Exercise: Elasticity of Substitution between Capital and Labor

The elasticity of substitution between capital and labor is defined as

$$\frac{d\ln(k/n)}{d\ln(MPL/MPK)}.$$

As we discussed in the lecture, this quantity measures the extent to which firms can substitute capital for labor as the relative productivity or the relative cost of the two factors changes. When this number is large, it means that firms can easily substitute between capital and labor. Geometrically, it measures the curvature of the isoquant. In general, the elasticity of substitution depends on the amount of capital and labor employed. But, for the CES production function, the elasticity turns out to be a constant, which is convenient in many applications.

To see this, consider the CES production function:

$$y = A\left[\alpha k^{1-\frac{1}{\phi}} + (1-\alpha)n^{1-\frac{1}{\phi}}\right]^{\frac{\phi}{\phi-1}},\quad 0<\alpha<1,\quad \phi>0,\quad A>0.$$

We have

$$MPK = \alpha A \Delta^{\frac{1}{1-\phi}} k^{-\frac{1}{\phi}} \quad \text{and} \quad MPL = (1-\alpha) A \Delta^{\frac{1}{1-\phi}} n^{-\frac{1}{\phi}},$$

where Δ denotes the expression in the bracket of the CES production function. Taking ratio to get

$$\frac{MPL}{MPK} = \frac{1-\alpha}{\alpha}\left(\frac{k}{n}\right)^{\frac{1}{\phi}}.$$

Taking log on both sides and rearranging, we have

$$\ln(k/n) = \text{constant} + \phi \ln(MPL/MPK).$$

It is easy to see that

$$\frac{d\ln(k/n)}{d\ln(MPL/MPK)} = \phi.$$