period 2π can be approximated to arbitrary accuracy by $\sum_n f_n e^{inx}$. By a change of variable (scaling of x) we have

$$f(x) = \sum_{n \in \mathbb{Z}} f_n e^{i2\pi nx/L}, \quad x \in [-L/2, L/2]. \quad (9.44)$$

Using

$$\int_{-L/2}^{L/2} dx e^{i2\pi mx/L} e^{i2\pi nx/L} = L \delta_{m+n}^0, \quad (9.45)$$

we find

$$f_n = \frac{1}{L} \int_{-L/2}^{L/2} dx e^{-i2\pi nx/L} f(x). \quad (9.46)$$

Now we take the limit $L \rightarrow \infty$ and expect that any fx. $f(x)$ defined on \mathbb{R} can be approximated to arbitrary accuracy by the expansion. To do so, we imagine that the fx $f_n : \mathbb{Z} \rightarrow \mathbb{C}$ defined on \mathbb{Z} is promoted to a fx. $\hat{f}(n) : \mathbb{R} \rightarrow \mathbb{C}$ defined on \mathbb{R} by interpolation, so that $\hat{f}(n) = f_n$ for $n \in \mathbb{Z}$. In the limit of large L , we focus on fx's f_n which are very smooth, so that they do not change much when $n \rightarrow n + 1$. That is, $f_{n+1} - f_n \rightarrow 0$ as $L \rightarrow \infty$.

Consider the change of variables

$$n \rightarrow k = 2\pi n/L, \quad (9.47)$$

$$f_n \rightarrow \tilde{f}(k) = \mathcal{N} \hat{f}(n). \quad (9.48)$$

The first equation implies that

$$\sum_n \simeq \int dn = \int dk \frac{L}{2\pi}, \quad (9.49)$$

$$\delta_{m+n}^0 \simeq \delta(m+n) = \delta\left(\frac{L}{2\pi}(k+k')\right) = \frac{2\pi}{L} \delta(k+k'). \quad (9.50)$$

Thus, in the limit $n \rightarrow \infty$, eqs. (9.44), (9.45) and (9.46) become

$$f(x) = \int_{-\infty}^{\infty} dk \frac{L}{2\pi \mathcal{N}} \tilde{f}(k) e^{ikx}, \quad (9.51)$$

$$\int_{-\infty}^{\infty} dx e^{ikx} e^{ik'x} = 2\pi \delta(k+k'), \quad (9.52)$$

$$\tilde{f}(k) = \frac{\mathcal{N}}{L} \int_{-\infty}^{\infty} dx e^{-ikx} f(x). \quad (9.53)$$

To get a well-defined limit for $L \rightarrow \infty$, the ratio L/\mathcal{N} should be a finite number. On the other hand, the choice of the finite number is merely a convention.

Fourier transform can be viewed as a limit of Fourier series. As we can never really measure anything with infinite spatial or temporal extension, the conceptual difference between $f(x)$ defined on \mathbb{R} and periodic functions defined on a finite interval is often an illusion, although for most applications of Fourier transform, Fourier series is simply impractical.

A normalization factor \mathcal{N} is inserted here. A scaling of $\hat{f}(n)$ will all us to scale the value of \mathcal{N} , but \mathcal{N} should be chosen such that $\hat{f}(n)$ is finite in the limit $n \rightarrow \infty$.

To make the connection between the Kronecker delta and Dirac delta fx., we examine a sum/integral of them

$$\begin{aligned} & \sum_n f_n \delta_{m+n}^0 \\ &= f_{-m} \simeq \hat{f}(-m) \\ &= \int dn \hat{f}(n) \delta(m+n). \end{aligned}$$

An often used convention is $L/\mathcal{N} = \sqrt{2\pi}$:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \tilde{f}(k) e^{ikx}, \quad (9.54)$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx} f(x). \quad (9.55)$$

A function $f(x, y)$ of 2 variables is also a function of one variable when the other variable is fixed. We can first do Fourier transform on x for any given y

$$f(x, y) = \int dk_x g(k_x, y) e^{ik_x x},$$

and then do Fourier transform on each $g(k_x, y)$ for given k_x

$$g(k_x, y) = \int dk_y h(k_x, k_y) e^{ik_y y}.$$

Together we have the Fourier transform on both variables of f .

Repeating this decomposition for n -variables, we get

$$f(x) = \int_{-\infty}^{\infty} \frac{d^n k}{(2\pi)^{n/2}} \tilde{f}(k) e^{ikx}, \quad (9.56)$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{d^n x}{(2\pi)^{n/2}} e^{-ikx} f(x). \quad (9.57)$$

Note that the distinction between Fourier transform and inverse Fourier transform is a mere convention. (The two eq's above are interchanged when we interchange $x \leftrightarrow k$.)

9.7 Exercises

1. Plot (schematically) the analogues of Fig. 9.16 and Fig. 9.17 in Griffiths for $n_1 > n_2$.
2. EM wave of angular frequency ω passes from medium 1, through a slab of medium 2, and into medium 3. The permittivity and permeability parameters are ϵ_i, μ_i ($i = 1, 2, 3$). We choose the coordinate system such that the plane of incidence is the $x - z$ plane, the interface between medium 1 and medium 2 is at $z = 0$, and the interface between medium 2 and medium 3 is at $z = d$. The electric field of the incident wave is given by

$$\tilde{\mathbf{E}}_I = \hat{\mathbf{y}} \tilde{E}_{0I} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)}, \quad (9.58)$$

where the wave vector of the incident wave is

$$\mathbf{k}_I = k_1 (\hat{\mathbf{x}} \sin \theta_1 + \hat{\mathbf{z}} \cos \theta_1) \quad (9.59)$$

and $k_1 = \omega \sqrt{\epsilon_1 \mu_1}$.

The constants in the Fourier transform and its inverse

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{\mathcal{N}_1} \tilde{f}(k) e^{ikx},$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dx}{\mathcal{N}_2} e^{-ikx} f(x)$$

always satisfy

$$\mathcal{N}_1 \mathcal{N}_2 = 2\pi.$$

- (a) Find the reflection and transmission coefficients R and T , assuming that $n_1 < n_2$ and $n_1 < n_3$. (The transmitted wave is defined to be the wave in medium 3.)
- (b) If $n_2 < n_1 < n_3$ such that

$$n_1 \sin \theta_1 > n_2, \quad (9.60)$$

how will the answer for part (a) be different?

3. Consider a wave guide with a uniform circular cross section of radius R lying along the z -axis. (This is not a coaxial cable as there is no conducting wire inside the wave guide.) Find the electric field for a monochromatic TM wave propagating down the z axis with rotation symmetry along the z -axis.
4. The boundary of two linear media is on the $y-z$ plane. The permittivity and permeability are ϵ_1, μ_1 and ϵ_2, μ_2 for $x > 0$ and $x < 0$, respectively. Find the electric field for $x < 0$ if the electric field for $x > 0$ is given by

$$\tilde{\mathbf{E}} = \hat{\mathbf{y}} \tilde{E}_0 e^{i(kz - \omega t)} \quad (9.61)$$

for $\epsilon_1 \mu_1 < \epsilon_2 \mu_2$.

5. A solution of the differential equation

$$\frac{d^2 f}{ds^2} + \frac{1}{s} \frac{df}{ds} + k^2 f = 0 \quad (9.62)$$

is the Bessel function of order 0

$$f(s) = J_0(ks). \quad (9.63)$$

- (a) Use this result to find a solution of the wave equation

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} \quad (9.64)$$

of the form

$$\psi(s, z, t) = f(s) e^{i(k_z z - \omega t)}, \quad (9.65)$$

where

$$k_z = \frac{\omega}{c} \cos \alpha \quad (9.66)$$

for some number α for given ω .

- (b) Find the phase velocity and group velocity

$$v_p = \frac{\omega}{k_z}, \quad v_g = \frac{d\omega}{dk_z} \quad (9.67)$$

in the z -direction for fixed α .

(c) Use the identity

$$J_0(s) = \frac{1}{2\pi} \int_{\beta}^{2\pi+\beta} e^{is \cos \theta} d\theta \quad (9.68)$$

to argue that the solution above does not imply that the speed of a signal can travel faster than c .

6. Griffiths Probs. 9.5, 9.6, 9.8, 9.11, 9.12, 9.16, 9.17, 9.18, 9.19, 9.20, 9.25, 9.30, 9.33, 9.34, 9.36, 9.37, 9.38.

Chapter 10

Potentials and Fields

Maxwell's equations are

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B}, \quad (10.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial}{\partial t} \mathbf{D}. \quad (10.2)$$

The first and last equations involve sources (charges and currents). The 2nd and 3rd equations are about basic properties of electric and magnetic fields. The 3rd equation implies (see Griffiths (1.104)) that

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (10.3)$$

for some \mathbf{A} at least locally. Plugging this into the 2nd equation gives

$$\nabla \times \left(\mathbf{E} + \frac{\partial}{\partial t} \mathbf{A} \right) = 0. \quad (10.4)$$

Thus locally there always exists a function V (see Griffiths (1.103)) such that

$$\mathbf{E} + \frac{\partial}{\partial t} \mathbf{A} = -\nabla V. \quad (10.5)$$

In other words, the 2nd and 3rd Maxwell's equations can be replaced by the definitions of \mathbf{E} and \mathbf{B} in terms of the potentials V and \mathbf{A}

$$\mathbf{E} = -\nabla V - \frac{\partial}{\partial t} \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (10.6)$$

Given a pair of potentials (V, \mathbf{A}) , one can compute the corresponding fields (\mathbf{E}, \mathbf{B}) via the expressions above (10.6). However, the map from (V, \mathbf{A}) to (\mathbf{E}, \mathbf{B}) is not 1-1, but infinitely many to one. In general, if a given pair of (V_0, \mathbf{A}_0) is mapped to $(\mathbf{E}_0, \mathbf{B}_0)$ via (10.6), then (V_1, \mathbf{A}_1) defined below is mapped to the same pair of fields $(\mathbf{E}_0, \mathbf{B}_0)$ via (10.6)

$$V_1 = V_0 - \frac{\partial}{\partial t} \lambda, \quad \mathbf{A}_1 = \mathbf{A}_0 + \nabla \lambda, \quad (10.7)$$

where λ is an arbitrary function of space-time. This is called *gauge transformation*.

Electromagnetism can be described in terms of V and \mathbf{A} . We can replace \mathbf{E} and \mathbf{B} in the 1st and last Maxwell's equations in (10.1)–(10.2) by the expressions (10.6), and we get the wave equations for V and \mathbf{A} . In vacuum, these equations are simplified in the *Lorentz gauge*

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial}{\partial t} V = 0. \quad (10.8)$$

(You can figure out the case of linear media by yourself.) The wave equations become

$$\square^2 V = -\rho/\epsilon_0, \quad \square^2 \mathbf{A} = -\mu_0 \mathbf{J}, \quad (10.9)$$

where \square^2 is the d'Alembertian defined by

$$\square^2 \equiv \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}. \quad (10.10)$$

Assuming that the fields vanish in the past infinity, the general solution to the wave equations (10.9) is called the *retarded potential*

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int d\tau' \frac{\rho(\mathbf{r}', t_r)}{z}, \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int d\tau' \frac{\mathbf{J}(\mathbf{r}', t_r)}{z}, \quad (10.11)$$

where the *retarded time* is defined by

$$t_r \equiv t - \frac{z}{c}. \quad (10.12)$$

Since the wave equations in Lorentz gauge (10.9) are invariant under the inversion of time $t \rightarrow -t$, the replacement of t_r by $t_a \equiv t + \frac{z}{c}$ in (10.11) must also give solutions of the same equations. These are called “advanced solutions”. Causality is violated in these solutions. There are infinitely many other solutions to the same equations. Adding any solution to the homogeneous equations gives you a new solution with a different initial condition or boundary condition.

The fields \mathbf{E} , \mathbf{B} corresponding to the retarded potentials (10.11) can be computed via (10.6) and are called *Jefimenko's equations*.

For a point charge with the trajectory given by

$$\mathbf{r} = \mathbf{w}(t) \quad \text{at time } t, \quad (10.13)$$

the corresponding charge density and current are

$$\rho(\mathbf{r}, t) = q\delta(\mathbf{r} - \mathbf{w}(t)), \quad \mathbf{J}(\mathbf{r}, t) = \rho(\mathbf{r}, t)\mathbf{v}(t), \quad (10.14)$$

One can check that this solution is in the Lorentz gauge (Griffiths Prob. 10.8).

In the appendix, you can find out how to solve the d'Alembertian equations to get the retarded solutions. (But these are not the only solutions — there are advanced solutions and more.) For this course, you are only required to understand how to check that these are indeed solutions to (10.9).

where

$$\mathbf{v}(t) \equiv \frac{d}{dt} \mathbf{w}(t). \quad (10.15)$$

Plugging them into (10.11), we find the retarded potential for a point charge

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\boldsymbol{z}c - \boldsymbol{z} \cdot \mathbf{v})}, \quad (10.16)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t). \quad (10.17)$$

This solution, the retarded potential for a point charge, is called the *Liénard-Wiechert potential*. The corresponding fields are

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\hat{\boldsymbol{z}}}{(\boldsymbol{z} \cdot \mathbf{u})^3} [(c^2 - v^2)\mathbf{u} + \boldsymbol{z} \times (\mathbf{u} \times \mathbf{a})], \quad (10.18)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \hat{\boldsymbol{z}} \times \mathbf{E}(\mathbf{r}, t), \quad (10.19)$$

where

$$\boldsymbol{z} \equiv \mathbf{r} - \mathbf{w}(t_r), \quad \mathbf{u} \equiv c\hat{\boldsymbol{z}} - \mathbf{v}, \quad \mathbf{a} \equiv \frac{d}{dt} \mathbf{v} = \frac{d^2}{dt^2} \mathbf{w}. \quad (10.20)$$

You are not expected to memorize the two equations above. But note the following features of these solutions.

- The first term in (10.18) is called the *generalized Coulomb field* or the *velocity field*. The 2nd term is called the *radiation field* or the *acceleration field*. For very large z (looking at the point charge from far away), the first term falls off as z^{-2} , like the static Coulomb field, and the 2nd term falls off (slower) as z^{-1} , just like radiation should. (The intensity of radiation is inversely proportional to z^2 , and so the field should be proportional to z^{-1} .)
- The magnetic field \mathbf{B} is locally perpendicular to both \mathbf{E} and \boldsymbol{z} .
- All the quantities \boldsymbol{z} , \mathbf{u} , \mathbf{v} in these expressions are evaluated at the retarded time t_r .
- Note that the definition of \boldsymbol{z} in (10.20) is actually an equation you need to solve in order to obtain an explicit expression of \boldsymbol{z} , because \mathbf{w} depends on t_r and t_r depends on \boldsymbol{z} .

As an exercise, find the explicit expression of \boldsymbol{z} for $\mathbf{r} = x\hat{\mathbf{x}}$ and $\mathbf{w} = vt\hat{\mathbf{x}}$. In some cases, the solution of \boldsymbol{z} does not exist, or we can say that the solution is $\boldsymbol{z} = \infty$. See Prof. 10.15 in Griffiths.

10.1 About Gauge Transformation

10.1.1 Gauge vs. Global Symmetry

The gauge transformation (10.7) is a symmetry transformation. If (V_0, \mathbf{A}_0) satisfies Maxwell's equations (i.e., the fields $(\mathbf{E}_0, \mathbf{B}_0)$ defined via (10.6) satisfy the 1st and last Maxwell's equations in (10.1)–(10.2)), then (V_1, \mathbf{A}_1) automatically satisfies Maxwell's equations in the same sense.

We have seen other symmetry transformations of Maxwell's equations. For example, translation of coordinates

$$\mathbf{E}_1(\mathbf{r}) = \mathbf{E}_0(\mathbf{r} + \mathbf{a}), \quad \mathbf{B}_1(\mathbf{r}) = \mathbf{B}_0(\mathbf{r} + \mathbf{a}) \quad (10.21)$$

is also a symmetry transformation. If $(\mathbf{E}_0, \mathbf{B}_0)$ satisfies Maxwell's equations, $(\mathbf{E}_1, \mathbf{B}_1)$ is guaranteed to satisfy Maxwell's equations, too. (The sources ρ and \mathbf{J} should also be translated, but they are omitted above just to be brief.)

However, there is a fundamental difference between these two symmetry transformations. In general, there are two kinds of symmetries: *gauge symmetry* and *global symmetry*. (The gauge transformation (10.7) is not the only gauge transformation in physics. There are gauge transformations in many other theories for fields other than electromagnetic fields. Similarly, there are many other examples of global symmetries besides translation of coordinates.)

The key question that distinguishes gauge symmetry from global symmetry is: “Does the transformation change the physical state?”. Gauge transformations do not change the physical state, but global transformations do.

When you move your experiment table 10 cm to the left, the experiments conducted on the table will not look any different, but having the experiment table (and the experiments on it) 10 cm to the left is physically distinguishable (different) from the original state. You can use a ruler to measure the distance from the table to a wall and the reading of the ruler is a physical observable that can tell the difference.

On the other hand, there is no experiment (no physical quantities that can be measured) that can tell the difference between (V_0, \mathbf{A}_0) and (V_1, \mathbf{A}_1) when they are related by a gauge transformation (10.7).

local vs. global

Another comparatively superficial difference between the translation and the gauge transformation is that the transformation of translation has a finite number of constant parameters, while the gauge transformation (10.7) uses a function of spacetime to parametrize the transformation. The fact that the former has *constant* parameters which do not change over spacetime (i.e., \mathbf{a} is independent of \mathbf{r} or t) is the reason of the name “global” symmetry. The fact that the latter uses a local function as transformation parameter leads

to a name “local” symmetry, which is nowadays less popular than “gauge” symmetry.

This difference is comparatively more superficial than the difference explained earlier because a transformation which has only a finite number of constant parameters can sometimes be a gauge transformation. In fact, the translation is considered a gauge transformation if it is carried over the whole universe in the context of general relativity. In general relativity, any coordinate transformation is a gauge transformation. Another possibility is that if the periodic boundary condition is imposed on a spatial direction with period L for a certain physical system (say, an extra dimension of the universe), translation by an integral number of L (this transformation is parametrized by a single integer) is now a gauge transformation.

physical observables

By “physical observables” we mean quantities that we can measure. Two configurations are considered (physically) equivalent if all physical observables are the same, i.e. there is no way to distinguish them experimentally. If two configurations (V, \mathbf{A}) and (V', \mathbf{A}') are related by a gauge transformation, all physical observables defined by the potentials must be the same. In particular, the fields (\mathbf{E}, \mathbf{B}) must agree.

Conversely, if the fields (\mathbf{E}, \mathbf{B}) are exactly the same for two configurations, can we say that the two configurations are the same (equivalent)? (Of course it is possible that the configurations differ in matters that are electromagnetically decoupled. Here we are focusing on the electromagnetic content of the configurations.) A related question is: can we say that all solutions of potentials (V, \mathbf{A}) for given fields (\mathbf{E}, \mathbf{B}) are related to each other by gauge transformations?

Aharonov-Bohm effect (AB effect) is a famous example telling you that the answer is no. As a result, (\mathbf{E}, \mathbf{B}) are not sufficient to specify a physical state. Consider an infinitely long solenoid with magnetic field inside but $\mathbf{E} = \mathbf{B} = 0$ outside the solenoid. In the space outside the solenoid, all the fields vanish, but the potential does not. By taking an electron to move around the solenoid, one can experimentally distinguish the physical states of the region outside the solenoid for different magnetic fluxes inside the solenoid.

The physical observable measured in the AB effect is called a *Wilson loop*. It is defined as

$$W(C) = \exp(i \oint_C \mathbf{A} \cdot d\mathbf{r}) \quad (10.22)$$

for an arbitrary closed curve C . You can check that $W(C)$ is invariant under gauge transformations. If the closed curve C is the boundary of a surface S ,

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \int_S \mathbf{B} \cdot d\mathbf{a} = \Phi \quad (10.23)$$

In general the Wilson loop is

$$W(C) = \exp(i \oint_C A_\mu dx^\mu)$$

for a curve C in space-time.

is the magnetic flux over S . In this case, the Wilson loop is completely determined by \mathbf{B} . If \mathbf{B} is the same for two configurations, all Wilson loops are the same. However, in a space with nontrivial topology (e.g. the space outside the solenoid), there are closed curves which cannot shrink to a point. These curves are not boundaries of surfaces. Wilson loops on these closed curves are physical observables independent of the fields (\mathbf{E}, \mathbf{B}) , and two configurations can be different for having different values in some Wilson loops, even when all the fields (\mathbf{E}, \mathbf{B}) are exactly the same.

10.1.2 Gauge fixing

Since the correspondence between the potential and the field is many to one, one can choose a *gauge fixing condition* to select a representative out of the many potentials corresponding to the same field. Usually, what we do is to first find the potentials and then compute the fields from the potentials using (10.6), and the purpose of gauge fixing condition is to help us find solutions for the potential. Therefore a *good* gauge fixing condition should satisfy the following two properties:

1. **existence** There exists at least one solution of the potential for all physical states (all possible configurations of \mathbf{E}, \mathbf{B}). We do not want the gauge fixing condition to exclude any physical state.
2. **uniqueness** There should not be too many solutions of the potential corresponding to the same physical state. In order to simplify the job of solving for the potential, one should utilize the gauge symmetry as much as possible. Allowing too many solutions for the same physical state means that one could impose a stronger gauge fixing condition to further simplify the calculation. For the sake of communication, we also hope that there is a single potential corresponding to each physical state.

For example, $\mathbf{A} = 0$ is a bad gauge fixing condition. It excluded all states with nonvanishing magnetic field, although it greatly simplified the job of finding solutions of the potential.

Another bad example is $\int d\tau V = 0$. This still allows all gauge transformations for λ satisfying $\int d\tau \dot{\lambda} = 0$. Apparently there are too many such functions λ because for every λ that does not satisfy this condition we can find another gauge transformation with λ' given by $\lambda' = \lambda - ct$. By suitably choosing the constant c , we can achieve $\int d\tau \dot{\lambda}' = 0$. In other words, this condition only eliminates a single parameter from the function λ which includes infinitely many parameters. (You can think of the coefficients of each

This is my definition of a *good* gauge fixing condition.

term in the Taylor expansion or Fourier transform of λ as the infinitely many parameters included in λ .)

An ideal gauge fixing condition leads to a 1-1 correspondence between potentials (V, \mathbf{A}) and fields (\mathbf{E}, \mathbf{B}) . Usually the gauge fixing condition leaves a relatively small number of potentials corresponding to the same field. One-to-one correspondence is possible only after imposing suitable boundary conditions.

Examples of good gauge fixing conditions:

1. Lorentz gauge $\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial}{\partial t} V = 0$.
2. Coulomb gauge $\nabla \cdot \mathbf{A} = 0$.
3. temporal gauge $V = 0$.
4. axial gauge $A_3 = 0$.

To show that a gauge fixing condition is good, one should carry out the following two tasks.

1. Check that for any pair of given (V_0, \mathbf{A}_0) that does not satisfy the gauge fixing condition, there exists a gauge transformation λ such that after transformation, (V_1, \mathbf{A}_1) satisfies the gauge fixing condition.
2. Check that if (V_0, \mathbf{A}_0) satisfies the gauge fixing condition, the only choice of λ that will allow (V_1, \mathbf{A}_1) to continue satisfying the gauge fixing condition is $\lambda = 0$ (after suitable boundary conditions are imposed on λ).

10.2 Exercises

1. Which of the following is not a “good” gauge fixing condition?

$$(1) A_x = 0, \quad (2) \frac{A_x}{A_y} = \sqrt{3}, \quad (3) \partial_x A_x + \partial_y A_y = 0, \quad (4) \partial_x A_y - \partial_y A_x = 0. \quad (10.24)$$

2. Suppose (V_0, \mathbf{A}_0) is the potential corresponding to a given physical configuration of the fields \mathbf{E}, \mathbf{B} in the temporal gauge ($V_0 = 0$). Find
 - (a) the gauge transformation λ such that the new potentials (V_1, \mathbf{A}_1) satisfy Coulomb gauge fixing condition $\nabla \cdot \mathbf{A}_1 = 0$.
 - (b) (V_1, \mathbf{A}_1) .

For this problem, you only need to check whether it is always possible to find (V, \mathbf{A}) satisfying the gauge fixing condition for given (\mathbf{E}, \mathbf{B}) .

3. Similar to Example 10.2 in Griffiths, but with the current replaced by

$$I(t) = \begin{cases} 0, & \text{for } t \leq 0, \\ at^2 & \text{for } t > 0 \end{cases} \quad (10.25)$$

for some constant a , find the resulting fields \mathbf{E} and \mathbf{B} . Specify clearly the range of parameters (t, s) for which each expression in your final answer is valid.

4. An infinite straight wire carries the current

$$I(t) = \begin{cases} 0, & \text{for } t \leq t_0, \\ I_0, & \text{for } t_0 \leq t \leq t_1, \\ 0, & \text{for } t_1 \leq t. \end{cases} \quad (10.26)$$

That is, a constant current I_0 is turned on abruptly at $t = t_0$ and then turned off abruptly at $t = t_1$.

- (a) Find the resulting vector potential \mathbf{A} using the formula of retarded potential.
- (b) Check that \mathbf{A} diverges in the limit $t_0 \rightarrow -\infty$. Does it mean that this result is simply wrong, or how should we interpret the result?

5. Determine the Liénard-Wiechert potentials for a charge in hyperbolic motion (the trajectory of a particle subject to a constant force in special relativity)

Griffiths Prob. 10.16.

$$\mathbf{w}(t) = \sqrt{b^2 + (ct)^2} \hat{\mathbf{x}}. \quad (10.27)$$

Assume the point \mathbf{r} of observation is on the x -axis and to the right of the charge.

6. (a) Suppose a point charge q is constrained to move along the x -axis. Find the fields at points on the axis to the right of the charge.

Part (a) of the problem is Griffiths Prob. 10.18.

(b) For

$$\mathbf{w}(t) = \frac{1}{2}at^2 \quad (10.28)$$

with constant a , find the *explicit* expression of \mathcal{Z} that you should use to plug into the expression of fields you obtained for part (a). Is there something wrong with it?

The definition of \mathcal{Z} as $\mathbf{r} - \mathbf{w}(t_r)$ is *implicit* because t_r depends on \mathcal{Z} .

7. You are also encouraged to study the following problems in Griffiths: Probs. 10.9 — 10.13, and 10.19.

10.3 Appendix: Derivation of the retarded potential

For both V and \mathbf{A} , we want to solve the following partial differential equation

$$\square^2 \phi(x) = -\rho(x), \quad (10.29)$$

where $\rho(x)$ is given. For the sake of discussion, we will refer to ϕ as the potential and ρ as the charge density. Due to superposition principle (the fact that this equation is linear), the solution of ϕ can be obtained by superposing the potential generated by a unit point charge at a given point in space-time

$$\square^2 G(x, x') \equiv (\nabla^2 - \partial_t^2)G(x, x') = \delta^{(4)}(x - x'), \quad (10.30)$$

where $G(x, x')$ represents the potential generated by a point charge at the space-time point x' . If $G(x, x')$ is known, $\phi(x)$ is just

$$\phi(x) = - \int d^4x' G(x, x') \rho(x'). \quad (10.31)$$

Therefore, solving $G(x, x')$ reduces the problem of solving $\phi(x)$ for arbitrary charge distributions $\rho(x)$ to a problem of integration.

Due to translation symmetry, $G(x, x') = G(x - x')$. Its Fourier expansion is

$$G(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{G}(k) e^{ik \cdot x}, \quad (10.32)$$

where $k \cdot x \equiv k_0 t - \mathbf{k} \cdot \mathbf{r}$, and $\tilde{G}(k)$, the Fourier transform of $G(x)$,

$$(k_0^2 - \mathbf{k}^2) \tilde{G}(k_0, \mathbf{k}) \equiv k^2 \tilde{G}(k) = 1. \quad (10.33)$$

We find

$$G(x) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} e^{ik \cdot x}. \quad (10.34)$$

However, this expression is ill-defined, because the integrand diverges at $k^2 \equiv k_0^2 - |\mathbf{k}|^2 = 0$. One can try to modify the integral so that it becomes finite, but there are many ways to do it. The ambiguity involved corresponds to the ambiguity in choosing the boundary/initial condition for the differential eq. The equation (10.30) has infinitely many solutions unless we specify the boundary/initial condition.

Let us single out the integral of k_0

$$G(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{r}} g(\mathbf{k}), \quad g(\mathbf{k}) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{1}{k_0^2 - \mathbf{k}^2} e^{ik_0 t}. \quad (10.35)$$

The ambiguity now resides in $g(\mathbf{k})$.

This derivation uses Residue Theorem and Fourier transform. Here we use x to represent (t, \mathbf{r}) , and for simplicity we will set $c = 1$.

In general, solutions of a PDE of the form

$$\mathcal{D}G(x, x') = \delta(x - x')$$

is called *Green's functions*

It turns out that the problem is better understood in terms of complex analysis. The divergence of the integrand can be circumvented in the complex plane of k_0 . That is, we imagine that k_0 is originally a complex number, although the integral of interest is an integral over the real line in the complex plane of k_0 . The divergences of the integrand include a simple pole at $k_0 = -|\mathbf{k}|$ and another simple pole at $k_0 = |\mathbf{k}|$. We can circumvent the poles by passing around them either from above or from below. There are 4 possible combinations of choices for the 2 poles, each giving a different but well-defined (finite) integral.

For a given choice of the (re)definition of the integral by deforming the contour or the poles, we can evaluate the integral using the residue theorem. To use the residue theorem we have to first pick a closed contour in the complex plane of k_0 . The contour we need here must include the real axis of k_0 (except infinitesimal deviations around the poles) because that is the integral we want to compute. To make a complete contour we need to add half a circle at infinity either in the upper half plane (UHP) or the lower half plane (LHP). At infinity in the UHP, $k_0 = \pm\infty + i\infty$, while in the LHP, $k_0 = \pm\infty - i\infty$. The difference between the choices of UHP vs. LHP is thus $e^{ik_0t} = e^{\pm i\infty t - \infty t}$ (for UHP) vs. $e^{ik_0t} = e^{\pm i\infty t + \infty t}$ (for LHP). Therefore, for $t > 0$, this exponential factor is 0 at infinity in the UHP, and for $t < 0$, it is 0 at infinity in the LHP.

Since what we want to compute is the integral over the real axis, the residue theorem helps us only if the half circle at infinity has no contribution to the integral. This criterion determines whether we should choose the half circle in the UHP or LHP for the contour.

For example, what will we get if we decide to circumvent both poles of k_0 by passing them from below (deforming the contour from the real axis to the LHP around the poles)? Let us consider what the Green's fx. is for $t > 0$ and $t < 0$ separately. For $t > 0$, we should choose the contour to be the real axis plus a half circle at infinity in the UHP, so that the contour integral is the same as the integral over the real axis. Then the residue theorem tells us that

$$g(\mathbf{k}) = \frac{2\pi i}{2\pi} \left(\frac{1}{2|\mathbf{k}|} e^{i|\mathbf{k}|t} - \frac{1}{2|\mathbf{k}|} e^{-i|\mathbf{k}|t} \right). \quad (10.36)$$

Similarly, for $t < 0$, we choose the contour to be the real axis plus a half circle at infinity in the LHP, so that the contour integral is the same as the integral over the real axis. The residue theorem now gives

$$g(\mathbf{k}) = 0 \quad \Rightarrow \quad G(x - x') = 0 \quad \text{for} \quad t - t' < 0 \quad (10.37)$$

because neither of the poles is inside the contour. The fact that $G(x - x') = 0$ for $t < t'$ tells us that this is the *retarded* Green's fx.

We can redefine the path of integration by an infinitesimal deformation or change the location of poles by an infinitesimal shift away from the real axis. In the end we take the limit that the integration contour approaches to the real axis or the poles approaches to their original locations on the real axis.

If we had chosen to circumvent both poles from above, we would have got the *advanced* Green's fx. Other choices of circumventing the poles give other types of solutions, e.g. retarded potential for positive freq. modes and advanced potential for neg. freq. modes, which is what Feynman chose for QED.

Having explained the conceptual problems, let us now evaluate the retarded Green's fx. more explicitly. Plugging (10.36) into (10.35), we get

$$G(x) = \int \frac{dk d\cos\theta d\phi k^2}{(2\pi)^3} e^{-ikr \cos\theta} \frac{i}{2k} (e^{ikt} - e^{-ikt}), \quad (10.38)$$

where we denoted $|\mathbf{k}|$ by k , and defined θ w.r.t. \mathbf{r} . One can easily integrate over ϕ and θ , and find

$$G(x) = \int \frac{dk}{(2\pi)^2} \frac{1}{2r} (e^{ikr} - e^{-ikr}) (e^{ikt} - e^{-ikt}). \quad (10.39)$$

Finally, integrating over k and recalling that this is for $t > 0$, the retarded Green's fx. is

$$G(\mathbf{r}, t) = \begin{cases} -\frac{1}{4\pi r} \delta(t - r) & (t > 0), \\ 0 & (t < 0). \end{cases} \quad (10.40)$$

The (retarded) sol. to the PDE

$$(\nabla^2 - \partial_t^2)\phi = -\rho \quad (10.41)$$

is therefore

$$\phi(\mathbf{r}, t) = - \int d^4x' G(x - x') \rho(x') = \int d\tau' \frac{1}{4\pi} \frac{1}{\mathbf{z}} \rho(\mathbf{r}', t - \mathbf{z}). \quad (10.42)$$

Chapter 11

Radiation

The word “radiation” refers to the part of the electromagnetic field that is detached from the source and propagates to infinity. For a localized source of size a near the origin, the energy flux in radiation that passes through a sphere of radius r should be independent of r for large r (i.e., $r \gg a$) due to energy conservation. Thus the magnitude of the Poynting vector is proportional to $1/r^2$, implying that \mathbf{E} and \mathbf{B} are proportional to $1/r$.

How do we define radiation if the universe is “finite” and so there is no “infinity” for the field to propagate?

On the other hand, the part of the electromagnetic field that is attached to the source, which is called the “Coulomb field”, does not propagate to infinity and stays around the source (although the distribution has a variable size and thus it may temporarily expand or shrink). When you look at the source from far away, it looks like a static charge and has \mathbf{E} and \mathbf{B} proportional to $1/r^2$.

This chapter is composed of two parts: Part 1 is the multiple expansion for the radiation field; Part 2 is the radiation of a point charge.

11.1 Multiple Expansion of Radiation

For an arbitrary source localized near the origin,

Here θ is the angle between \mathbf{p} and \mathbf{r}

$$\mathbf{E}(\mathbf{r}, t) \simeq \frac{\mu_0}{4\pi r} [\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \ddot{\mathbf{p}})] = \frac{\mu_0 \ddot{p}(t_0)}{4\pi} \left(\frac{\sin \theta}{r} \right) \hat{\theta}, \quad (11.1)$$

$$\mathbf{B}(\mathbf{r}, t) \simeq -\frac{\mu_0}{4\pi r c} [\hat{\mathbf{r}} \times \dot{\mathbf{p}}] = \frac{\mu_0 \dot{p}(t_0)}{4\pi c} \left(\frac{\sin \theta}{r} \right) \hat{\phi}, \quad (11.2)$$

where t_0 is the retarded time for the origin,

$$t_0 \equiv t - \frac{r}{c}. \quad (11.3)$$

Interpretation of the expressions above:

- **direction**

When we put the dipole to oscillate in the $\hat{\mathbf{z}}$ -direction, the radiation

wave measured at \mathbf{r} has a wave vector \mathbf{k} in the $\hat{\mathbf{r}}$ -direction, and \mathbf{E} and \mathbf{B} should be in the transverse directions. The \mathbf{E} -field should be in the $\hat{\theta}$ -direction, because this is what you get by projecting \mathbf{z} onto the plane perpendicular to $\hat{\mathbf{r}}$. Then \mathbf{B} is determined to be in the $\hat{\phi}$ -direction. (We should have $(\mathbf{E} \times \mathbf{B})$ parallel to $\hat{\mathbf{r}}$.)

- $1/r$

\mathbf{E} and \mathbf{B} of the radiation field must be $\propto 1/r$ so that the energy is conserved.

- $\sin \theta$

The projection of \mathbf{z} onto the plane perpendicular to $\hat{\mathbf{r}}$ involves a factor of $\sin \theta$.

- \ddot{p}

Charge at constant velocity can not radiate, and radiation is generated only when there is acceleration. The second time derivative on \mathbf{p} corresponds to the second time derivative on the position of charges.

- **relative $1/c$ factor between E and B**

Just like the magnitudes of \mathbf{E} and \mathbf{B} in the plane wave solution, we have $|\mathbf{B}| = |\mathbf{E}|/c$.

- $\mu_0/4\pi$

Finally, the only thing that remains to be explained is the factor $\frac{\mu_0}{4\pi}$. The factor μ_0 can be argued to be there based on dimensional analysis. You should not feel embarrassed to admit that you don't know where the factor of $\frac{1}{4\pi}$ comes from.

The factor $\frac{1}{4\pi}$ comes from the surface area of a unit sphere, which shows up whenever you apply the Gauss law to a charge at the origin. But one may not be so sure that there is no additional factors of -1 or $1/2$ in the calculation.

The Poynting vector and the total radiated power are

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \simeq \frac{\mu_0}{16\pi^2 c} [\ddot{p}(t_0)]^2 \left(\frac{\sin^2 \theta}{r^2} \right) \hat{\mathbf{r}}, \quad (11.4)$$

$$P = \int \mathbf{S} \cdot d\mathbf{a} \simeq \frac{\mu_0 \ddot{p}^2}{6\pi c}. \quad (11.5)$$

These expressions are good approximations provided that

$$b \ll \lambda \ll r, \quad (11.6)$$

where b is the characteristic size of the arbitrary source, λ is the wavelength of the radiation field for harmonic oscillations, and r is the distance from the source to the point of observation.

If the source is not in harmonic oscillation, λ should be replaced by all characteristic length scales of the radiation field

$$\lambda \sim \frac{c}{|\dot{\rho}/\rho|}, \frac{c}{|\ddot{\rho}/\rho|^{1/2}}, \frac{c}{|\dddot{\rho}/\rho|^{1/3}}, \dots \quad (11.7)$$

The condition (11.6) has to be satisfied for all possible choices of λ in order for the result to be a good approximation.

The approximation involves an expansion of b . The electric dipole moment gives the leading order contribution. When the 2nd time derivative of the electric dipole moment vanishes, we need to compute the next order terms due to magnetic dipole moment and electric quadrupole moment, etc.

11.2 Point Charge Radiation

In this section we are concerned with (1) the power of radiation from a point charge, and (2) the exchange of energy between a point charge and the EM field it generates.

The point charge radiates and losses energy into the radiation field, and there is back-reaction due to radiation. The point charge also exchanges energy with the Coulomb field. Despite its name, “radiation reaction” includes effects due to both radiation field and Coulomb field.

Apart from the change in the energy of the Coulomb field, there can be difference between the power of radiation and the rate of energy loss of the charge into the radiation field, because the length of time in which a given amount of energy leaving a charge is in general different from the length of time in which the same energy passes through a large sphere of radius r .

The total power radiated from a point charge is given by the Liénard’s generalization of the Larmor formula

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c} \left(a^2 - \left| \frac{\mathbf{v} \times \mathbf{a}}{c} \right|^2 \right), \quad (11.8)$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \quad (11.9)$$

For $v \ll c$, this can be approximated by the Larmor formula

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}, \quad (11.10)$$

which agrees with (11.5).

For $v \ll c$, the radiation reaction force is approximately given by the Abraham-Lorentz formula

$$\mathbf{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c} \dot{\mathbf{a}}. \quad (11.11)$$

This is consistent with the Larmor formula for radiation (11.10) in the sense that energy is conserved over a complete period for a periodic process,

See Figure 11.13 on p. 462 and related discussions in Griffiths.

For a particle with given charge and mass, a characteristic time scale τ is defined via $\mathbf{F}_{\text{rad}} = m\tau\dot{\mathbf{a}}$.

for which the energy of the Coulomb field does not change (due to the periodic conditions) and so all the energy loss of the charge can only be attributed to the radiation.

The relativistic version of the Abraham-Lorentz force is the Abraham-Lorentz-Dirac force

$$F_\mu = \frac{\mu_0 q^2}{6\pi m c} \left(\frac{d^2 p_\mu}{d\tau^2} + \frac{p_\mu}{m^2 c^2} \left(\frac{dp_\nu}{d\tau} \frac{dp^\nu}{d\tau} \right) \right). \quad (11.12)$$

This equation is beyond the scope of this course.

Comments on the radiation reaction force:

Here p_μ is the energy-momentum 4-vector.

- The force corresponds to the energy exchange between the charge and the field, including both radiation field and Coulomb field.
- The force vanishes if \mathbf{a} is constant, even though there is still radiation whenever $\mathbf{a} \neq 0$. (This has to do with the *equivalence principle*.)
- The force involves higher derivative of time and demands more information for the initial condition.
- The force leads to solutions with *preacceleration* and thus violation of causality.
- The force also leads to *runaway* solutions.
- One can choose to avoid the *runaway* solution or the *preacceleration* solution, but not both, if the force can change abruptly.
- Energy is still conserved if the energy of the field and that of the charge are properly taken into account.
- The runaway solutions should be excluded by hand (just like how the advanced potentials are excluded by hand), because they are never observed.
- The preacceleration solution would be acceptable if causality is violated only at a very small time scale where classical electrodynamics and/or the point-like description of charges are/is expected to be invalid.
- In principle, in a theory valid only for length and time scales larger than $\Delta x, \Delta t$, everything should be specified only up to those scales. For example, a force involving the step function is not acceptable because the change of the force should take at least the time of Δt .

These comments are made for the Abraham-Lorentz force, but can also be suitably modified to apply to the Abraham-Lorentz-Dirac force.

For example, at microscopic scale quantum effect may be important.

- The modification of Newton's 2nd law $F = ma$ by $F = m(a - \tau\dot{a})$ should be considered as a small modification when τ is extremely small. That is, the modified theory should approach to the original theory when $\tau \rightarrow 0$. This would not be true if we allow runaway solutions.

See the Appendix for generic considerations on modifications on a physical law.

11.3 Exercises

1. Use eq.(10.65) for the electric field of a moving point charge in Griffiths to re-derive eq.(11.18) for an oscillatory electric dipole moment. (Hint: Consider the motion of the point charge q_0 given by $\mathbf{w}(t) = d \cos(\omega t)\hat{\mathbf{z}}$.)
2. Find the leading order correction to eq.(11.19) in Griffiths for the magnetic field in radiation. (The leading order correction is of order $d\omega/c$ smaller than (11.19).)
3. Find the electric field in radiation observed at $\mathbf{r} = \hat{\mathbf{z}}r \cos \theta + \hat{\mathbf{y}}r \sin \theta$ (that is, $\phi = 0$) due to the simultaneous presence of two electric dipole moments $\mathbf{p} = \pm p_0 \cos(\omega t)\hat{\mathbf{z}}$ located at $\mathbf{r}' = \pm(b/2)\hat{\mathbf{y}}$, respectively.
4. Find the total time-averaged power of radiation due to the electric dipole

$$\mathbf{p} = p_1 \cos(\omega t)\hat{\mathbf{x}} + p_2 \sin(\omega t)\hat{\mathbf{y}} \quad (11.13)$$

at the origin. (Hint: Note that the two dipoles are 90° out of phase and we only need the time-averaged power.)

5. Find the electric and magnetic fields at a height x above the $y - z$ plane on which a uniform time-dependent surface current is given by

$$\mathbf{K}(t) = K_0 \delta(t)\hat{\mathbf{z}}, \quad (11.14)$$

where K_0 is a constant.

6. (a) A particle of charge q moves in a circle of radius R at a constant speed v . To sustain the motion, you must provide a centripetal force mv^2/R . What additional force \mathbf{F}_e must you exert, in order to counteract the radiation reaction? (Express your answer in terms of the instantaneous velocity v .)
 (b) Repeat part (a) for a particle in simple harmonic motion with amplitude A and angular frequency ω ($\mathbf{w}(t) = A \cos(\omega t)\hat{\mathbf{z}}$).
7. For the dumbbell model of two charges q separated by a fixed distance d considered in Sec. 11.2.3 in Griffiths, find the term linear in d in the self-force \mathbf{F}_{self} in eq.(11.95).

This problem is similar to Prob. 11.24 in Griffiths.

Prob. 11.17 (a) and (b) in Griffiths

8. Newton's 2nd law for a charged point particle is

$$a = \tau \dot{a} + f, \quad (11.15)$$

where

$$\tau \equiv \frac{\mu_0 q^2}{6\pi m c}, \quad f \equiv F/m. \quad (11.16)$$

- (a) How should one specify the initial condition for a point charge in order for the solution to be unique?
- (b) In the absence of external force ($F = 0$), this equation implies the runaway solution, for which the kinetic energy of the particle increase exponentially with time. Where is the energy from?
- (c) If the runaway solution is excluded for $t > T$ in Prob. 11.19, check that the energy is conserved.
- (d) Is it possible to exclude runaway solutions consistently without violation of causality? Is causality more seriously violated for larger τ or smaller τ ?

See Prob. 11.27 in Griffiths for Prob. (8c).

9. You are also encouraged to study the following problems in Griffiths: Probs. 3, 4, 5, 7, 9, 12, 13, 14, 16, 19, 20, 24, 27, 29, 30, 31.

11.4 Appendix: Modifying A Physical Law by Higher Derivative Terms

Well established physical laws may still be modified when the precision of experiments gets higher and higher, or when theoretical considerations become more complete. We saw that for a charged point particle, Newton's 2nd law $F = ma$ is modified to be

$$F = m(a + \tau \dot{a}). \quad (11.17)$$

In practice τ is small, and the effect of the radiation reaction should not drastically change our understanding of Newton physics for a point particle. We take this as an example to illustrate the idea about how one should treat a small modification to a well established theory.

Despite the smallness of τ , $\tau \dot{a}$ is a higher derivative term that changes the property of the differential equation in a fundamental way. While the phase space of the original equation is 2 dimensional (corresponding to the initial condition of (x, v)), the phase space of the modified equation is 3 dimensional (corresponding to the initial condition of (x, v, a)). However, a small modification is not supposed to drastically change the number of the degrees of

The phase space can be identified with the space of initial conditions, and with the space of all solutions to the equation of motion.

freedom for a well established physical theory. The modified theory should be understood in such a way that as $\tau \rightarrow 0$, the original theory is obtained.

A simple way to achieve this goal is to treat the modified equation *perturbatively*. We expand the solution as

$$x = x_0 + \tau x_1 + \tau^2 x_2 + \dots . \quad (11.18)$$

Correspondingly we have

$$a = a_0 + \tau a_1 + \tau^2 a_2 + \dots , \quad (11.19)$$

where $a_i = \ddot{x}_i$. Plugging this into the modified equation of motion, and match the terms of the same order in the τ -expansion, we get

$$F = ma_0, \quad (11.20)$$

$$a_1 = -\dot{a}_0, \quad (11.21)$$

$$a_2 = -\dot{a}_1, \quad (11.22)$$

$$\vdots = \vdots \quad (11.23)$$

For example, when $F = 0$, we get $a_0 = 0$, which implies that all $a_i = 0$ iteratively. Runaway solutions are excluded automatically, and the phase space is 2 dimensional (because the initial condition that uniquely determines the solution is the initial values of (x, v)).

Another way to understand the proper physical interpretation of (11.17) is to rewrite it as

$$\left(1 + \tau \frac{d}{dt}\right)^{-1} F = ma. \quad (11.24)$$

The left hand side involves an operator that can be defined as an infinite expansion of time derivatives to all orders. This operator has a well defined action on F only if F is smooth (F has finite time derivatives to all orders). That is, the equation of motion rules out the possibility that F is a step function. In a consistent theory of physical interactions, there cannot be any other interactions that allows us to exert a force on a point charge that changes abruptly.

In some other examples, the new degrees of freedom introduced by higher derivative correction terms actually exist. Consider a mass m attached to the end of a spring with spring constant k . The equation of motion is $m\ddot{x} = -kx$. Now imagine that the other end of the spring, which was thought to be fixed to a rigid structure, can actually oscillate because the structure is not absolutely rigid. We use a large spring constant K and an effective mass M to describe the natural frequency of the structure and the equations of motion are now

$$M\ddot{y} = -Ky, \quad m\ddot{x} = -k(x - y). \quad (11.25)$$

They can be combined into a single higher order differential equation

$$(m - kM/K)\ddot{x} = -kx - mMx^{(4)}/K, \quad (11.26)$$

where $x^{(4)}$ is the 4-th time derivative of x . For very large K , this is a small modification to the original equation $m\ddot{x} = -kx$. But the fact that now we have a $x^{(4)}$ term implies that there exist two natural frequencies, as we should expect for a system of two coupled springs. One of the two natural frequencies would be close to $\sqrt{k/m}$, but the other would be very large. In low energy experiments, the almost rigid structure is never significantly deformed, and a precision measurement would only reveal the lower natural frequency. The perturbative treatment of the modified equation via an expansion in powers of $1/K$ would give you solutions corresponding to this low energy oscillation mode and ignore the high frequency mode.

Chapter 12

Electrodynamics and Relativity

12.1 Special Relativity

In special relativity, the spacetime is a 3+1 dimensional Minkowski space. That is, the notion of distance is defined by the line element ds as

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu. \quad (12.1)$$

$$x^0 \equiv ct, \\ \mu, \nu = 0, 1, 2, 3.$$

The symmetry of this definition of distance is called *Poincare symmetry*, which is composed of translations and Lorentz transformations

$$x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu + a^\mu, \quad (12.2)$$

where $\Lambda^\mu{}_\nu, a^\mu \in \mathbb{R}$, and

$$\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}. \quad (12.3)$$

This condition is a symmetric equation of 4×4 matrices, and is thus composed of 10 independent equations. As a result, the matrix $\Lambda^\mu{}_\nu$ should have $4 \times 4 - 10 = 6$ independent free parameters.

Lorentz symmetry includes two types of transformations: rotation and boost; each type of transformations is labelled by 3 independent parameters.

A rotation can be specified by giving a unit vector $\hat{\mathbf{n}}$ (a point on the unit sphere) as the axis of rotation and an angle θ of rotation along the $\hat{\mathbf{n}}$ -axis. We can use the vector $\theta\hat{\mathbf{n}} = (\theta_x, \theta_y, \theta_z)$ to label such a rotation. Another way to specify rotation is to perform first a rotation along the x -axis by θ'_x , and then a rotation along the y -axis by θ'_y , followed by a rotation along the z -axis by θ'_z . Either way, it takes three independent parameters to specify a rotation.

A boost is also specified by three parameters. You can boost by a velocity vector $\mathbf{v} = (v_x, v_y, v_z)$, or you can first boost along the x -axis by v'_x , and then

along the y -axis by v'_y , followed by a boost along the z -axis by v'_z .

A generic Lorentz transformation is a combination of rotation and boost. You can first rotate by $\theta \hat{\mathbf{n}}$ and then boost by \mathbf{v} , or vice versa. Either way, you need 6 free parameters to specify a generic Lorentz transformation.

With a suitable choice of the axis, we can always have $\mathbf{v} = \hat{\mathbf{x}}v$ for the boost. The y and z coordinates are invariant under the boost in the x -direction, and so the corresponding matrix $\Lambda^\mu{}_\nu$ must be of the form

$$\Lambda^\mu{}_\nu = \begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{12.4}$$

or more conveniently, we can omit the y, z coordinates and simply write

$$\Lambda^\mu{}_\nu = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{12.5}$$

The condition (12.3) then implies that

$$\Lambda^T \eta \Lambda = \eta. \tag{12.6}$$

There are 4 elements (A, B, C, D) in Λ subject to 3 independent conditions (12.6) and so there is only one free parameter in the boost, and the general solution is

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}, \tag{12.7}$$

Or equivalently,

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh \chi & \sinh \chi \\ \sinh \chi & \cosh \chi \end{pmatrix}.$$

χ is called *rapidity*.

where

$$\beta = v/c, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}. \tag{12.8}$$

As a result, we no longer have consensus on simultaneity. Instead we have

- Time dilation
Everyone else looks dumber than you.
- Lorentz contraction
Running helps you look slimmer.

Q: Explain time dilation and Lorentz contraction by (1) drawing a space-time diagram and (2) algebraic calculation using Lorentz transformation law.

12.2 Minkowski spacetime

As a set of points, the Minkowski spacetime is the same as \mathbb{R}^4 . The difference between Minkowski spacetime and \mathbb{R}^4 is that the former has the “metric” (12.1) which defines the “distance” between two points (“events”) in spacetime with coordinates x^μ and $x^\mu + dx^\mu$. The phrase “distance” here is obviously an extension of the usual meaning of the word “distance”. We could have invented a different name for it. The key point is that the geometry of Minkowski spacetime has some symmetries (the Poincare symmetry, as generalizations of rotation and translation symmetry in 3D) and there is this quantity (which is called “distance”) that is invariant under these symmetry transformations. All transformations that keep this quantity “distance” invariant is a symmetry of the Minkowski space.

The idea of Special Relativity is that the spacetime, as a 4-dimensional “space” (set of points) probed by all sorts of objects available in the universe, has the geometric property of a Minkowski space mentioned above. More precisely, there is a universal feature shared by all sorts of physical phenomena (Poincare symmetry) that we can choose to attribute to the geometric nature of spacetime. The notion of distance stands out naturally as an invariant quantity of the symmetry transformations.

12.3 Covariance and Invariance

The material in this section is a straightforward generalization of the notion of vectors and tensor in a 3 dimensional space which we are familiar with.

Often people use the convention in which they distinguish upper (contravariant) indices from lower (covariant) indices. Upper indices transform like the infinitesimal displacement 4-vector

$$dx^\mu \rightarrow dx'^\mu = \Lambda^\mu{}_\nu dx^\nu. \quad (12.9)$$

Therefore, A^μ is a *contravariant* 4-vector if it transforms like

$$A^\mu \rightarrow A'^\mu = \Lambda^\mu{}_\nu A^\nu. \quad (12.10)$$

An object A_μ that transforms like

$$A_\mu \rightarrow A'_\mu = A_\nu \Lambda^{-1\nu}{}_\mu. \quad (12.11)$$

is called a *covariant* 4-vector.

Conventionally we use $\eta^{\mu\nu}$ to denote the inverse of $\eta_{\mu\nu}$, and they can be used to raise and lower Lorentz indices to turn a contravariant index into a

Even if you decide to label 4 quantities as A^0, A^1, A^2, A^3 , whether they form a contravariant vector is a physical question that has to be checked experimentally.

covariant index, and vice versa. Hence a 4-vector can be represented either as a contravariant 4-vector or a covariant 4-vector. It is merely a matter of notation. For example,

$$A^\mu = \eta^{\mu\nu} A_\nu, \quad A_\mu = \eta_{\mu\nu} A^\nu. \quad (12.12)$$

The inner product of a covariant vector A_μ and a contravariant vector B^μ

$$A_\mu B^\mu \rightarrow A'_\mu B'^\mu = A^T \Lambda^{-1} \Lambda B = A_\mu B^\mu \quad (12.13)$$

is invariant.

We can also define the norm squared of a 4-vector A as $A^2 \equiv A_\mu A^\mu$. A 4-vector A is

- *time-like* if $A^2 < 0$.
- *light-like* if $A^2 = 0$.
- *space-like* if $A^2 > 0$.

On the trajectory of something moving at the speed of light (say, a massless particle), the displacement 4-vector dx^μ is light-like.

Recall that a vector $\mathbf{v} = \hat{\mathbf{x}}^i v_i$ is invariant under rotations in 3 dimensional space. One can check that the components v_i and vector basis $\hat{\mathbf{x}}^i$ transform simultaneously under a rotation such that \mathbf{v} is invariant. The reason behind is that the vector \mathbf{v} is a single entity that has an invariant meaning. Similarly, we should think of a 4-vector A as a single entity with some invariant meaning, and its components A_μ (or A^μ) a particular way to represent it when a reference frame is chosen.

In general, a tensor with m contravariant indices and n covariant indices is denoted as

$$T^{\mu_1 \dots \mu_m}_{\nu_1 \dots \nu_n}. \quad (12.14)$$

We can use the metric $\eta_{\mu\nu}$ and its inverse to raise or lower the indices to express the tensor with an arbitrary combination of k contravariant and $m + n - k$ covariant indices.

In special relativity, all physical laws are invariant under Lorentz transformations.

12.3.1 Lorentz transformation as matrix multiplication

Some of the calculations involving Lorentz transformations can be simplified if we use the notation of matrix multiplication. For objects with two indices, such as $\Lambda^\mu{}_\nu$, we can use the notation in which the index on the left labels the

A set of equations of motion representing a physical law can be “covariant” under Lorentz transformations. That is, the equations of motion may transform into linear combinations of others like individual components of a tensor.

row number and the index on the right labels the column number, so that a matrix multiplication AB is implemented by contracting the neighboring indices between A and B . That is

$$(AB)_{ij} = A_{ik}B_{kj}. \quad (12.15)$$

This rule is applied regardless of whether the indices are upper or lower.

An object with one index can be either a column or a row depending on how its index is to be contracted. For example, applying two Lorentz transformations (first Λ_1 and then Λ_2) to a contravariant vector gives

$$A^\mu \rightarrow (\Lambda_1 A)^\mu \rightarrow (\Lambda_2 \Lambda_1 A)^\mu = (\Lambda_2 \Lambda_1)^\mu{}_\nu A^\nu. \quad (12.16)$$

Similarly, for a covariant vector, it is

$$A_\mu \rightarrow (A\Lambda_1^{-1})_\mu \rightarrow (A\Lambda_1^{-1}\Lambda_2^{-1})_\mu = (A(\Lambda_2\Lambda_1)^{-1})_\mu. \quad (12.17)$$

This notation allows us to manipulate Lorentz transformations as matrices and can sometimes simplify calculations.

12.3.2 Lorentz transformation of fields

We will use x to represent (x^0, x^1, x^2, x^3) . The Lorentz transformations of a scalar field $\phi(x)$ is given by

$$\phi(x) \rightarrow \phi'(x) \ni \phi'(x'(p)) = \phi(x(p)), \quad (12.18)$$

where $x'(p)$ and $x(p)$ are the coordinates of the same spacetime point p . This relation is usually abbreviated as

$$\phi'(x') = \phi(x). \quad (12.19)$$

It says that the value of the scalar field at a point p is independent of the coordinate system. This definition can be extended to arbitrary coordinate transformations.

For a contravariant vector field A^μ and covariant vector field A_μ , we have

$$A^\mu \rightarrow A'^\mu(x'(p)) = \Lambda^\mu{}_\nu A^\nu(x(p)), \quad (12.20)$$

$$A_\mu \rightarrow A'_\mu(x'(p)) = A_\nu(x(p))(\Lambda^{-1})^\nu{}_\mu \quad (12.21)$$

under a Lorentz transformation.

Relativistic equations of motion for a physical system must be covariant under Lorentz transformations such that the complete set of equations have the same set of solutions before and after the transformation. Or equivalently we can say that given any solution of the equations of motion, you get another solution via Lorentz transformation.

Q: Which of the following can not be a relativistic equation of motion?

- (1) $A_\mu = B_\mu{}^\nu C_\nu$, (2) $A_\mu = C_\nu B_\mu{}^\nu$, (3) $A_\mu = B_\nu{}^\lambda C_\lambda$, (4) $A_\mu = B_\nu C^\nu$.

Occasionally it is even simplified as $\phi' = \phi$.

Q: How does a rank-2 tensor $T_{\mu\nu}(x)$ transform?

12.4 Relativistic Mechanics

Define *proper time* by

$$d\tau = ds/c = \gamma dt \quad (12.22)$$

for a time-like trajectory in spacetime.

For a particle with rest mass m , define 4-velocity, 4-momentum, Minkowski force and 4-acceleration as

$$\eta^\mu = \frac{dx^\mu}{d\tau}, \quad (12.23)$$

$$p^\mu = m \frac{dx^\mu}{d\tau}, \quad (12.24)$$

$$K^\mu = \frac{dp^\mu}{d\tau}, \quad (12.25)$$

$$\alpha^\mu = \frac{d^2x^\mu}{d\tau^2}. \quad (12.26)$$

$$p^0 = m\gamma c = E/c,$$

$$p^i = m\gamma v^i,$$

$$K^0 = \frac{1}{c} \frac{dE}{d\tau},$$

$$K^i = \frac{dp^i}{d\tau}.$$

Q: Check the following identities

$$\eta^2 \equiv \eta^\mu \eta_\mu = -c^2, \quad (12.27)$$

$$p^2 \equiv p^\mu p_\mu = -m^2 c^2, \quad (12.28)$$

$$v^\mu \alpha_\mu = 0. \quad (12.29)$$

For the first two equations, our convention is $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

12.5 Relativistic Formulation of Electrodynamics

In this section, let us use the convention

$$\eta = \text{diag}(-1, 1, 1, 1). \quad (12.30)$$

This is the same conventions as Griffiths.

The gauge potentials V and \mathbf{A} are unified in a 4-vector

$$A^\mu = (V/c, A_x, A_y, A_z). \quad (12.31)$$

The gauge transformation of A_μ is given by

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \lambda. \quad (12.32)$$

The field strength is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (12.33)$$

which is antisymmetric

$$F_{\mu\nu} = -F_{\nu\mu}, \quad (12.34)$$

and thus there are 6 independent components in $F_{\mu\nu}$, and it is invariant under gauge transformations (12.32). The electric field \mathbf{E} and magnetic field \mathbf{B} are unified in $F_{\mu\nu}$ as

$$F^{0i} = E^i/c, \quad F^{ij} = \epsilon^{ijk} B_k. \quad (12.35)$$

The charge density ρ and current density \mathbf{J} are combined into a current density 4-vector J_μ as

$$J^0 = c\rho, \quad J^i = (J_x, J_y, J_z), \quad (12.36)$$

and the continuity equation can be expressed as

$$\partial_\mu J^\mu = 0. \quad (12.37)$$

Maxwell's equations can be rewritten as

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu, \quad (12.38)$$

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0. \quad (12.39)$$

The 2nd equation is called *Bianchi identity*. It is trivially valid if $F_{\mu\nu}$ is given by (12.33). Conversely, for any $F_{\mu\nu}$ that satisfies the Bianchi identity, one can always find A_μ locally such that $F_{\mu\nu}$ is given by (12.33).

The Lorentz force law is

$$K^\mu = qF^{\mu\nu}\eta_\nu. \quad (12.40)$$

It should be clear from the presence of the source term how these two equations are matched with the usual expressions of Maxwell's equations.

12.6 Comments

12.6.1 Assumptions about special relativity

There are three logically possible levels of relativity:

1. There is an absolute reference frame, but some physical laws happen to look the same in other frames that move at constant velocity.
2. There is no absolute reference frame. All inertial frames moving at constant velocity with respect to each other are equally good, but no velocity has the same value in all inertial frames.

Traditional view of classical mechanics before Einstein.

This is the $c \rightarrow \infty$ limit of special relativity, a logically possible interpretation of Newtonian physics.

3. All inertial frames are equally good, and there is a (finite) velocity that has the same value in all inertial frames.

Often we say that special relativity follows from two postulates. The first postulate is the *principle of relativity*. The principle of relativity states that all physical laws apply equally well in all inertial frames. The first logically possible level of relativity violates this principle. The other two observe it. The second logically possible level of relativity respects Galilean symmetry and the third Lorentz symmetry. Thus it should be obvious that one can not deduce special relativity if we do not assume the second postulate: there is a *universal speed* — a (finite) speed at which all inertial frames agree on its value.

On the other hand, are these two postulates sufficient to logically deduce special relativity and nothing else? Let me give a counter-example. The transformation (12.2) still preserves the universal speed c if we relax the condition (12.3) to

$$\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \lambda^2\eta_{\alpha\beta}, \quad (12.41)$$

where $\lambda \in \mathbb{R}$ is an arbitrary real number. This can be viewed as the composition of Lorentz transformation with a scaling transformation

$$x^\mu \rightarrow \lambda x^\mu. \quad (12.42)$$

In the literature, there are various discussions on how to derive special relativity from the two postulates. Implicit assumptions, such as spatial homogeneity, isotropy, etc. are almost always involved. For example, some people argue that the parameter λ in (12.41) must be 1 because the boost by $-\mathbf{v}$ should be the inverse transformation of the boost by \mathbf{v} . But in fact the desired relation between boosts by $\pm\mathbf{v}$ can still hold if λ depends on, say, the x -component of \mathbf{v} in such a way that $\lambda(-v_x) = \lambda^{-1}(v_x)$. You need to assume isotropy to eliminate this possibility.

Therefore, if we assume

- Scaling is not a symmetry.
- Isotropy.
- Homogeneity.

We can probably reasonably rigorously deduce special relativity from the two postulates.

This is the viewpoint of special relativity. The universal velocity is assumed to be the speed of light.

If all physical laws look the same in all inertial frames, there is no way to distinguish which is the absolute one.

Verify that the postulate on universal speed is not violated by this generalization.

Of course, the scaling is not found to be a (manifest) symmetry.

12.6.2 About reference frames

What does it mean to say that the spacetime is a Minkowski space? Mathematically it means that there is a notion of distance given by (12.1), which is invariant under Lorentz transformations. Physically, we might also want to say in addition that physical laws are universally invariant under Lorentz transformations, and thus it is convenient to think about the Minkowskian structure as a fundamental property of the spacetime itself, although spacetime itself is just a platform in which things happen.

To understand this point, let us consider an inertial frame \mathcal{S} with infinitely many clocks at every point in space to define time t in this coordinate system at each point (x, y, z) . A theorist moving at velocity $\mathbf{v} = \hat{\mathbf{x}}v$ in \mathcal{S} can insist on using the same clocks to record his/her observations to label an event by $(t, x' = x - vt, y' = y, z' = z)$ in his reference frame, but the physical laws he/she deduces from the observations will have a different form from those verified in \mathcal{S} .

On the other hand, if the moving theorist takes a clock with him, due to the Lorentz symmetry of the physical laws that govern the machinery of the clock, the time t' defined by the clock will allow him/her to verify the same physical laws as those verified in \mathcal{S} . This is certainly a more reasonable way to define a reference frame (a system of labeling events by (t', x', y', z')). This is what we mean when we say that Lorentz symmetry is a symmetry of spacetime, and why we define reference frames in a particular way.

Here are some tips for understanding reference frames in special relativity:

- Think of everything in terms of “events”.
Special relativity suggests that the notion of “events” is more fundamental than the notion of spacetime. When you think of everything in spacetime as sequences of events, it is easier to understand special relativity and Lorentz transformation.
- Each inertial frame is composed of infinitely many observers at every point in space with their own clocks in their hands.
Each observer is given a spatial coordinate and they are relatively at rest. Their clocks can be synchronized in an intuitive way.
- The spacetime coordinates of an event observed in an inertial frame is determined by the location of the coincident observer and the reading on his/her clock.
For every inertial frame, there is an observer at the same place where the event occurs, and each of them records the event in his/her own way.

A reference frame is a system of labeling each point in space by spatial coordinates (x, y, z) together with a clock fixed at each point for t .

12.7 Exercises

- Let $B(\mathbf{v})$ denote the Lorentz transformation of a boost by the velocity \mathbf{v} (the new reference frame moves at the velocity \mathbf{v} with respect to the old reference frame). $B(\mathbf{v}_2)B(\mathbf{v}_1)$ means the Lorentz transformation of first boosting by \mathbf{v}_1 and then boosting by \mathbf{v}_2 . Check that

$$B(-\hat{\mathbf{y}}v_2)B(-\hat{\mathbf{x}}v_1)B(\hat{\mathbf{y}}v_2)B(\hat{\mathbf{x}}v_1) \quad (12.43)$$

is approximately equivalent to an infinitesimal rotation when $|v_1|, |v_2| \ll c$. Express the rotation as a 4×4 Lorentz transformation matrix.

- Find the trajectory of a point charge q in the background of a constant electric field $\mathbf{E} = \hat{\mathbf{x}}E$. (At $t = 0$, the charge is at rest at the origin.)
- For the electric and magnetic fields in the inertial frame \mathcal{S} given by

$$\mathbf{E}(x^\mu) = \hat{\mathbf{x}}E_0 \cos(kz - \omega t), \quad (12.44)$$

$$\mathbf{B}(x^\mu) = \hat{\mathbf{y}}\frac{E_0}{c} \cos(kz - \omega t), \quad (12.45)$$

where $k = \omega/c$, find the electric and magnetic fields in the reference frame \mathcal{S}' which is moving at the velocity $\hat{\mathbf{x}}v$ with respect to \mathcal{S} .

- Suppose we have a constant electric field \mathbf{E} and a constant magnetic field \mathbf{B} perpendicular to each other. Find the condition on \mathbf{E} and \mathbf{B} such that it is possible to find an inertial frame in which $\mathbf{B} = 0$. Find also the condition on \mathbf{E} and \mathbf{B} such that it is possible to find another inertial frame in which $\mathbf{E} = 0$.

- A stationary magnetic dipole $\mathbf{m} = m\hat{\mathbf{z}}$ is situated above an infinite uniform surface current $\mathbf{K} = \hat{\mathbf{x}}K$.

(1) Find the torque of the dipole, using $\mathbf{N} = \mathbf{m} \times \mathbf{B}$.

(2) Suppose that the surface current consists of a uniform surface charge σ , moving at velocity $\mathbf{v} = \hat{\mathbf{x}}v$, so that $\mathbf{K} = \sigma\mathbf{v}$, and the magnetic dipole consists of a uniform line charge λ , circulating at speed v (same v) around a square loop of side l , so that $m = \lambda vl^2$. Examine the same configuration from the point of view of system $\bar{\mathcal{S}}$, moving in the x direction at speed v . In $\bar{\mathcal{S}}$ the surface charge is at rest, so it generates no magnetic field. Show that in this frame the current loop carries an electric dipole moment, and calculate the resulting torque.

- A point charge q moves with the velocity $\mathbf{v} = v\hat{\mathbf{x}}$ in a constant electric field $\mathbf{E} = E\hat{\mathbf{y}}$ and magnetic field background $\mathbf{B} = B\hat{\mathbf{z}}$.

Carry out the calculation to the first nontrivial order in v_1 and v_2 .

Griffiths Prob. 12.63.

- (a) Find the condition on E and B such that the magnetic field is absent in an inertial frame \bar{S} moving in the x direction with respect to the original frame S .
- (b) Assuming that the condition in the previous question is satisfied, find the relative velocity of \bar{S} with respect to S .
- (c) Find the Lorentz force on the charge in \bar{S} .
- (d) Show that the force in \bar{S} is related to that in S through suitable Lorentz transformations.

7. Griffiths Probs. 12.8, 12.10, 12.11, 12.16, 12.27, 12.33, 12.44, 12.46, 12.47, 12.50, 12.53, 12.62, 12.64, 12.66.

12.8 Appendix

12.8.1 Riemannian Geometry

The primitive notion of space is just a set of points, and perhaps also the definitions about how points are connected with each other. The notion of distance can be considered as an extra construction (structure) on a space. The same space (as a set of points) can be equipped with all sorts of different definitions of distance (length). In General Relativity, for example, the notion of distance is determined dynamically by physical fields (the metric) over the 3+1 dimensional spacetime.

With the notion of distance, one can define *geodesics* as the curve/line connecting two given points on the space with the shortest length. Depending on the choice of the definition of distance,

The notion of distance is given in the form of a line element ds

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \quad (12.46)$$

in Riemannian geometry.

Examples:

1. Minkowski space:

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -(dx^0)^2 + (dx^i)^2. \quad (12.47)$$

2. S^2 :

$$ds^2 = \frac{dx^2 + dy^2}{(1 + r^2)^2}, \quad (12.48)$$

where $r^2 = x^2 + y^2$.

This is the stereographic projection of a sphere, i.e., the projection of the sphere onto the tangent plane of the south pole from the north pole.

3. Pseudosphere:

$$ds^2 = \frac{dx^2 + dy^2}{(1 - r^2)^2}, \quad (12.49)$$

where $r^2 = x^2 + y^2 \leq 1$.

On the 2 dimensional Euclidean space \mathbb{R}^2 , outside a straight line L , there is a unique straight line passing through a given point A that has no intersection with L . This fact is a postulate of the *Euclidean geometry*. For S^2 , there is no straight line (geodesic) passing through A that has no intersection with a given straight line L . This is an example of the *elliptic geometry*. As an example of the *hyperbolic geometry*, the pseudosphere has infinitely many geodesics passing through A that has no intersection with a given geodesic L .

Some more terminology: A space (or a manifold) with a metric $g_{\mu\nu}$ that is positive definite is called a Riemannian space (Riemannian manifold), or a space with Euclidean signature. A space with a metric that has 1 positive and 3 negative (or 3 positive and 1 negative, depending on your convention) eigenvalues is called a pseudo-Riemannian space, or a space with a Lorentzian signature.

All three examples, \mathbb{R}^2 , S^2 and the pseudosphere, have Euclidean signature. The Minkowski space has the Lorentzian signature, of course.

A Riemannian space can be curved, like S^2 and the pseudosphere, or flat, like \mathbb{R}^n and the Minkowski space. Often we imagine that a space is curved by visualizing the embedding of the curved space in a higher dimensional flat space, like how we picture S^2 in \mathbb{R}^3 . An important concept is that, after having the notion of distance on a space, it is unnecessary (and sometimes bad for your understanding) to imagine a curved space embedded in a flat space. The notion of distance itself is sufficient to tell whether and how the space is curved. The corresponding notion is called *curvature*. This is why we can talk about curved spacetime in General Relativity without having to think of a higher dimensional flat space in which the spacetime is embedded. By definition the spacetime is the whole arena where anything physical can happen. The higher dimensional flat space has no physical meaning and can be completely forgotten.

12.8.2 The notion of inertial frame – – from special to general relativity

Usually, the term *inertial frame* refers to those reference frames in which Newton's first law holds: objects at rest remain at rest, objects in motion move at constant velocities, assuming that the objects are not subject to external forces.

This is the Poincaré disk model or Lobacheski disk model of the pseudosphere.

Physicists refer to pseudo-Riemannian space just as Riemannian space.

A *manifold* is a space that is locally the same kind of space as \mathbb{R}^n (for a fixed n), that is, a small patch of the space has the same definition as an open set of points and the connectivity among points as a small patch of \mathbb{R}^n .

But how do we know whether there is external force or not? If you ask a friend this question (without bringing up the definition of inertial frame first), very likely his/her answer is this: You know there is external force if the object is not moving at constant velocity. Is this tautology?

If there is only one object in your universe, this is tautology, but then your universe is apparently not a good example of classical mechanics. In classical mechanics we always assume infinitely many observers and infinitely many tiny particles they can use as probes. In such a universe, you can ask your infinitely many experimentalist friends to carry out experiments for you. If they find that tiny (uncharged) particles always move at constant velocities when they are very very far away from everything else (so that interactions with other objects can be ignored), you are justified to believe that your reference frame is inertial.

From human experience, it is possible to define inertial frames with good approximations.

Let us take one step further and assume that the notion of inertial frame can be defined for an infinitesimal chunk of spacetime. (Imagine that we can have infinitely many infinitesimal particles and observers to check whether Newton's first law holds in an arbitrarily small space over an arbitrarily short period of time.) Physical laws should look the same in all infinitesimal inertial frames. The minor technical issue involved is that you have to know how to carry out the transition from one infinitesimal local patch of spacetime to the next patch.

Einstein realized that an infinitesimal region around an observer in free fall is an inertial frame because Newton's first law is valid. Thus we can "undo" gravity by choosing the reference frames of free falls. But one has to understand how to "patch" the infinitesimal regions of inertial frames together to cover the whole spacetime, because we also want to describe physical phenomena of finite size.

For example, a particle free from external force should move along a straight line at constant velocity in an infinitesimal inertial frame. But as it moves, it keeps getting out of one inertial frame and into another. How do we put infinitely many these infinitesimal trajectories of free fall together to constitute a trajectory of finite length for the particle in free fall? This is analogous to the following question in geometry: how do we define "straight lines" in general? We know how to define straight lines in a flat space. If the space is not flat, but locally each infinitesimal patch of the space is approximately flat, can we extend the notion of straight lines to such curved space? The answer is yes, and they are called "geodesics". The great circles on the surface of a sphere, for example, can be viewed as curves composing of infinitely many infinitesimal

Is it possible that there are unknown nonlocal forces that can not be ignored at large distances? If there are such forces, we would not have found the earth a good approximation of an inertial frame.

straight lines in each infinitesimal (approximately flat) region on the sphere.