We first present a simple dynamic pure exchange economy with two infinitely lived consumers engaging in intertemporal trade.

We demonstrate the connection between competitive equilibria and Pareto optimal equilibria, and how this connection can exploited to compute equilibria by solving a social planner’s problem.

1 The Model Economy

Need to describe the environment of the model by specifying technology, preferences, endowments and the information structure.

Time is discrete and indexed by $t = 0, 1, 2, \ldots$ There are 2 infinitely-lived consumers in this deterministic pure exchange economy (no production).
In each period the two agents trade a non-storable consumption good. Hence there are (countably) infinite number of commodities. An allocation is a sequence \((c^1, c^2) = \{(c^1_t, c^2_t)\}_{t=0}^{\infty}\) of consumption in each period for each individual.

Each individual have preferences over consumption allocations:

\[
u(c^i) = \sum_{t=0}^{\infty} \beta^t \ln(c^i_t)
\]

where \(\beta \in (0, 1)\).

Agents have deterministic endowment streams \((e^1, e^2) = \{(e^1_t, e^2_t)\}_{t=0}^{\infty}\) of the consumption goods given by

\[
e^1_t = \begin{cases} 2 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases}, \quad e^2_t = \begin{cases} 0 & \text{if } t \text{ is even} \\ 2 & \text{if } t \text{ is odd} \end{cases}.
\]

There is no uncertainty in this economy and all information is publicly known.

At period 0, before endowments are received and consumption decisions are made, the two agents meet at a central market place and sign contracts to trade all commodities,
i.e. trade consumption for all future dates.

Let $p_t$ denote the price, determined in period 0, of one unit of consumption to be delivered in period $t$, in terms of a numeraire (for example, say, $p_0 = 1$).

Assume that both consumers take the sequence of prices $\{p_t\}_{t=0}^{\infty}$ as given when making their consumption decisions. All trades take place in period 0 and agents are committed in future periods to what they have agreed upon in period 0 (no bargaining or strategic behavior). There is perfect enforcement of these contracts signed in period 0.

1.1 Definition of Competitive Equilibrium

Given a sequence of prices $\{p_t\}_{t=0}^{\infty}$ consumers solve the following optimization problem

$$\max_{\{c_t^i\}_{t=0}^{\infty}} u(c^i) = \sum_{t=0}^{\infty} \beta^t \ln (c_t^i) \tag{1}$$

s.t. $\sum_{t=0}^{\infty} p_t c_t^i \leq \sum_{t=0}^{\infty} p_t e_t^i,
\quad c_t^i \geq 0, \forall t.$
Note that the since agents’ period utility function is strictly increasing, the budget constraint of both agents will hold with equality.

An Arrow-Debreu competitive equilibrium is prices $\{\hat{p}_t\}_{t=0}^{\infty}$ and allocations $\{\hat{c}_t^i\}_{t=0}^{\infty}$, such that

(1) Given prices $\{\hat{p}_t\}_{t=0}^{\infty}$, the allocations $\{\hat{c}_t^i\}_{t=0}^{\infty}, i = 1, 2$, solve (1). (Note that the prices in the budget constraint now are replaced by the equilibrium prices.)

(2) Market clearing condition

$$\hat{c}_t^1 + \hat{c}_t^2 = e_t^1 + e_t^2, \text{ for all } t.$$  

Note that we do not allow free disposal of goods, as the market clearing condition is stated as an equality.
1.2 Solving for the Equilibrium

For arbitrary prices $\{p_t\}_{t=0}^{\infty}$ the first order necessary conditions are

$$\frac{\beta^t}{c^i_t} = \lambda^i p_t, \forall t.$$  \hfill (2)

The Euler conditions are,

$$\frac{c^i_{t+1}}{c^i_t} = \frac{\beta p_t}{p_{t+1}}, \forall t, i = 1, 2.$$  \hfill (3)

Equations (3), together with the budget constraint, can be solved for the optimal sequence of consumption of consumer $i$ as a function of the infinite sequence of prices $c^i_t = c^i_t (\{p_t\}_{t=0}^{\infty})$. Together with the budget constraint, we can solve for the equilibrium prices

$$c^i_t = c^i_t (\{p_t\}_{t=0}^{\infty})$$

Combined with the goods market clearing condition, we can solve for the equilibrium
prices

\[ c_t^1 (\{ \hat{p}_t \}_{t=0}^\infty) + c_t^2 (\{ \hat{p}_t \}_{t=0}^\infty) = e_t^1 + e_t^2, \forall t. \]

This is a system of infinite equations with an infinite number of unknowns \( \{p_t\}_{t=0}^\infty \). In general, this is difficult to solve. We will discuss Negishi’s method later to solve for the general case.

For our simple economy considered here, we can solve for the equilibrium directly by summing (3) across agents and then using the goods market clearing condition, we have

\[ \frac{e_t^1 + e_t^2}{e_t^1 + e_t^2} = \frac{\beta p_t}{p_{t+1}}, \forall t. \]

That is,

\[ p_{t+1} = \beta p_t. \]

Thus, the equilibrium prices are given by

\[ p_t = \beta^t p_0, \forall t. \]
Set \( p_0 = 1 \), i.e. taking date 0 consumption good as the numeraire, the equilibrium prices are

\[
\hat{p}_t = \beta^t, \forall t.
\]

Since \( \beta < 1 \), the price for period \( t \) consumption is lower than the price for period 0 consumption. This reflects the impatience of both agents.

Given the equilibrium prices, (3) gives the allocations

\[
c_{t+1}^i = c_t^i = c_0^i, \forall t, i = 1, 2,
\]

i.e., consumption is perfectly smooth across time for both agents.

Starting at date 0, consumer 1 has an endowment of 2, by the budget constraint, his consumption allocations are

\[
\sum_{t=0}^{\infty} \hat{p}_t c_1^t = c_0^1 \sum_{t=0}^{\infty} \hat{p}_t = c_0^1 \frac{1}{1 - \beta}
\]

\[
= \sum_{t=0}^{\infty} \hat{p}_t e_1^t = 2 \sum_{t=0}^{\infty} \beta^{2t} = \frac{2}{1 - \beta^2}.
\]
Thus,

$$\hat{c}_t^1 = \hat{c}_0^1 = \frac{2}{1 + \beta} > 1.$$

For consumer 2,

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^2 = c_0^2 \sum_{t=0}^{\infty} \hat{p}_t = c_0^2 \frac{1}{1 - \beta}$$

$$= \sum_{t=0}^{\infty} \hat{p}_t c_t^2 = 2\beta \sum_{t=0}^{\infty} \beta^{2t} = \frac{2\beta}{1 - \beta^2}.$$

Thus,

$$\hat{c}_t^2 = \hat{c}_0^2 = \frac{2\beta}{1 + \beta} < 1.$$

Note that the first agent is rich first makes him consume more in all the periods.

There is substantial trade each period, and the trade is mutually beneficial. Without trade both agents just consume their endowments and receive lifetime utility $u(c_t^i) = -\infty$. 
Due to the strict concavity of the utility function (so that the objective is strictly concave and the constraint set is convex), the equilibrium is unique. We next show that the unique competitive equilibrium allocation is socially optimal.

2 Pareto Optimality and the First Welfare Theorem

We first define what socially optimal means. Our notion of optimality will be Pareto efficiency (or Pareto optimality). An allocation is Pareto efficient if it is feasible and if there is no other feasible allocation that makes no consumer worse off and at least one consumer strictly better off.

An allocation \( \{(c_t^1, c_t^2)\}_{t=0}^{\infty} \) is feasible if

1. \( c_t^i \geq 0 \) for all \( t \), for \( i = 1, 2 \);
2. \( c_t^1 + c_t^2 = e_t^1 + e_t^2 \), for all \( t \) (resource constraint).

An allocation \( \{(c_t^1, c_t^2)\}_{t=0}^{\infty} \) is Pareto efficient if it is feasible and there is no other feasible
allocation \( \{(\tilde{c}^1_t, \tilde{c}^2_t)\}_{t=0}^{\infty} \) such that

\[
\begin{align*}
    u(\tilde{c}^i_t) & \geq u(c^i_t) \quad \text{for all } i, \\
    u(\tilde{c}^i_t) & > u(c^i_t) \quad \text{for at least one } i = 1, 2.
\end{align*}
\]

Note that Pareto efficiency has nothing to do with fairness.

### 2.1 The First Welfare Theorem

**Proposition** Every competitive equilibrium allocation for the economy \( \{(\tilde{c}^i_t)_{t=0}^{\infty}\}_{i=1,2} \) is Pareto efficient (note: The only assumption required is that the consumers’ preferences are locally nonsatiated so that each consumer exhausts her budget).

Proof: Suppose not. Then by the definition of Pareto efficiency there exists another feasible allocation \( \{(\tilde{c}^i_t)_{t=0}^{\infty}\}_{i=1,2} \) such that

\[
\begin{align*}
    u(\tilde{c}^i_t) & \geq u(c^i_t) \quad \text{for all } i, \quad (4) \\
    u(\tilde{c}^i_t) & > u(c^i_t) \quad \text{for at least one } i = 1, 2. \quad (5)
\end{align*}
\]

Without loss of generality we assume that the strict inequality holds for \( i = 1 \).
Then we will show that (1) $\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 > \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1$; (2) $\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 \geq \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2$; (3) and $(\tilde{c}_t^2)_{t=0}^{\infty}$ violates the feasibility constraint.

(1) Suppose not, i.e., $\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 \leq \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1$, then for agent 1 the allocation $(\{\tilde{c}_t^1\}_{t=0}^{\infty})$ is strictly better off for him (by(5)), and not more expensive (affordable). but this violates that fact the allocation $(\{\tilde{c}_t^1\}_{t=0}^{\infty})$ is a competitive equilibrium (i.e. maximizes agent 1’s utility given equilibrium prices).

(2) Suppose not, i.e., $\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 < \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2$. Then there exists $\delta > 0$ such that $\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 + p_0 \delta \leq \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2$. We can find another allocation for consumer 2 $(\{\tilde{c}_t^2\}_{t=0}^{\infty})$: 

$$\tilde{c}_0^2 = \tilde{c}_0^2 + \delta, \text{ for } t = 0,$$

$$\tilde{c}_t^2 = \tilde{c}_t^2, \text{ for } t \geq 1,$$

so that

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 = \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 + p_0 \delta \leq \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2,$$

and

$$u(\tilde{c}_t^2) > u(\tilde{c}_t^2) \geq u(\tilde{c}_t^2),$$
where the former inequality is due to (6) and the latter by (4). Thus, this new allocation is affordable and can achieve higher utility for consumer 2. This contradicts the fact that the allocation \( \{\tilde{c}_t^2\}_{t=0}^{\infty} \) is a competitive equilibrium.

(3) Summing \( \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 > \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 \) and \( \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 \geq \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 \), we have

\[
\sum_{t=0}^{\infty} \hat{p}_t (\tilde{c}_t^1 + \tilde{c}_t^2) > \sum_{t=0}^{\infty} \hat{p}_t (\tilde{c}_t^1 + \tilde{c}_t^2) .
\]

But since both allocations \( \{\tilde{c}_t^1\}_{t=0}^{\infty} \) and \( \{\tilde{c}_t^2\}_{t=0}^{\infty} \) are feasible (the former by assumption, and the latter is a competitive equilibrium allocation), we must have

\[
\tilde{c}_t^1 + \tilde{c}_t^2 = e_t^1 + e_t^2 = \tilde{c}_t^1 + \tilde{c}_t^2 , \forall t,
\]

which leads the inequality (8) to be a contradiction. \textit{QED}

3 Negishi's (1960) Method to Compute Equilibria

We now describe a method to compute the competitive equilibrium for more general models. But still we confine ourselves to those economies in which the welfare theorem hold.
If the first welfare theorem holds, then competitive equilibrium allocations are Pareto optimal; by solving for all Pareto optimal allocations we have then solved for all potential equilibrium allocations.

Negishi’s method provides an algorithm to compute all Pareto optimal allocations and to isolate those who are in fact competitive equilibrium allocations.

Thus, we can solve a simple social planner’s problem (which does not involve any prices) and use the welfare theorem to argue that we have solved for the allocations of competitive equilibria, and then find equilibrium prices (from the Lagrange multipliers) that support these allocations.

### 3.1 The Social Planner’s Problem

Consider the following social planner’s problem with a Pareto weight $\alpha \in [0, 1]$,

$$\max_{\{(c^1_t, c^2_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[ \alpha \ln (c^1_t) + (1 - \alpha) \ln (c^2_t) \right]$$

subject to

$$c^1_t + c^2_t = e^1_t + e^2_t = 2, \forall t,$$

$$c^i_t \geq 0, \forall t, \forall i.$$
The social planner maximizes the weighted average of utilities of all the agents, subject to the feasibility constraint.

**Proposition** An allocation $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ is Pareto efficient if and only if it solves the social planner’s problem (9) for some $\alpha \in [0, 1]$.

Note that the optimal consumption allocation depends on the choice of the Pareto weights. The first order conditions, with Lagrange multipliers for resource constraints $\mu_t/2$, are (Note that the non-negativity constraints on $c_t^i$ never bind, because the period utility function satisfies the Inada conditions, $\lim_{c \to 0} U'(c) = \infty$ and $\lim_{c \to \infty} U'(c) = 0.$):

$$\frac{\alpha \beta^t}{c_t^1} = \frac{\mu_t}{2}, \quad \frac{(1 - \alpha) \beta^t}{c_t^2} = \frac{\mu_t}{2}. \quad (10)$$

Then, $\frac{c_t^1}{c_t^2} = \frac{\alpha}{1 - \alpha}$, i.e., the ratio of consumption between the two agents equals the ratio of the Pareto weights.
Using the resource constraint, we have
\[ c_t^1 = 2(1 - \alpha), \quad c_t^2 = 2\alpha, \]
for some \( \alpha \in [0, 1] \), i.e. the social planner divides the total resources in every period according to the Pareto weights. The allocations are constant over time and are independent of the agents’ endowments in a particular period.

The Lagrange multipliers are
\[ \mu_t = \beta^t. \]

### 3.2 The Correspondence between Competitive Equilibria and Pareto Optimal Allocations

How are the Pareto optimal allocations correspond to the allocations and prices of competitive equilibria?

Comparing the FOCs of competitive equilibrium (2) and of Social Planner’s problem
(10),

\[
\frac{\beta^t}{c^1_t} = \frac{\beta^t}{c^2_t} = \lambda_1 p_t, \quad \frac{\beta^t}{c^1_t} = \frac{\beta^t}{c^2_t} = \lambda_2 p_t,
\]

\[
\frac{\beta^t}{c^1_t} = \frac{\mu_t}{2\alpha}, \quad \frac{\beta^t}{c^2_t} = \frac{\mu_t}{2(1 - \alpha)}.
\]

By picking

\[
\lambda_1 = \frac{1}{2\alpha}, \quad \lambda_2 = \frac{1}{2(1 - \alpha)}, \quad p_t = \mu_t = \beta^t,
\]

then the FOCs are identical.

Thus, by appropriate choices of the individual Lagrange multipliers \( \lambda_i \) and prices \( p_t \), the optimality conditions for the social planner’s problem and for the consumer maximization problems coincide.

Given that the competitive equilibrium is unique but there are a lot of Pareto efficient allocations (for each \( \alpha \)), there must be some additional requirement that a competitive equilibrium imposes which the planners problem does not require.
Note that the resource feasibility in the planner’s problem is the goods market clearing condition in the competitive equilibrium, but the individual budget constraint is not required for the Pareto efficient allocations.

We next ask which Pareto efficient allocation(s) would be affordable for all consumers (satisfy individual budget constraints).

### 3.3 The Transfer Functions

We define transfer functions $t^i(\alpha)$ which is the amount of the numeraire good (in terms of date 0 consumption good) that agent $i$ need to receive or give away in order to be able to afford the Pareto efficient allocation for a given $\alpha$:

$$t^i(\alpha) = \sum_{t=0}^{\infty} \mu_t \left[ c^i_t(\alpha) - e^i_t \right], \; i = 1, 2,$$

where prices are Lagrange multipliers on the resource constraints in the planner’s prob-
lem. Thus,

\[
\begin{align*}
t^1(\alpha) &= \sum_{t=0}^{\infty} \mu_t c^1_t(\alpha) - \sum_{t=0}^{\infty} \mu_t c^2_t = 2\alpha - \frac{2}{1 - \beta - \beta^2}, \\
t^2(\alpha) &= \frac{2(1 - \alpha)}{1 - \beta} - \frac{2\beta}{1 - \beta^2}.
\end{align*}
\]

To find the competitive equilibrium allocation, we must find the Pareto weight \( \alpha \) such that \( t^1(\alpha) = t^1(\alpha) = 0 \), so that the Pareto efficient allocations satisfy the budget constraints \( \sum_{i=0}^{\infty} p_t c^i_t = \sum_{i=0}^{\infty} p_t e^i_t \). We have

\[
\hat{\alpha} = \frac{1}{1 + \beta}.
\]

Given this particular \( \alpha \), the Pareto efficient allocations are

\[
c^1_t(\alpha) = \frac{2}{1 + \beta}, \quad c^2_t(\alpha) = \frac{2\beta}{1 + \beta},
\]

which is equivalent to the competitive equilibrium allocation.
The Negishi method reduces the computation of equilibrium to a finite number of equations in a finite number of unknowns in finding the Pareto weight $\alpha$.

4 Sequential Markets Equilibrium

In this section we show that the Arrow-Debreu (A-D) equilibrium would arise in the Sequential Markets (SM) equilibrium in which we allow agents to trade one-period assets in each period.

Let $r_{t+1}$ denote the (net) interest rate on one period bond from period $t$ to period $t+1$. A one period bond is a promise (contract) to pay 1 unit of the consumption good in period $t+1$ in exchange for $1/(1 + r_{t+1})$ units of the consumption good in period $t$.

Thus, $q_t \equiv 1/(1 + r_{t+1})$ represents the relative price of a bond traded in period $t$ that pays one unit of the consumption good in period $t+1$, and can also be interpreted as the relative price of one unit of the consumption good in period $t+1$ in terms of the period $t$ consumption good.

Let $a^i_{t+1}$ denote the amount of such bonds purchased or sold by agent $i$ in period $t$. 

If $a_{t+1}^i < 0$, then the agent is a borrower.

Given a sequence of prices $\{r_{t+1}\}_{t=0}^\infty$ consumers solve the following optimization problem

$$\max_{\{c_t^i,a_{t+1}^i\}_{t=0}^\infty} u(c^i) = \sum_{t=0}^\infty \beta^t \ln(c_t^i)$$

subject to

$$c_t^i + \frac{a_{t+1}^i}{1 + r_{t+1}} \leq e_t^i + a_t^i,$$

$$c_t^i \geq 0, \forall t,$$

$$a_{t+1}^i \geq -\overline{A}^i.$$

The FOCs are

$$\frac{\beta^t}{c_t^i} = \pi_t^i, \quad \pi_t^i = \frac{\pi_t^i}{(1 + r_{t+1})} = \cdots = \frac{\pi_0^i}{\prod_{s=0}^t (1 + r_{s+1})},$$

where $\pi_t^i$ is the Lagrangian multiplier for the budget constraint. The Euler conditions

...
are

\[ \frac{c_{i+1}^t}{c_i^t} = \frac{\beta}{1 + r_{t+1}}, \forall t, i = 1, 2. \]

A **Sequential Markets equilibrium** is prices \( \{ \hat{r}_{t+1} \}_{t=0}^{\infty} \) and allocations \( \{ \{ \hat{c}_t^i, \hat{a}_{t+1}^i \}_{t=0}^{\infty} \}_{i=1,2} \), such that

(1) Given prices \( \{ \hat{r}_{t+1} \}_{t=0}^{\infty} \), the allocations \( \{ \{ \hat{c}_t^i, \hat{a}_{t+1}^i \}_{t=0}^{\infty} \}_{i=1,2} \), solve (11).

(2) Market clearing conditions

\[
\begin{align*}
\sum_{i=1}^{2} \hat{c}_t^i &= \sum_{i=1}^{2} e_t^i, \\
\sum_{i=1}^{2} \hat{a}_{t+1}^i &= 0, \text{ for all } t.
\end{align*}
\]

The constraint \( a_{t+1}^i \geq -\bar{A}^i \) is to rule out the Ponzi scheme, which says that the agent never repays his debt (or, equivalently, postpones repayment indefinitely). Without a limit on borrowing, the agent can always be better off by borrowing an extra small amount at date 0, consume, and then roll over the debt forever.
Note that the no-Ponzi scheme is usually added to part of the budget constraints of a consumer, which imposes a restriction on the choices of the agent.

Even though we specify a borrowing limit to rule out Ponzi schemes, we allow $\overline{A}^i$ to be large enough so that agents will never be credit constrained, i.e.,

$$a^i_{t+1} > -\overline{A}^i.$$

### 4.1 Sequential Markets equilibrium and the Arrow-Debreu Equilibrium

Now we consider the correspondence of the equilibrium between (1) and (11).

**Proposition** Let allocations prices $\{\hat{P}_t\}_{t=0}^\infty$ and allocations $\{\{\hat{c}^i_t\}_{t=0}^\infty\}_{i=1,2}$ be an Arrow-Debreu equilibrium. Then there exists a corresponding sequential markets equilibrium with prices $\{\hat{r}_{t+1}\}_{t=0}^\infty$ and the allocations $\{\{\tilde{c}^i_t, \tilde{a}^i_{t+1}\}_{t=0}^\infty\}_{i=1,2}$, together with $\left(\overline{A}^i\right)_{i=1,2}$, such that $\tilde{c}^i_t = \hat{c}^i_t$, $\forall t, \forall i$.

Conversely, given a sequential markets equilibrium with prices $\{\hat{r}_{t+1}\}_{t=0}^\infty$ and the alloca-
junctions \( \{\tilde{c}_t^i, \tilde{a}_t^i\}_{t=0}^\infty \) \( i=1,2 \), together with \( \tilde{a}_{t+1}^i \geq -\tilde{A}_t^i, \forall t, \forall i \), there exists a corresponding Arrow-Debreu equilibrium \( \{\tilde{p}_t\}_{t=0}^\infty \) and \( \{\tilde{c}_t^i\}_{t=0}^\infty \) \( i=1,2 \) such that \( \tilde{c}_t^i = \tilde{c}_t^i, \forall t, \forall i \).

Proof:

(I) S-M \( \rightarrow \) A-D

Begin with the sequence of sequential markets equilibrium with prices \( \{\tilde{r}_{t+1}\}_{t=0}^\infty \) and the allocations \( \{\tilde{c}_t^i, \tilde{a}_t^i\}_{t=0}^\infty \) \( i=1,2 \), together with \( \tilde{a}_{t+1}^i \geq -\tilde{A}_t^i, \forall t, \forall i \). We want to show that there is a corresponding A-D equilibrium \( \{\tilde{p}_t\}_{t=0}^\infty \) and \( \{\tilde{c}_t^i\}_{t=0}^\infty \) \( i=1,2 \).

Note that the A-D allocation \( \{\tilde{c}_t^i\}_{t=0}^\infty \) \( i=1,2 \) will satisfy the market clearing condition in S-M equilibrium (because it satisfies the feasibility constraint in A-D equilibrium).

Define \( \tilde{p}_0 = 1 \), and \( \tilde{p}_t = \tilde{p}_{t-1}/(1 + \tilde{r}_t) \), for \( t \geq 1 \), then

\[
\tilde{p}_t = \frac{1}{\prod_{s=1}^t (1 + \tilde{r}_s)}, \quad t = 1, 2, \ldots
\]  

(13)

Now set

\[
\lambda_i = \pi_0^i,
\]
which turns out that by (13) and \( \pi^i_{t+1} = \frac{\pi^i_0}{\Pi_{s=0}^t (1+r_{s+1})} \) in (12), we have

\[
\lambda_i = \pi^i_0 = \frac{\pi^i_t}{p_t}, \quad t = 1, 2, \ldots
\]

Then we can see that the allocation will satisfy the FOCs of the A-D equilibrium in (2).

Then we need to show their budget constraints are the same. Let us begin with the sequence of sequential markets budget constraints (in equalities), \( c^i_t + \frac{a^i_t}{1+r_{t+1}} \leq e^i_t + a^i_t \), \( t = 0, 1, 2 \ldots \) Assume \( a^i_0 = 0 \). We repeatedly substituting for \( a^i_{t+1} \), \( t = 0, 1, 2 \ldots \). We have

\[
\sum_{t=0}^T \frac{c^i_t}{\Pi_{j=1}^t (1 + \hat{r}_j)} + \frac{a^i_{T+1}}{\Pi_{j=1}^{T+1} (1 + \hat{r}_j)} = \sum_{t=0}^T \frac{e^i_t}{\Pi_{j=1}^t (1 + \hat{r}_j)}
\]

Taking limits w.r.t. \( t \) on both sides and using (13),

\[
\sum_{t=0}^\infty \hat{p}_t c^i_t + \lim_{T \to \infty} \frac{a^i_{T+1}}{\Pi_{j=1}^{T+1} (1 + \hat{r}_j)} = \sum_{t=0}^\infty \hat{p}_t e^i_t.
\]
By assumption,

\[
\lim_{T \to \infty} \frac{a^i_{T+1}}{\prod_{j=1}^{T+1} (1 + \tilde{r}_j)} \geq \lim_{T \to \infty} \frac{-\overline{A}^i}{\prod_{j=1}^{T+1} (1 + \tilde{r}_j)} = 0. \tag{14}
\]

Thus, the no-Ponzi scheme condition \(a^i_{t+1} \geq -\overline{A}^i\) implies (14), which says that in present-value terms, the agent cannot have a negative asset holdings (being a net debtor), otherwise he would borrow and keep deferring repayment indefinitely. Thus, we obtain the A-D budget constraints, \(\sum_{t=0}^{\infty} \hat{p}_t c^i_t \leq \sum_{t=0}^{\infty} \hat{p}_t e^i_t\).

(II) A-D \to SM

Begin with the A-D equilibrium \(\{\hat{p}_t\}_{t=0}^{\infty}\) and allocations \(\{\hat{c}^i_t\}_{t=0}^{\infty}\) \(i=1,2\). We want to show that there exist a S-M equilibrium with same consumption allocation, i.e., \(\{\hat{c}^i_t\}_{t=0}^{\infty}\) \(i=1,2\) = \(\{\hat{c}^i_t\}_{t=0}^{\infty}\) \(i=1,2\).

Firstly the S-M allocation \(\{\hat{c}^i_t\}_{t=0}^{\infty}\) \(i=1,2\) will satisfy the feasibility constraint in A-D equilibrium (because it satisfies the market clearing condition in S-M equilibrium).
We then define the asset demand

\[ \tilde{a}_{t+1}^i = \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau} (\hat{c}_{t+\tau}^i - e_{t+\tau})}{\hat{p}_{t+1}}, \]  

and the interest rates (let \( \hat{p}_0 = 1 \))

\[ \hat{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}} - 1. \]

By the definition (15), we can see that the allocation satisfies the SM budget constraints.

Now set

\[ \pi_t^i = \lambda_i \hat{p}_t. \]

Then we can see that \( \{\hat{p}_t\}_{t=0}^\infty \) and \( \{\hat{c}_t^i\}_{t=0}^\infty \) satisfy the FOCs of the A-D equilibrium in (2).

Note that in the SM equilibrium we have the additional borrowing constraints, \( \hat{a}_{t+1}^i \geq -\bar{A}_t^i \). By (15),
\[
\tilde{a}^i_{t+1} > - \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}e^i_{t+\tau}}{\hat{p}_{t+1}} = - \frac{1}{\hat{p}_{t+1}} \sum_{\tau=1}^{\infty} \hat{p}_{t+\tau}e^i_{t+\tau} \geq - \sum_{t=0}^{\infty} \hat{p}_t e^i_t > -\infty,
\]
where we have used \(\hat{p}_{t+1} < 1\) (because \(\hat{p}_0 = 1\)). Thus, we take

\[
\overline{A}^i = \sum_{t=0}^{\infty} \hat{p}_t e^i_t.
\]

Thus, these additional constraints are never binding.

(It remains to argue that the SM allocation \((\{\tilde{c}^i_t\}_{t=0}^{\infty})_{i=1,2}\) maximizes utility, subject to the sequential markets budget constraints and the borrowing constraints. We have showed that this allocation satisfies the A-D budget constraint. Suppose there is any other allocation that satisfy these constraints and can attain higher utility for the consumer than \((\{\tilde{c}^i_t = \tilde{c}^i_t\}_{t=0}^{\infty})_{i=1,2}\), this allocation would have been chosen to be an A-D equilibrium. But it was not.) QED