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Exposita Notes

Multilateral bargaining: conditional and unconditional offers*

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Summary. We present a game in which n persons split a cake, where a distinction between conditional and unconditional offers is made. This distinction sheds light on the contrasting results obtained in the previous literature of multilateral bargaining. By allowing the proposer to make both conditional and unconditional offers, we show that the game has a unique subgame perfect Nash equilibrium outcome.

Keywords and Phrases: Bargaining, Conditional and unconditional offers.

JEL Classification Numbers: C78, C72.

1 Introduction

In this paper we consider the problem of splitting a cake among n persons. When $n = 2$, Rubinstein [14] constructs a game where players make alternating offers. He shows that the proposer's ability to force the responder to choose between accepting his offer now and waiting to counter-propose one period later pins down the subgame perfect Nash equilibrium outcome uniquely.[1]

Attempts to generalize Rubinstein's result to $n > 2$ have been less successful. On one hand, Shaked (reported by Sutton [16]) defines a unanimity game.[2] Since

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[1] Throughout the paper, the equilibrium concept is the subgame perfect Nash equilibrium.

[2] The rules of the unanimity game in the case of 3 players are as follows. Player 1 first proposes a way to cut the cake. Players 2 and 3 are then called to answer yes or no either sequentially (in any order) or simultaneously. The proposal is adopted if both say yes, but is rejected if either says no. In the latter case, the game goes to period 2 with player 2 as the proposer and so on.

unanimous agreement is needed for a proposal to take effect, every partition of the cake can be supported as an equilibrium outcome when players are patient enough.[3] On the other hand, Krishna and Serrano[4] [12] propose a bargaining game with the possibility of "exit".[5] With the discount factor near 1, each player gets about $1/n$ in equilibrium when players have linear utilities. The sharp contrast between the uniqueness result of Krishna and Serrano and the multiplicity result of Shaked is perplexing.

This paper attempts to investigate these contrasting results further. To this effect, we make a novel distinction between conditional and unconditional offers.

Observe that an offer in Shaked's game is a *conditional* offer because even if the proposed responder agrees, it will only take effect conditional on other responders' acceptance. On the other hand, an offer in Krishna and Serrano is of *unconditional* nature since as long as the proposed responder agrees, he can take that amount away immediately, irrespective of the other responders' responses. In the pure bargaining situation, only the grand coalition can produce the value (the cake), and thus it is natural to interpret the conditional offer as an offer in terms of cake, for a player can take a piece of cake away only when unanimous agreement is reached. Following this, an unconditional offer should be interpreted as some monetary payment that the proposer can have complete control over. This explains why it only takes the proposed responder's acceptance to make an unconditional offer bind.

A priori, there is no theoretical ground why offers should take a conditional or unconditional form. Furthermore, Shaked gets multiplicity of equilibria by restricting offers to be conditional, while Krishna and Serrano get uniqueness of equilibrium by restricting offers to be unconditional. To sort out how robust these results are, it is natural to ask, in an environment where both conditional and unconditional offers can be made, what will happen in equilibrium? This precisely motivates the paper. We shall show that if the proposer can make either conditional or unconditional offers, uniqueness still results.

In some sense, this result provides additional support to the uniqueness result of Krishna and Serrano because it suggests that a restriction to unconditional offers is not really a restriction since the equilibrium outcome is not affected by

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[3] For the case of $n > 3$, Herrero [10] has demonstrated a similar result.

[4] Jun [11], Chae and Yang [4, 5] and Yang [18] have proposed games with the possibility of exit. However, their bargaining procedures are bilateral.

[5] For the case of $n = 3$, the game rules in the main model of Krishna and Serrano [12] are as follows. In period 1, player 1 proposes a way (x_1, x_2, x_3) to cut the cake, where $x_1+x_2+x_3 = 1$. Players 2 and 3 then simultaneously answer yes or no. If both say yes, the game ends with the proposed partition implemented. If only one player, for instance player 2, says yes, then it is assumed that player 2 exits with x_2, even *without* 3's consent, and the game goes to period 2, where players 1 and 3 will play Rubinstein's game for the remaining cake $1 - x_2$. If no one says yes, the game also goes to period 2 with 2 replacing 1's role as the proposer and so on. However, in Section 8, Krishna and Serrano provide another way to interpret their game. The difference is when 2 exits, players 1 and 3 play Rubinstein's game for the *entire* cake. Yet 1's payoff is adjusted by the pay-out, i.e., his final payoff is his share of the cake minus x_2.

the introduction of conditional offers.[6] However, as we shall argue below, there are some differences between these two approaches.

Suppose that $n = 3$ and the protocol is $1, 2, 3$ and so on. We postulate the following game. In period 1 player 1 makes two offers: one to player 2, another to player 3. A proposer can decide whether to use his outside money to buy responders out. If, for example, player 1 wants to buy out player 2, then the offer binds if 2 agrees. Player 3's consent is not needed because the offer is in terms of 1's outside money and thus it is a private transaction between 1 and 2. We call this an *unconditional* offer. On the other hand, if player 1 chooses not to use his outside money to buy 2 out, then his offer to 2 is a share of the cake and thus 3's consent will be needed to make the offer bind. We call this a *conditional* offer (the offer is conditional on player 3's acceptance).

After the offers by player 1 are posted, players 2 and 3 are called to answer either simultaneously or sequentially in any order. If both offers bind, the game ends. If neither binds, the game goes to period 2 with 2 replacing 1 as the proposer. If only one offer binds (for example, that between 1 and 2), then player 1 buys 2 out. When this happens, in later rounds of bargaining, 1 and 3 still bargain over the *entire* cake because the transaction between 1 and 2 is monetary. In return for the money he pays player 2, player 1 gets 2's *right-to-propose* in subsequent bargaining because a player's only valuable resource is his right to propose in different periods prescribed by the protocol.[7] Thus, from period 2 on, 1 and 3 play a *variant* of Rubinstein's bargaining game where the former gets 2 out of 3 turns to propose in a proposing cycle.[8] To motivate the idea that player 1 makes offers in succession, consider the following scenario. There are several parties in the Congress. Every representative may have equal chances to make proposals about a public policy. However, since a major party has more representatives, the party's proposal gets mentioned more often. Consider the situation where party A logrolls with party B. For an issue that party A strongly cares about, party B is essentially bought out by A and thus the representatives from B might just dutifully propose what party A might propose. From the aspect of modelling, it might look like party A can propose in consecutive periods. With this interpretation in mind[9], in Section 3, we shall show that the unique equilibrium outcome is where player 1 gets $\frac{1}{1+\delta+\delta^2}$, 2 $\frac{\delta}{1+\delta+\delta^2}$ and 3 $\frac{\delta^2}{1+\delta+\delta^2}$ while δ is the common discount factor.

[6] Note that it is not true vice versa. That is, a restriction to conditional offers is indeed a restriction. If only conditional offers are allowed, the multiplicity result of Shaked applies. However, this is not robust to the introduction of unconditional offers. As long as unconditional offers are allowed, uniqueness results.

[7] To appreciate this, consider the following variant of Rubinstein's game. Suppose there are two players, 1 and 2. The protocol is that 1 gets to propose in the first 99 periods (out of 100 periods) while 2 only proposes in the last period. In equilibrium, player 1 will then get 99% of the cake while 2 only 1% when the discount factor is large enough. This suggests that a player's bargaining position in Rubinstein's type of games is captured by his relative turns to make proposals in the protocol.

[8] To be precise, the protocol from period 2 on is changed from $2, 3, 1$ and so on to $1, 3, 1$ and so on.

[9] This interpretation is similar to the game rules in Baron [2].

To intuitively understand why we obtain uniqueness, notice that the driving force behind multiplicity of equilibria in Shaked's unanimity game is that every responder has veto power (or in our words, the offers are conditional) and *distinct* responders are used to reject different kinds of out of equilibrium offers. If we relax this assumption by allowing proposers to make unconditional offers, responders' veto power is limited and hence uniqueness restored. It thus suggests that the multiplicity result is not robust to the introduction of unconditional offers. The proof in Section 3 relies heavily on the use of unconditional offers and exactly reflects this intuition.

Although we get uniqueness as Krishna and Serrano did, the two games however are different in the following respects. First, unlike Krishna and Serrano, when an unconditional offer binds, the exiting responder confers on the proposer his right to make proposals at every node where the former would have had the right to make a proposal. Second, we interpret an unconditional offer as the proposer using his outside resource to buy out a responder. In the case where the proposer does not have money at hand, we can alternatively assume that he can borrow to do so, but at the interest rate of $\frac{1-\delta}{\delta}$ per period. On the other hand, Krishna and Serrano assume that the proposer can borrow at zero interest cost. Third, although the idea of the proof is similar to that of Lemma 3 in Krishna and Serrano, the equilibrium responding strategies are relatively simple because no matter what an existing binding unconditional offer is, the continuing subgame is exactly the same and thus the remaining responders use the same equilibrium responding strategies. This simplifies the proof greatly. Moreover, because the proof is simple enough, the analysis will be so general as to cover all the cases as long as the protocol is periodic. For instance, suppose $n = 3$ and the protocol is $1, 2, 1, 3$ and so on, the unique equilibrium outcome is where 1 gets $\frac{1+\delta^2}{1+\delta+\delta^2+\delta^3}$, 2 $\frac{\delta}{1+\delta+\delta^2+\delta^3}$ and 3 $\frac{\delta^3}{1+\delta+\delta^2+\delta^3}$.

The rest of the paper is organized as follows. Section 2 describes the setup of the game. Section 3 contains the proof in showing the uniqueness of the equilibrium outcome.

2 The setup of the game

Imagine a situation where a set N of n players is trying to split a cake of size 1. Following Rubinstein [14], we assume that at the beginning of the game, there is a fixed protocol. The protocol specifies who has the right to propose at what time. Let $p(t)$ be the proposer in period $t = 1, 2, 3, \ldots$. For our purpose, we are going to assume that the protocol is cyclic. Let the periodicity of the protocol be $p < \infty$ and assume that every player appears at least once in the protocol of any cycle.[10] From period to period, there is a common discount factor δ that

[10] Hence, $p \geq n$. If there is one player who has no right to propose, he will get 0 in equilibrium and we can simply ignore him. Allowing this only complicates the notation.

applies.[11] We are going to assume that a proposer could use his outside money to buy out responders.[12]

The following notations will be used. Suppose the game has evolved to period t. Let $C(t)$ denote the set of players still in the game in period t. For every player i in $C(t)$, let $B_i(t)$ be the set of players who have sold their right-to-propose *directly* to player i before period t. Let $D_i(t)$ denote the set of players whose right-to-propose is owned by i at the beginning of period t.[13]

Example 1 At $t = 1$, since everyone is still in the game, so $C(1) = N$, $B_i(1) = \emptyset$ and $D_i(1) = \{i\}$ for all $i \in N$.

If $B_i(t) \neq \emptyset$, for every $j \in B_i(t)$, player i must have bought out player j in some period $s_j < t$. Let $x_j(s_j)$ denote the amount that player i pays player j using his outside money in period s_j.[14] In period t, $C(t)$, $\{B_i(t)\}_{i\in C(t)}$, $\{D_i(t)\}_{i\in C(t)}$ and $\{x_j(s_j)\}_{j\in B_i(t)}$ and $_{i\in C(t)}$ summarize all the information about pervious binding offers.

In period t, the game is played as follows. Player $i \in C(t)$ has the right to propose in his own "endowment" periods, which are $\{s : p(s) = i$ and $s \geq t\}$ and those periods he "buys", which are $\{s : p(s) \in D_i(t)\backslash\{i\}$ and $s \geq t\}$. So there must exist a player, say player i^*, who has the right to propose in period t because $p(t) \in D_{i^*}(t)$.

Player i^* will make one offer to each player in $C(t)$ except himself. An offer could take an unconditional or conditional form. An *unconditional* offer made by player i^* to player j in period t is just a share denoted by $x_j(t)$, which should be interpreted as the price that i^* is willing to pay j in return for getting the latter's right-to-propose using his outside money in period t. A *conditional* offer made by i^* to j is also a share $x_j(t)$, but it comes with a condition which is a non-empty set $S \subseteq C(t)\backslash\{i^*, j\}$.[15,16] This should be read as the offer $x_j(t)$ will bind only if every player in S, in addition to player j, says yes.[17] In other words, j's offer is conditional on the acceptance of the players in S.

After the offers are made, all the responders $(C(t)\backslash\{i^*\})$, are called to answer either simultaneously or sequentially (in any order). An answer could be either yes or no. After the responders have answered, we can determine which offers bind.

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[11] We assume that $0 < \delta < 1$.

[12] Alternatively, we may assume that the proposer can borrow at the interest cost of $\frac{1-\delta}{\delta}$ per period to buy the responders out. Allowing this does not change the result.

[13] Note that it is possible i owns j's right-to-propose indirectly through buying out k, who has bought j out directly.

[14] We may as well interpret $x_j(s_j)$ as the amount that player i borrows (at the interest cost of $(1 - \delta)/\delta$ per period) in period s_j.

[15] Of course, an unconditional offer could be thought as a conditional one with the condition being empty and hence always valid. We separate the conditional and unconditional offers to make our point clearer.

[16] We certainly require that $0 \leq \sum_{j\in C(t)\backslash\{i^*\}} x_j(t) \leq 1$ and $x_j(t) \geq 0$ for all $j \in C(t)\backslash\{i^*\}$.

[17] Note that in the unanimity game, $S = C(t)\backslash\{i^*, j\}$ for every offer $x_j(t)$.

If j is offered an unconditional offer and he answers yes, then the offer between i^* and j binds in period t. Player j sells his right-to-propose to i^* and i^* pays j the amount of $x_j(t)$ immediately using his outside money. Player j's payoff is $x_j(t)\delta^{t-1} - \sum_{k \in B_j(t)} x_k(s_k)\delta^{s_k-1}$. On the other hand, if j answers no, the unconditional offer is void.

If j is offered a conditional offer and he answers yes, then the offer between i^* and j binds only if all the players in S answer yes. In that case, j's payoff is calculated the same way. If either j answers no or anyone in S answers no, the conditional offer is void.

Let A be the set of the responders whose offers bind in period t.

If $A = C(t)\backslash\{i^*\}$, then the game ends in this period. Player i^*'s utility is $(1 - \sum_{j \in C(t)\backslash\{i^*\}} x_j(t))\delta^{t-1} - \sum_{k \in B_{i^*}(t)} x_k(s_k)\delta^{s_k-1}$.

If $A = \emptyset$, then the game goes to period $t+1$ and $C(t+1) = C(t)$, since no offer binds in this period. Furthermore, nothing has changed, thus for $j \in C(t+1)$, $D_j(t+1) = D_j(t)$, $B_j(t+1) = B_j(t)$. In this case, the proposing protocol is not changed from period t to $t+1$, so the person who owns $p(t+1)$'s right-to-propose will be the next proposer.

If $A \neq C(t)\backslash\{i^*\}$ or $\emptyset$, then the game also goes to period $t+1$, but we have to update the information about the binding offers. In this case, $C(t+1) = C(t)\backslash A$ because player i has bought out all the players in A. For any responder whose offer does not bind, say $j \in C(t)\backslash(A \cup \{i^*\})$, we have $D_j(t+1) = D_j(t)$ and $B_j(t+1) = B_j(t)$. For player i^*, since he buys out the rights from the players in A, so $B_{i^*}(t+1) = B_{i^*}(t) \cup A$ with $x_j(s_j)$ for all $j \in B_{i^*}(t)$ and $x_j(t)$ for all $j \in A$. Moreover, since player i^* also "inherited" the rights from the players he buys out, then $D_{i^*}(t+1) = D_{i^*}(t) \cup \{\cup_{j \in A} D_j(t)\}$. The proposing protocol is updated according to the new ownership structure $D_j(t+1)$'s for all $j \in C(t+1)$. The person who owns $p(t+1)$'s right-to-propose will be the next proposer.

A player who never leaves the game gets the payoff of zero minus the discounted sum of money he pays others during the play of the game.

Notice that when $n = 2$ and the protocol is $1, 2$ and so on, since a conditional offer is the same as an unconditional offer, the game certainly reduces to Rubinstein's game.

3 The equilibrium

We will show that as long as the proposing cycle is periodic, the equilibrium outcome of the game is unique.

The following notations will be used in the proof.

For all $i \in C(t)$, let $A_i(t) = \{s \mid t \leq s \leq t+p-1 \text{ and } p(s) \in D_i(t)\}$. In words, $A_i(t)$ is the set of periods where player i is the proposer in the first cycle of the proposing protocol starting from period t.

Let $\triangle = \sum_{s=1}^{p} \delta^{s-1}$. Define

$$x_i^*(t) = \frac{\sum_{s \in A_i(t)} \delta^s}{\triangle * \delta^t}.$$

Let $x^*(t)$ be the vector where the i-th component is $x_i^*(t)$ for all $i \in C(t)$.

Notice that the set of periods where a remaining player i serves as the proposer, denoted by $A_i(t)$, depends on two things: the proposing protocol , $\{p(t)\}_{t=1}^{\infty}$, that we start the game with and the previous binding agreements $D_i(t)$. Thus, $x_i^*(t)$ certainly depends on these two things as well. We illustrate this dependence by the following example.

Example 2 Let $n = 3$ and the original protocol is $1, 2, 1, 3$ and so on. Suppose player 3 has sold his right to player 1 and no further binding agreement is reached. Let us look at $x^*(5)$. For the first proposing cycle starting from period 5, since player 1 has not only his own endowment turns to propose, but also the turns he bought from player 3, thus player 1 will be the proposer of periods 5, 7 and 8, player 2 of period 6. Thus, $A_1(5) = \{5, 7, 8\}$ and $A_2(5) = \{6\}$. Accordingly, $x_1^*(5) = \frac{1+\delta^2+\delta^3}{1+\delta+\delta^2+\delta^3}$ and $x_2^*(5) = \frac{\delta}{1+\delta+\delta^2+\delta^3}$. □

Intuitively, $A_i(t)$ and thus $x_i^*(t)$ reflect player i's proposing position in the first proposing cycle starting from period t. Our goal is to show that $x_i^*(t)$ is player i's equilibrium share in the subgame starting from period t and $x^*(1)$ is the unique equilibrium outcome of the game. We now state the main result.

Theorem 3 *The unique equilibrium outcome of the game is* $x^*(1)$ *and is reached in period 1.*

Before presenting the formal proof, we first explain the idea. It is based on the same argument in Lemma 3 of Krishna and Serrano.[18] Suppose to the contrary that there is an equilibrium where the proposer is getting less than his equilibrium payoff. Then, when a certain profile of unconditional offers is made and subsequently rejected, it must be unanimously rejected because otherwise the responders' expectations about the continuing equilibrium will be inconsistent. Since every responder rejects an unconditional offer, he must expect to get more in the continuing equilibrium. This further implies that the proposer is getting less because of discounting. Exploiting the stationary structure of the game, this leads us to conclude that if the proposer is getting less than his supposed equilibrium share, there is another equilibrium in which he is getting even less. Iterating this for enough times, there must be an equilibrium where the proposer is getting negative payoff, but that is impossible since no player's equilibrium payoff can ever be negative. Thus, the proposer must be getting at least his equilibrium payoff. Since every player serves as the proposer at different periods according to the proposing protocol, this puts a lower bound for every player's equilibrium payoff. The sum of the lower bounds is exactly one, and thus the equilibrium outcome is unique.

We first have the following observation.

Observation: For any *responder* i, $x_i^*(t) = x_i^*(t+1)\delta$ if $i \in C(t+1)$.

[18] A similar argument can also be found in Serrano [15].

The observation simply says that a responder's discounted equilibrium shares in different periods stay the same. This is intuitive because the proposer tries to keep the responder indifferent between accepting now or later.

We now state the induction hypothesis.

Induction hypothesis: Suppose upon reaching period t, at least one player has dropped out, so there are at most $n-1$ players remaining. The unique equilibrium outcome of the subgame from period t is $x^*(t)$ and it will be reached immediately.[19]

Before proving uniqueness, we first establish existence of an equilibrium.

Lemma 4 $x^*(1)$ *is an equilibrium outcome of the game and it is reached immediately in period 1.*

Proof. It is fairly easy to check that the following strategies constitute an equilibrium and hence $x^*(1)$ is reached in period 1.

Propose $x_j^*(t)$ (conditionally or unconditionally) to any other player j in $C(t)$. For any responder j, say yes if and only if the share in the offer is at least $x_j^*(t)$. □

The equilibrium strategies are relatively simple compared to those in Lemma 2 of Krishna and Serrano. In this paper, a responder is affected by the previously binding agreements only to the extent that he faces a proposer with more turns to propose. At what price the other responder was bought out has no effect on the continuing subgame. Thus, the responder's best response is a simple one-dimensional threshold instead of the complicated multi-dimensional acceptance rules used in Lemma 2 of Krishna and Serrano.

For simplicity, call $p(1)$ player 1. We first show that a certain profile of unconditional offers must be unanimously rejected if it is not unanimously accepted.[20]

Lemma 5 *If 1 makes* $n-1$ *unconditional offers such that every responder* $i \neq 1$ *is offered* $x_i^*(t)+\frac{l}{(n-1)\delta^{0.5(t-1)}}$ *for some* $l > 0$*, and they are not unanimously accepted, then they must be unanimously rejected.*

Proof. Offers of this particular form must be unanimously rejected if they are not unanimously accepted. For if the offers were neither unanimously accepted nor unanimously rejected, then there must be some players who accept and others reject. Take any rejecting player i. By the induction hypothesis, he will get $x_i^*(t+1)$ in $t+1$. However, $x_i^*(t+1)\delta = x_i^*(t) < x_i^*(t) + \frac{l}{(n-1)\delta^{0.5(t-1)}}$. Hence, i is better off accepting than rejecting. A contradiction. Thus, the profile of offers must be unanimously rejected. □

We argue next that given a particular equilibrium $\sigma(t)$ from period t, we can generate another equilibrium $\sigma(t+1)$ from period $t+1$.

[19] The induction hypothesis should read as: every player $i \in C(t)$ gets the equilibrium share $x_i^*(t)$ in period t. To calculate player i's utility, one needs to subtract the discounted sum player i pays to buy others out before period t and apply discounting accordingly.

[20] This is also a critical step in Lemma 3 of Krishna and Serrano.

Lemma 6 *Suppose upon reaching period t, no prior offer binds and an equilibrium $\sigma(t)$ is supposed to be played from period t onward. Suppose 1's equilibrium payoff in $\sigma(t)$ is strictly less than $(x_1^*(t) - \frac{l}{\delta^{0.5(t-1)}})\delta^{t-1}$ for some $l > 0$. Then we can construct an equilibrium $\sigma(t+1)$, played from period $t+1$ onward (with all players still in the game), where 1's payoff is strictly less than $(x_1^*(t+1) - \frac{l}{\delta^{0.5t}})\delta^t$.*

Proof. In period t, either player 1 is the proposer or he is not. We discuss the two possible cases.

1. Player 1 is the proposer in period t.

If 1 makes $n-1$ *unconditional* offers such that every responder $i \neq 1$ is offered $x_i^*(t) + \frac{l}{(n-1)\delta^{0.5(t-1)}}$, then the offers must not be unanimously accepted. For if they were unanimously accepted, player 1 will get

$$\begin{aligned} &(1 - \sum_{i \neq 1}(x_i^*(t) + \frac{l}{(n-1)\delta^{0.5(t-1)}}))\delta^{t-1} \\ = \quad &(x_1^*(t) - \frac{l}{\delta^{0.5(t-1)}})\delta^{t-1}, \end{aligned}$$

a contradiction to $\sigma(t)$ being an equilibrium.

By Lemma 5, offers of this particular form must be unanimously rejected. Since every responder rejects an *unconditional* offer, he must expect to get more in the future. Denote $\sigma(t+1)$ the continuing equilibrium.[21] Assume that player i leaves the game at period $s_i(t+1) > t$. Denote i's equilibrium payoff by $z_i(t+1)\delta^{s_i(t+1)-1}$.

Since every responder gets more than his period t unconditional offer in $\sigma(t+1)$, hence,

$$z_i(t+1)\delta^t \geq z_i(t+1)\delta^{s_i(t+1)-1} \geq (x_i^*(t) + \frac{l}{(n-1)\delta^{0.5(t-1)}})\delta^{t-1}, \ \forall i \neq 1.$$

Dividing by δ^t, we get

$$z_i(t+1) \geq \frac{1}{\delta}(x_i^*(t) + \frac{l}{(n-1)\delta^{0.5(t-1)}}) = x_i^*(t+1) + \frac{l}{(n-1)\delta^{0.5(t+1)}}, \ \forall i \neq 1$$

where the equality follows by the observation.

The sum of $z_i(t+1)$ across all the players is weakly less than 1,[22] so

$$z_1(t+1) \leq 1 - \sum\nolimits_{i \neq 1} z_i(t+1) < x_1^*(t+1) - \frac{l}{\delta^{0.5t}}$$

[21] We know that all responders get strictly more than zero in $\sigma(t+1)$, so they must leave the game at some point. Since all responders leave at some point, so does player 1.

[22] Note that $z_i(t+1)$ can be interpreted as the "effective share" player i gets in the period he drops out in $\sigma(t+1)$. This saves us the effort to keep track of any unconditional binding transfers that player i makes before period $s_i(t+1)$. The sum of the $z_i(t+1)$'s over all the players has to be weakly less than 1 because if unanimous agreement is reached in one period, then the sum of the effective shares $z_i(t+1)$'s is just one. In all other cases, the agreement has to be reached over two consecutive periods (by the induction hypothesis, once a player drops out, the others will drop out in the next period). Due to discounting, the sum of the effective shares is weakly less than one.

because $\sum_{i \in N} x_i^*(t+1) = 1$ by definition.

The earliest period 1 can leave the game is period $t+1$, and thus 1's equilibrium payoff in $\sigma(t+1)$ is at most $z_1(t+1)\delta^t$, strictly less than $(x_1^*(t+1) - \frac{l}{\delta^{0.5t}})\delta^t$.

2. Player 1 is not a proposer in period t.

In this case, if 1 rejects his offer, there will be no binding agreement reached in period t. To see this, suppose to the contrary that some binding agreement reached between the proposer and a responder other than 1. By the induction hypothesis, player 1 gets

$$x_1^*(t+1)\delta^t = x_1^*(t)\delta^{t-1} > (x_1^*(t) - \frac{l}{\delta^{0.5(t-1)}})\delta^{t-1},$$

where the first equality follows by the observation. This is clearly a contradiction to $\sigma(t)$ being an equilibrium. Thus, no agreement is reached in this period.

Denote $\sigma(t+1)$ the continuing equilibrium from period $t+1$. Since $\sigma(t+1)$ is generated from $\sigma(t)$ by 1's unilateral deviation, his payoff in $\sigma(t+1)$ must be no greater than that in $\sigma(t)$.

Because

$$(x_1^*(t) - \frac{l}{\delta^{0.5(t-1)}})\delta^{t-1} < (x_1^*(t+1) - \frac{l}{\delta^{0.5t}})\delta^t,$$

1's equilibrium payoff in $\sigma(t+1)$ is strictly less than $(x_1^*(t+1) - \frac{l}{\delta^{0.5t}})\delta^t$.

We now summarize cases 1 and 2. In both cases, given 1's payoff in $\sigma(t)$ is strictly less than $(x_1^*(t) - \frac{l}{\delta^{0.5(t-1)}})\delta^{t-1}$, we construct a new equilibrium $\sigma(t+1)$, where his payoff is strictly less than $(x_1^*(t+1) - \frac{l}{\delta^{0.5t}})\delta^t$. □

The following lemma shows that player 1 gets at least $x_1^*(1)$ in any equilibrium.

Lemma 7 *Player* 1 *gets at least* $x_1^*(1)$ *in any equilibrium.*

Proof. Suppose to the contrary that there exists an equilibrium, denoted by $\sigma(1)$, where 1's payoff is strictly less than $x_1^*(1)$. For some $k > 0$ small enough, 1's payoff is strictly less than $x_1^*(1) - k$.

Since 1's payoff is strictly less than $x_1^*(1) - k$ in $\sigma(1)$, by going through Lemma 6 for p times to complete a proposing cycle, we shall reach period $1+p$ with $\sigma(1+p)$. In $\sigma(1+p)$, 1's payoff is strictly less than $(x_1^*(1) - \frac{k}{\delta^{0.5p}})\delta^p$.

By stationarity of the game, we can treat $\sigma(1+p)$ as an equilibrium in period 1 and go through the argument all over again but replace k with $\frac{k}{\delta^{0.5p}}$ and $\sigma(1)$ with $\sigma(1+p)$. Iterating the process enough times[23], eventually 1's payoff in some equilibrium $\sigma(1+mp)$ has to be negative,[24] but that is impossible. Hence, 1's payoff in any equilibrium is at least $x_1^*(1)$. □

Similarly, any other player $i \neq 1$ gets at least $x_i^*(1)$ in any equilibrium.

[23] So that a sequence of equilibria, $\sigma(1)$, $\sigma(1+p)$, $\sigma(1+2p)$ and so on, is generated.

[24] A strict upper bound for player 1's payoff in $\sigma(1+mp)$ is $x_1^*(1) - \frac{k}{\delta^{0.5*pm}} \leq 0$ when m is big enough. Note that m is the number of cycles that we have gone through.

Lemma 8 *Player i gets at least $x_i^*(1)$ in any equilibrium, $\forall\ i \in N$.*

Proof. For any other player $i \neq 1$, denote s the first period he proposes according to the protocol. In $t < s$, if no binding agreement has been reached before period t, let i reject in that period. Eventually either no one leaves the game before period s or in some period $t < s$ some agreement binds between the proposer and a responder not i. In the first case, by a similar argument, i could guarantee himself $x_i^*(s)\delta^{s-1}$, which is just $x_i^*(1)$ by the observation. In the second case, by the induction hypothesis, i gets $x_i^*(t+1)\delta^t$, which is also $x_i^*(1)$ by the observation. □

Since every player i gets at least $x_i^*(1)$ and the sum of them is exactly one, the uniqueness of the equilibrium outcome is implied. Thus, Lemmas 4 through 8 together complete the proof of Theorem 3.[25]

Remark 1 We wish to emphasize the importance of unconditional offers in pinning down the equilibrium outcome uniquely. In Lemma 4, it does not matter whether conditional or unconditional offers are made. Thus, there exists an equilibrium where the proposer makes only unconditional offers. Moreover, Lemma 5 relies only on the use of unconditional offers. These two observations together suggest that provided unconditional offers are allowed, all the results go through. *Conditional offers play no role in deriving uniqueness*. Thus, as far as the equilibrium outcome is concerned, conditional offers are irrelevant when unconditional offers are present. It is not true vice versa. If only conditional offers can be made, Shaked's multiplicity result applies. However, the multiplicity result is not robust to the introduction of unconditional offers.

References

1. Baliga, S., Serrano R.: Multilateral bargaining with imperfect information. Journal of Economic Theory **67,** 578–89 (1995)
2. Baron, D.: Government formation and endogenous parties. American Political Science Review **87**, 34–47 (1993)
3. Binmore, K. G., Rubinstein, A., Wolinsky, A.: The Nash bargaining solution in economic modelling. Rand Journal of Economics **17**, 176-88 (1986)
4. Chae, S., Yang, J. -A.: The unique perfect equilibrium of an N-person bargaining game. Economics Letters **28**, 221–223 (1988)
5. Chae, S., Yang, J. -A.: An N-person pure bargaining game. Journal of Economic Theory **62**, 86–102 (1994)
6. Chatterjee, K., Dutta, B., Ray, D., Sengupta, K.: A noncooperative theory of coalitional bargaining. Review of Economic Studies **60**, 463–477 (1993)
7. Gul, F.: Bargaining foundations of Shapley value. Econometrica **57**, 81–95 (1989)
8. Haller, H.: Non-cooperative bargaining of N≥3 players. Economics Letters **22**, 11–13 (1986)
9. Hart, S., Mas-Colell, A.: Bargaining and value. Econometrica **64**, 357–380 (1996)
10. Herrero, M.: N-person bargaining and involuntary underemployment. Ph. D. dissertation, London School of Economics, London (1985)

[25] Strictly speaking, for the induction to go through, we need to establish the theorem for $n = 2$ first. All the lemmas go through for $n = 2$ easily without appealing to the induction hypothesis (because there are only two players), so we do not write out the proof for $n = 2$ explicitly.

11. Jun, B. H.: A Structural consideration on 3-person bargaining. Ph. D. dissertation, University of Pennsylvania, Pennsylvania (1987)
12. Krishna, V., Serrano, R.: Multilateral bargaining. Review of Economic Studies **63**, 61–80 (1996)
13. Osborne, M., Rubinstein, A.: Bargaining and markets. New York: Academic Press 1990
14. Rubinstein, A.: Perfect equilibrium in a bargaining model. Econometrica **50**, 97–109 (1982)
15. Serrano, R.: Non-cooperative implementation of the nucleolus: the 3-player case. International Journal of Game Theory **22**, 345-3-57 (1993)
16. Sutton, J.: Non-cooperative bargaining theory: An introduction. Review of Economics Studies **53**, 709–724 (1986)
17. Vannetelbosch, V.: Rationalizability and equilibrium in N-person sequential bargaining. Economic Theory **14**, 353–371 (1999)
18. Yang, J.: Another n-person bargaining game with a unique perfect equilibrium. Economics Letters **38**, 275–277 (1992)
19. Shaked, A., Sutton, J.: Involuntary unemployment as a perfect equilibrium in a bargaining model. Econometrica **52**, 1351–1364 (1984)