Credit and Business Cycles

Kiyotaki (1998)

1 Introduction

Why do theories of credit are important in understanding the mechanism of business cycles? Where does RBC fail in explaining the observed facts in business cycles?

1. RBC is less successful in explaining price movements, either relative or nominal.

2. RBC needs large, persistent and exogenous aggregate productivity shocks to generate observed business fluctuations: it lacks a powerful propagation mechanism by which the effects of small shocks amplify, persist and spread.

This paper considers models in which credit limits are endogenously determined, and acts as a propagation mechanism so as to generate large, persistent fluctuations in aggregate productivity, output and asset prices given small and temporary shocks on technology or wealth distribution.

2 The Basic Model

Consider a discrete-time economy with a single good and a continuum of homogeneous agents with preferences $\sum_{\tau=0}^{\infty} \beta^t \ln c_{t+\tau}$. At each date t, there is a competitive one-period credit market, in which the real (gross) interest rate is r_t from date t to date $t+1$.

At each date, some agents are productive and the others are unproductive, where the ratio of population is $n:1$. The productivity of each type is

$$y_{t+1} = \alpha x_t, \tag{1}$$

if productive, and

$$y'_{t+1} = \gamma x'_t, \tag{2}$$

if unproductive, where $1 < \gamma < \alpha$.

Each agent's productivity is public information. Agents shift stochastically between productive and unproductive states accord to Markov process:

$t \diagdown t+1$	Productive	Unproductive
Productive	$1-\delta$	δ
Unproductive	$n\delta$	$1-n\delta$

The purpose of specifying recurrent shifts in productivity of an individual agent is to analyze how the credit system affects the dynamic interaction between distribution of wealth and productivity.

(A1) $n\delta + \delta < 1$ (productivity of each agent is positively correlated between t and t+1, i.e. productivity shift is not too large).

2.1 The Bargaining Problem

We assume that
(i) The production technology is specific to each producer: only the producer who invested at t has the necessary skill to obtain the full returns at date $t+1$. Without the skill of the producer who invested, creditors can recover θ fraction of returns.

(ii) Each producer is free to walk away from the production between t and $t+1$ and from any debt obligations.

As a consequence, if a producer owes a lot of debt, he may be able to renegotiate with the creditor for a smaller debt before harvesting time. If the debtor-producer has strong bargaining power, then he can reduce his debt repayment to a fraction θ of the total returns (the fraction of returns that can be recovered by creditors, i.e., collateral value of the investment).

Anticipating the possibility of the default between dates t and $t+1$, the creditor limits the amount of credit at date t, so that the *debt repayment* of the debtor-producer in the next period b_{t+1} will not exceed the value of the collateral θy_{t+1}:

$$b_{t+1} \leq \theta y_{t+1}. \tag{3}$$

To see this, suppose the farmer repudiates between t and $t+1$, and the renegotiation proceeds. Let η be the bargaining power of the farmer, and $1-\eta$ that of the gatherer. Both parties bargain over the repayment d_t if an agreement is reached.

	Farmer	Gatherer
Agreement reached	$y_{t+1} - d_t$	d_t
Default	0	θy_{t+1}

The bargaining problem maximize

$$\max_{d_t} \left[y_{t+1} - d_t\right]^{\eta} \left[d_t - \theta y_{t+1}\right]^{1-\eta}$$

FOC implies

$$\frac{\eta}{1-\eta} = \frac{y_{t+1} - d_t}{d_t - \theta y_{t+1}}.$$

Then, we have

$$d_t = (1 - \eta + \theta\eta)\, y_{t+1}.$$

The incentive constraint for *no-renegotiation* requires that the farmer's value of no default should be greater than or equal to the value of default and an agreement is reached, i.e.,

$$y_{t+1} - b_{t+1} \geq y_{t+1} - d_t,$$

i.e.,

$$b_{t+1} \leq d_t = (1 - \eta + \theta\eta)\, y_{t+1}.$$

If the farmer has all the bargaining power, $\eta = 1$, then the collateral constraint will be (3)

(A2) $\theta\alpha < \gamma$ (We will see later that in equilibrium $r_t = \gamma$. Thus, this assumption says that if $\theta\alpha$ (the fraction of output that can be promised to creditors) is too high, i.e., $\theta\alpha > r_t$, then productive agents will borrow unlimited amount).

2.2 The Equilibrium

Each agent maximizes his expected lifetime utility

$$\begin{aligned} &\max_{\{c_t, x_t, b_{t+1}\}} E_t \sum_{\tau=0}^{\infty} \beta^{\tau} \ln c_{t+\tau} \\ &s.t. \qquad c_t + x_t = y_t + b_{t+1}/r_t - b_t \\ &\qquad\qquad b_{t+1} \leq \theta y_{t+1}, \end{aligned} \tag{4}$$

and (1) and (2).

The goods market equilibrium is

$$C_t + C_t' + X_t + X_t' = Y_t + Y_t', \tag{5}$$

which says that the aggregate consumption and investment of productive and unproductive agents are equal to the aggregate output of the two types. The credit market also clear by Walras Law.

2.3 Full Enforceability

Suppose the financial contract is fully enforceable. Thus, there will be no credit constraint.
Let $a_t = y_t - b_t$ (net worth at date t). The flow of funds constraint becomes

$$\begin{aligned} c_t + x_t &= (y_t - b_t) + \frac{(y_{t+1} - b_{t+1})}{r_t} + \frac{y_{t+1}}{r_t} \\ &= a_t - \frac{a_{t+1}}{r_t} + \frac{y_{t+1}}{r_t}. \end{aligned}$$

Without credit constraint, the productive will compete for credit and push the real interest rate up to the level the marginal productivity of the productive agent, α. Thus,

$$r_t = \alpha.$$

Thus, the unproductive agents will drops out of the credit market and become lenders.

For *productive* agents, the flow of funds constraint is

$$c_t + x_t = a_t - \frac{a_{t+1}}{\alpha} + \frac{\alpha x_t}{\alpha}$$
$$\Rightarrow \; a_t = c_t + \frac{1}{\alpha} a_{t+1}. \tag{6}$$

Also the FOC (**??**) becomes

$$c_{t+1} = \alpha\beta c_t,$$

which also implies

$$c_{t+k} = (\alpha\beta)^k c_t, k \geq 1. \tag{7}$$

Then, by (6),

$$\begin{aligned} c_t &= a_t - \frac{1}{\alpha} a_{t+1} \\ &= \alpha_t - \frac{1}{\alpha} c_{t+1} - \frac{1}{\alpha^2} a_{t+2} + \cdots \\ &= a_t - \frac{1}{\alpha} c_{t+1} - \frac{1}{\alpha^2} c_{t+2} - \cdots - \frac{1}{\alpha^k} a_{t+k}, \end{aligned}$$

where $\lim_{k \to \infty} \frac{1}{\alpha^k} a_{t+k} = 0$ since $\alpha > 1$. Thus, by (7),

$$c_t = a_t - \beta c_t - \beta^2 c_t - \cdots .$$

This, every period the agent consumes a fixed fraction of his net worth,

$$c_t = (1 - \beta) a_t,$$

which is because of the log utility. Thus the saving of productive agents is a fraction β of the net worth.

For the *unproductive*, they will not invest, $x'_t = 0$. Their consumption is still a fixed fraction of net worth, $c'_t = (1 - \beta) a'_t$, where $a'_t = y'_t - b'_t$. Thus, the unproductive each lend the amount

$$b'_{t+1} = -\alpha\beta a'_t < 0.$$

Thus, the borrowing of each productive agent (on average) is

$$b_{t+1} = \frac{-b'_{t+1}}{n} = \frac{\alpha\beta a'_t}{n}.$$

The investment of each productive agent is

$$x_t = \beta a_t + \frac{b_{t+1}}{\alpha} = \beta a_t + \frac{\alpha\beta a'_t}{\alpha n} = \beta\left(a_t + \frac{a'_t}{n}\right).$$

In aggregate,

$$\begin{aligned} B_t + B'_t &= nb_t + b'_t = 0, \\ C_t + C'_t &= (1-\beta)Y_t. \end{aligned}$$

Thus the aggregate investment is

$$X_t = nx_t = \beta\left(na_t + a'_t\right) = \beta Y_t = \beta W_t,$$

where $W_t \equiv Y_t + Y'_t$ is the aggregate wealth.

Since $Y_t = \alpha X_{t-1}$, we have $X_t = \beta Y_t = \alpha\beta X_{t-1}$. The aggregate wealth evolves according to

$$\frac{W_{t+1}}{W_t} = \frac{Y_{t+1}}{Y_t} = \frac{X_t}{X_{t-1}} = \alpha\beta.$$

Aggregate investment, output, and the growth rate aggregate wealth are independent of distribution of wealth between productive and unproductive agents.

2.4 Imperfect Enfoceability

Assume that the probability of a productive agent becoming unproductive in the next period δ is large enough and that the ratio of population of productive to unproductive agents (n) is small:
(A3) $\delta > \theta\frac{\alpha-\gamma}{\gamma}\frac{\gamma-\theta\alpha}{1-\theta\alpha-n\theta\gamma}$.

In HW, you are asked to solve for the maximization problem of the productive agents when there is imperfect enfoceability, by deriving consumption and investment (c_t, x_t) in terms of a_t, and verify that $r_t = \gamma$.

The investment of the productive is

$$x_t = \frac{1}{1 - \theta\alpha/\gamma}\beta a_t.$$

Unproductive agents are indifferent between lending and investing because the real interest rate is the same as the rate of return on their investment, $r_t = \gamma$.
Their saving is also a fraction β of their net worth. Then the aggregate lending and investment of unproductive agents are determined by the market-clearing condition (5).

Since c_t, b_{t+1}, and x_t are linear function of net worth, we can easily aggregate over agents.

The law of motion of aggregate worth (W_t), and

$$\begin{aligned} W_{t+1} &= Y_{t+1} + Y'_{t+1} = \alpha X_t + \gamma X'_t \\ &= \alpha \frac{\beta A_t}{1-\theta\alpha/\gamma} + \gamma(\underbrace{\beta W_t}_{\text{aggregate savings}} - \frac{\beta A_t}{1-\theta\alpha/\gamma}) \\ &= \gamma\beta W_t + (\alpha-\gamma)\frac{1}{1-\theta\alpha/\gamma}(\frac{A_t}{W_t})\beta W_t. \end{aligned} \tag{8}$$

Note that, for a given present aggregate wealth, the aggregate net worth next period W_{t+1} is increasing in the share of net worth of productive agent, $s_t = \frac{A_t}{W_t}$.

The aggregate net worth of *productive* agents (A_t) are

$$A_{t+1} = (1-\delta)(Y_{t+1} - B_{t+1}) + n\delta(Y'_{t+1} - B'_{t+1}). \tag{9}$$

Since $B'_{t+1} = -B_{t+1}$, $Y_{t+1} - B_{t+1} = A_{t+1}$, and by (28), $A_{t+1} = \alpha^{+}\beta A_t$, we have

$$\begin{aligned} Y'_{t+1} - B'_{t+1} &= W_{t+1} - (Y_{t+1} - B_{t+1}) = \gamma\beta W_t + (\alpha - \gamma)\frac{1}{1 - \theta\alpha/\gamma}(\frac{A_t}{W_t})\beta W_t - \alpha^{+}\beta A_t \\ &= \gamma\beta W_t + \frac{\alpha - \gamma - \alpha(1-\theta)}{1 - \theta\alpha/\gamma}\beta A_t = \gamma\beta W_t + \frac{-1 + \alpha\theta/\gamma}{1 - \theta\alpha/\gamma}\gamma\beta A_t \\ &= \gamma\beta\left(W_t - A_t\right). \end{aligned}$$

Thus, we have

$$A_{t+1} = (1 - \delta)\alpha^{+}\beta A_t + n\delta\gamma\beta(W_t - A_t). \tag{10}$$

The growth rate of aggregate worth is also increasing in the share of net worth of productive agents

$$\frac{W_{t+1}}{W_t} = \beta(\gamma + \frac{\alpha - \gamma}{1 - \theta\alpha/\gamma}s_t),$$

which is smaller than the growth rate of aggregate worth without credit constraint, $\alpha\beta$.

By (8) and (10), the share of net worth of productive agents evolves according to

$$s_{t+1} = \frac{(1-\delta)\alpha^{+}s_t + n\delta\gamma(1-s_t)}{\alpha^{+}s_t + \gamma(1-s_t)} \equiv f(s_t), \tag{11}$$

which implies that the share of net worth of productive agents monotonically converges to a unique steady-state $s^* \in [1-\delta, n\delta]$, where s^* solves $s^* = f(s^*)$.

In HW you are asked to verify that in equilibrium $r_t = \gamma$.

The intuition is that the probability of shift from productive to unproductive is large (δ is large) and the population of productive agents is small (n is small), as required by (A3), then the share of the net worth of productive agents is small in the steady state (A/W is small); then, given that the fraction of the collateralized returns is not too large (θ is not large, as required by (A2)), aggregate saving is larger than the investment of productive agents ($\beta W_t > X_t$), and unproductive agents will invest too.

2.5 Dynamics

Consider on unexpected once-and-for-all negative productive shocks at beginning of t, by (9) the aggregate net worth of productive agents becomes

$$\begin{aligned} A_t &= (1-\delta)\left[(1+\triangle)\alpha X_{t-1} - B_t\right] + n\delta\left[(1+\triangle)\gamma X'_{t-1} + B_t\right] \\ &= (1+\triangle)[(1-\delta)\alpha X_{t-1} + n\delta\gamma X'_{t-1}] - \underbrace{(1-\delta-n\delta)\theta\alpha X_{t-1}}_{\text{Existing debt}}. \end{aligned}$$

Note that in steady state

$$A^* = (1-\delta)\alpha X^* + n\delta\gamma X'^* - (1-\delta-n\delta)\theta\alpha X^*.$$

Then, when a shock hits,

$$\frac{(A_t - A^*)}{A^*} = \frac{(1-\delta)\alpha X^* + n\delta\gamma X^*}{A^*}\left|\triangle\right| > \left|\triangle\right|.$$

Since productive agents have a net debt in aggregate, the debt obligation carried from the last period is fixed, but now the revenue is lower. Thus, the net worth of productive agents decreases proportionately more than the aggregate productivity as a result of leverage effect of the debt ($X_t = \frac{1}{1-\theta\alpha/\gamma}\beta A_t$ and $\alpha\theta < \gamma$).

Also, because aggregate wealth decrease in the same proportion as the aggregate productivity, the share of net worth of productive agents (A_t/W_t) decreases at t. Then the growth rate of the economy is lower.

One possible impact of monetary policy may be considered as the unanticipated redistribution of wealth between debtors and creditors. For example, if the debt is nominal and is not indexed, the unanticipated lower inflation redistributes wealth from debtors to creditors. Then the share of net worth of productive agents decreases and the growth rate will decrease (debt-deflation).

3 The Model with Fixed Asset: Propagation and Persistence

We now introduce fixed asset to analyze the interaction between value of fixed asset, credit, and production over the business cycle.

There are two modifications from the basic model

(I) Technologies: land and goods as inputs

The technologies of the productive and unproductive are respectively

$$y_{t+1} = \alpha(\frac{k_t}{\sigma})^\sigma(\frac{x_t}{\sigma})^{1-\sigma}, \tag{12}$$

$$y'_{t+1} = \gamma(\frac{k'_t}{\sigma})^\sigma(\frac{x'_t}{\sigma})^{1-\sigma}, \quad 0 < \gamma < \alpha, \tag{13}$$

where $\sigma \in (0, 1)$ is the share of land in costs of input. There is a spot market for land, in which one unit of land is exchange for q_t units of goods.

(II) Assume that if the agent who has invested at date t with land k_t withdraws his labor between t and $t+1$, there would be *no output* at $t+1$. Thus, the value of collateral is the value of land. Now both parties bargain over the repayment d_t if an agreement is reached.

	Farmer	Gatherer
Agreement reached	$y_{t+1} + q_{t+1}k_t - d_t$	d_t
Default	0	$q_{t+1}k_t$

The incentive constraint for *no-renegotiation* will lead to the collateral constraint

$$b_{t+1} \leq q_{t+1}k_t. \tag{14}$$

(Recall that b_{t+1} is debt repayment, not borrowing)

This also implicitly implies a variable (in contrast to the constant θ) fraction of collateralized return

$$\theta_{t+1} = \frac{q_{t+1}k_t}{y_{t+1} + q_{t+1}k_t}.$$

The flow of funds constraint is

$$c_t + x_t + q_t(k_t - k_{t-1}) = y_t - b_{t+1}/r_t - b_t. \tag{15}$$

Each agent maximizes his expected lifetime utility, subject to (14), (15), (12) and (13). Besides the goods market, the market for land clears

$$K_t + K_t' = 1,$$

where we have normalized the total supply of land to be 1.

3.1 Without Borrowing Constraint

Without borrowing constraint (14), only productive agent invest in goods and use land. Thus, $K_t = 1$, and the competitive equilibrium corresponds to an efficient allocation, which maximize the weighted average of utility of production and unproductive, subject to resource constraint,

$$C_t + C_t^{'} + X_t = Y_t = \frac{\alpha}{\sigma^\sigma}(\frac{X_{t-1}}{1-\sigma})^{1-\sigma}.$$

Note that the aggregate investment in land is zero. The marginal products of land and investment are

$$\begin{aligned} MP_k &= \sigma \frac{Y_t}{K_{t-1}} = \sigma Y_t, \\ MP_x &= (1-\sigma)\frac{Y_t}{X_{t-1}}. \end{aligned}$$

Using Bellman's equation, we can solve for the investment of goods, which is proportional to output,and related to its marginal productivity

$$X_t = (1-\sigma)\beta Y_t,$$

and thus,

$$Y_{t+1} = \frac{\alpha}{\sigma^\sigma}(\frac{X_t}{1-\sigma})^{1-\sigma} = \frac{\alpha}{\sigma^\sigma}(\beta Y_t)^{1-\sigma}. \tag{16}$$

The land price is the present value of the future marginal product of land

$$q_t = \frac{\beta}{1-\beta}(\sigma Y_t) = \frac{\beta\sigma}{1-\beta+\beta\sigma} W_t.$$

where aggregate wealth

$$W_t = Y_t + q_t.$$

Land price is proportional to aggregate output and aggregate wealth, owing to the log utility function.

Without borrowing constraint, aggregate output, investment, and land price do not depend upon distribution of wealth.

3.2 With Borrowing Constraint

Assume that the turnover rate of productive agents is relatively high
(A4) $\delta > \frac{\alpha-\gamma}{\gamma}$.

Given (A4), we can show that, in the neighborhood of the steady state with a small enough ratio of proportion of productive to unproductive agents, the real rate of interest is equal to the return on investment of unproductive agents (MP_{x_t}),

$$r_t = \gamma u_t^{-\sigma}.$$

By the maximization problem of the unproductive,

$$\begin{aligned}
c_t' &: \frac{\beta^t}{c_t'} = \lambda_t, \\
b_{t+1}' &: \frac{\lambda_t}{r_t} = \lambda_{t+1}, \\
k_t' &: \lambda_t q_t = \lambda_{t+1}(q_{t+1} + \gamma(\frac{k_t'}{\sigma})^{\sigma-1}(\frac{x_t'}{1-\sigma})^{1-\sigma}) = \lambda_{t+1}(q_{t+1} + MP_{k'}), \\
x_t' &: \lambda_t = \lambda_{t+1}\gamma(\frac{k_t'}{\sigma})^{\sigma}(\frac{x_t'}{1-\sigma})^{-\sigma} = \lambda_{t+1} MP_{x'}.
\end{aligned}$$

The FOCs imply

$$
\begin{aligned}
c'_{t+1} &= \beta r_t c'_t, \\
q_t &= \frac{q_{t+1}}{r_t} + \frac{MP_{k'}}{r_t} \qquad (17) \\
\frac{\sigma}{1-\sigma}\frac{x'_t}{k'_t} &= q_t - \frac{1}{r_t} q_{t+1} \equiv u_t, \qquad (18)
\end{aligned}
$$

where u_t is the opportunity cost or user cost of holding land. This also implies

$$
\frac{x'_t}{k'_t} = \frac{1-\sigma}{\sigma/u_t}, \qquad (19)
$$

i.e., the ratio of inputs for goods and land is $(1-\sigma) : \sigma/u_t$.

Note also that the marginal products of land and goods investment are, using (18),

$$
\begin{aligned}
MP_k &= \gamma \left(\frac{x'_t}{k'_t}\right)^{-\sigma} \sigma^{-\sigma} (1-\sigma)^{\sigma} = \gamma u_t^{-\sigma}, \\
MP_x &= \gamma \left(\frac{x'_t}{k'_t}\right)^{1-\sigma} \sigma^{1-\sigma} (1-\sigma)^{\sigma-1} = \gamma u_t^{1-\sigma},
\end{aligned}
$$

where $\gamma u_t^{-\sigma}$ is the rate of return on investment of the unproductive agent. Thus,

$$\frac{MP_{k'}}{MP_{x'}} = u_t = \frac{x'_t}{k'_t}\frac{\sigma}{1-\sigma}.$$

Productive agents borrow up to the credit limit, because their rate of return on investment exceeds the real interest rate. Substituting the borrowing constraint and the flow of constraint into the objective function. The FOCs are

$$x_t \;:\; \frac{1}{c_t} = \beta \frac{1}{c_{t+1}}(1-\sigma)\frac{y_{t+1}}{x_t},$$

$$k_t \;:\; \frac{1}{c_t}\left(q_t - \frac{q_{t+1}}{r_t}\right) = \beta \frac{1}{c_{t+1}}\sigma\frac{y_{t+1}}{k_t},$$

which implies that the ratio of inputs for goods and land is $(1-\sigma) : \sigma/u_t$, same as (19):

$$\frac{x_t}{k_t} = \frac{1-\sigma}{\sigma/u_t},$$

By the flow of funds constraint of the productive agents (15), the investment in goods is

$$\begin{aligned} x_t &= (y_t + q_t k_{t-1} - b_t - c_t) + \frac{q_{+1t} k_t}{r_t} - q_t k_t \\ &= (a_t - c_t) - u_t k_t \\ &= (a_t - c_t) - u_t \frac{\sigma}{1-\sigma} \frac{1}{u_t} x_t \end{aligned}$$

where $a_t = y_t + q_t k_{t-1} - b_t$ is the net worth of the productive at time t. Thus, we have

$$x_t = (1-\sigma)(a_t - c_t), \tag{20}$$

which says that the productive agent spends $(1-\sigma)$ fraction of his net worth (after consumption) on investment goods.

The investment in land of the productive is

$$k_t = \frac{a_t - c_t - x_t}{q_t - \frac{q_{t+1}}{r_t}} = \frac{\sigma (a_t - c_t)}{q_t - \frac{q_{t+1}}{r_t}}, \tag{21}$$

which says that the productive agent spends σ fraction of his net worth (after consumption) to finance the difference between land value $q_t k_t$ and the amount he can borrow against land $q_{+1t}k_t/r_t$.

Thus, $q_t - \frac{q_{t+1}}{r_t}$ equals the down payment required to purchase one unit of land, which happens to be equal to the user cost of land u_t.

Again, by the productive agents' flow of fund constraint, the net worth evolves according to (using (20) and (21)),

$$\begin{aligned} a_{t+1} &= y_{t+1} + q_{t+1}k_t - b_{t+1} = y_{t+1} \\ &= \alpha(\frac{k_t}{\sigma})^{\sigma}(\frac{x_t}{1-\sigma})^{1-\sigma} \\ &= \alpha u_t^{-\sigma}(a_t - c_t), \end{aligned} \tag{22}$$

where $\alpha u_t^{-\sigma}$ is the return on savings (MP_x) for the productive agents.

Similarly, the rate of return on investment of the unproductive agent is $\gamma u_t^{-\sigma}$. Thus, when $r_t = \gamma u_t^{-\sigma}$, the unproductive is indifferent between investing and lending.

Due to log utility function, both productive and unproductive agents consume $(1-\beta)$ fraction of their net worth. To see this, by (22), the FOC implies

$$c_{t+1} = \alpha u_t^{-\sigma}\beta c_t.$$

Using (22),

$$\begin{aligned} c_t &= a_t - \frac{1}{\alpha u_t^{-\sigma}} a_{t+1} = a_t - \frac{1}{\alpha u_t^{-\sigma}} \left[c_{t+1} + \frac{1}{\alpha u_{t+1}^{-\sigma}} a_{t+2} \right] \\ &= a_t - \frac{1}{\alpha u_t^{-\sigma}} \left[\alpha u_t^{-\sigma} \beta c_t + \frac{1}{\alpha u_{t+1}^{-\sigma}} \left(c_{t+2} + \frac{1}{\alpha u_{t+2}^{-\sigma}} a_{t+3} \right) \right] \\ &= a_t - \left[\beta c_t + \beta^2 c_t + ... \right] \end{aligned}$$

Then, we have

$$c_t = (1 - \beta) a_t.$$

Given this, the investment in goods and land are

$$\begin{aligned} x_t &= (1 - \sigma) \beta a_t, \\ k_t &= \frac{\sigma}{u_t} \beta a_t. \end{aligned}$$

In aggregate, the total investment of the productive is

$$X_t + \underbrace{u_t K_t}_{\text{downpayment}} = (1-\sigma)\beta A_t + \sigma\beta A_t = \beta A_t$$

By definition we have

$$A_t = Y_t + q_t K_{t-1} - B_t,$$

then

$$A_t + A_t' = Y_t + Y_t' + q_t \equiv W_t.$$

Thus, by the market clearing condition for goods

$$C_t + C_t^{'} + X_t + X_t' = Y_t + Y_t',$$

we have

$$\begin{aligned} (1-\beta)(A_t + A_t') + \tfrac{1-\sigma}{\sigma} u_t (K_t + K_t') = &\ Y_t + Y_t', \\ \Rightarrow \quad (1-\beta)[Y_t + Y_t' + q_t] + \tfrac{\sigma}{1-\sigma} u_t = &\ W_t - q_t, \\ \Rightarrow \quad (1-\beta) W_t + \tfrac{1-\sigma}{\sigma} u_t = &\ W_t - q_t, \end{aligned} \tag{23}$$

where we have used $W_t = Y_t + Y_t' + q_t$, $K_t + K_t' = 1$, $K_t = \frac{\sigma}{u_t}\beta A_t$, and $X_t = (1-\sigma)\beta A_t = \frac{\sigma}{1-\sigma} u_t K_t$.

By (22), $A_{t+1} = \alpha u_t^{-\sigma}\beta A_t$,and $A'_{t+1} = \gamma u_t^{-\sigma}\beta A'_t$.
Then, the law of motion of aggregate worth is (by aggregating the production functions)

$$\begin{aligned} W_{t+1} &= Y_{t+1} + Y'_{t+1} + q_{t+1} = (Y_{t+1} + q_{t+1}K_t - B_t) + (Y'_{t+1} + q_{t+1}K'_t - B'_t) \\ &= A_{t+1} + A'_{t+1} = \alpha u_t^{-\sigma}\beta A_t + \gamma u_t^{-\sigma}\beta(W_t - A_t) \\ &= \underbrace{[\alpha u_t^{-\sigma} s_t + \gamma u_t^{-\sigma}(1 - s_t)]}_{\text{rate of returns on savings}} \quad \underbrace{\beta W_t}_{\text{aggregate saving}} \quad . \end{aligned} \tag{24}$$

The aggregate net worth of the productive is

$$\begin{aligned} A_{t+1} &= (1-\delta)A_{t+1} + n\delta A'_{t+1} \\ &= (1-\delta)\alpha u_t^{-\sigma}\beta A_t + n\delta\gamma u_t^{-\sigma}\beta(W_t - A_t) + \\ &= \beta u_t^{-\sigma}[(1-\delta)\alpha s_t + n\delta\gamma(1 - s_t)]W_t. \end{aligned}$$

Thus, the share of net worth of productive agents is

$$s_{t+1} = \frac{(1-\delta)\alpha s_t + n\delta\gamma(1-s_t)}{\alpha s_t + \gamma(1-s_t)} \equiv g(s_t). \tag{25}$$

Finally, the land price satistics

$$q_t = u_t + \frac{q_{t+1}}{r_t} = u_t + \frac{u_t^{\sigma}}{\gamma} q_{t+1} \tag{26}$$

In the definition of perfect foresight equilibrium, we need to rule out bubbles

$$\lim_{t\to\infty} E_0(\frac{q_t}{r_0 r_1 \cdots r_{t-1}}) = 0$$

Unlike the basic model, the economy does not grow in the steady state, because the land is the fixed factor of production.

In order to verify that $r_t = \gamma u_t^{-\sigma}$, we need only show that aggregate land holding of unproductive agents is positive:

$$K_t^{'} = 1 - K_t = 1 - \sigma\beta s_t W_t / u_t > 0,$$

if assumption (A4) holds for a small enough n in the NBHD of the steady state.

3.3 Dynamics

Solve for u_t by using (23), then substitute into (24), (25), and (26) to obtain the system for $\{q_t, W_t, s_t\}$. We then linearize the system and solve for the eigenvalues.

Consider the impact of a small, unanticipated, temporary productivity shock $\Delta < 0$, at date t.
(1a) The land price falls proportionately more than the temporary productivity shock itself at date t (collateral value falls).
(1b) The net worth of productive agents falls proportionately more than the aggregate wealth owing to the leverage effect of debt (wealth distribution matters).
(1c) Thus, it takes time for the aggregate wealth and the balance sheet of productive agents to recover through saving and investment. Then the user cost of land is expected to continue to be low in dates t, $t+1$, $t+2$, ...

(2a) This anticipated, persistent decline in the user cost in future dates is reflected by a significant fall in the land price at date t.

(2b) The fall in land price at date t further reduces the aggregate wealth, and particularly the net worth of productive agents who have outstanding debts.
(2c) The decrease in the aggregate wealth and the share of net worth of productive agents in turn further reduces the land price.

This says that the introduction of the fixed asset makes the equilibrium system both history-dependent and forward-looking, highlighting the interaction between *persistence* and *amplification*.