Part IV: Production and Supply

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Chapter 9
Production Functions

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Marginal Productivity

- **Production function.** The firm's production function for a particular good (q), $q = f(k, l)$, shows the maximum amount of the good that can be produced using alternative combinations of capital (k) and labor (l).

Marginal physical product

- **Marginal physical product.** The *marginal physical product* of an input is the additional output that can be produced by using one more unit of that input while holding other inputs constant.

$$MP_k = \frac{\partial q}{\partial k} = f_k$$

$$MP_l = \frac{\partial q}{\partial l} = f_l$$

Diminishing marginal productivity

- The marginal physical product depends on how much of that input is used.
- Diminishing marginal productivity is about the second-order derivatives

$$\frac{\partial MP_k}{\partial k} = \frac{\partial^2 f}{\partial k^2} = f_{kk} < 0, \text{ for high enough } k,$$
$$\frac{\partial MP_l}{\partial l} = \frac{\partial^2 f}{\partial l^2} = f_{ll} < 0, \text{ for high enough } l.$$

- The assumption of diminishing marginal productivity was originally proposed by 19th-century economist Thomas Malthus, who worried that rapid increases in population would result in lower labor productivity.
- Malthus' gloomy predictions led economics to be called the "dismal science."

- Changes in the marginal productivity of labor also depend on changes in other inputs such as capital.
- We need to consider f_{lk},

$$f_{lk} = \frac{\partial MP_l}{\partial k}$$

which is often the case $f_{lk} > 0$.

- It appears that labor productivity has risen significantly since Malthus' time, primarily because increases in capital inputs (along with technical improvements) have offset the impact of decreasing marginal productivity alone.

Average productivity

- In common usage, the term *labor productivity* often means *average productivity*.
- We define the average product of labor (AP_l) to be

$$AP_l = \frac{\text{output}}{\text{labor input}} = \frac{q}{l} = \frac{f(k,l)}{l}$$

Notice that AP_l also depends on the level of capital used.

Example 9.1 A Two-Input Production Function

- Suppose the production function for flyswatters during a particular period can be represented by

$$q = f(k, l) = 600k^2l^2 - k^3l^3.$$

To construct MP_l and AP_l, we must assume a value for k. Let $k = 10$, then the production function becomes

$$q = 60,000l^2 - 1,000l^3$$

- The **marginal productivity** function is

$$MP_l = \frac{\partial q}{\partial l} = 120,000l - 3,000l^2,$$

which diminishes as l increases. q has a maximum value when $MP_l = 0$,

$$120,000l - 3000l^2 = 0, \quad l = 40$$

- To find **average productivity**, we hold $k = 10$ and solve

$$AP_l = \frac{q}{l} = 60,000l - 1000l^2$$

AP_l reaches its maximum where

$$\begin{aligned} \frac{\partial AP_l}{\partial l} &= 60,000 - 2000l = 0 \\ l &= 30 \end{aligned}$$

When $l = 30$, $AP_l = MP_l = 900,000$. When AP_l is at its maximum, AP_l and MP_l are equal.

- This result is general. Because

$$\frac{\partial AP_l}{\partial l} = \frac{\partial\left(\frac{q}{l}\right)}{\partial l} = \frac{l \cdot MP_l - q \cdot 1}{l^2},$$

at a maximum $l \cdot MP_l = q$ or $MP_l = AL_l$.

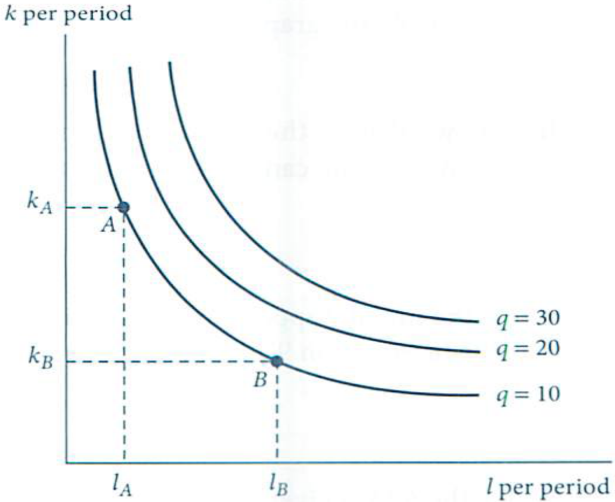
Isoquant Maps and the Rate of Technical Substitution

- To illustrate the possible substitution of one input for another in a production function, we use its *isoquant map*.
- **Isoquant,** An *isoquant* shows those combinations of k and l that can produce a given quantity of output (say, q_0). Mathematically, an isoquant records the set of k and l that satisfies

$$f(k,l) = q_0$$

- There are infinitely many isoquants in the $k - l$ plane. Each isoquant represents a different level of output.

Figure 9.1 An Isoquant Map



Marginal rate of technical substitution (RTS)

- **Marginal rate of technical substitution.** The *marginal rate of technical substitution (RTS)* shows the rate at which, having added a unit of labor, capital be decreased while holding output constant along an isoquant.

$$RTS(l \text{ for } k) = -\frac{dk}{dl}\bigg|_{q=q_0}$$

RTS and marginal productivities

- The equation for an isoquant is

$$q_0 = f(k(l), l),$$

Total differentiate the equation with q_0 constant gives

$$0 = f_k \frac{dk}{dl} + f_l = MP_k \frac{dk}{dl} + MP_l.$$

Therefore,

$$RTS(l \text{ for } k) = -\left.\frac{dk}{dl}\right|_{q=q_0} = \frac{MP_l}{MP_k}$$

RTS is given by the ratio of the inputs' marginal productivity, and because MP_l and MP_k are both nonnegative, RTS will be positive.

Reasons for a diminishing RTS

- It is not possible to derive a diminishing RTS from the assumption of diminishing marginal productivity alone.
- To show that isoquants are convex, we would like to show that $\frac{dRTS}{dl} < 0$. Because $RTS = \frac{f_l}{f_k}$, we have

$$\frac{dRTS}{dl} = \frac{d(f_l/f_k)}{dl}$$

- Using the fact that $\frac{dk}{dl} = -\frac{f_l}{f_k}$ along an isoquant and Young's theorem ($f_{kl} = f_{lk}$), we have

$$\begin{aligned}\frac{dRTS}{dl} &= \frac{f_k(f_{ll} + f_{lk} \cdot dk/dl) - f_l(f_{kl} + f_{kk} \cdot dk/dl)}{(f_k)^2} \\ &= \frac{f_k^2 f_{ll} - 2 f_k f_l f_{kl} + f_l^2 f_{kk}}{(f_k)^3} < 0 \text{ if } f_{kl} > 0.\end{aligned}$$

$$\frac{dRTS}{dl} = \frac{f_k^2 f_{ll} - 2f_k f_l f_{kl} + f_l^2 f_{kk}}{(f_k)^3}$$

- The denominator is positive because we have assumed $f_k > 0$.
- The ratio will be negative if f_{kl} is positive because f_{ll} and f_{kk} are both assumed to be negative.
- Intuitively, it seems reasonable that $f_{kl} = f_{lk}$ should be positive. If workers have more capital, they will be more productive.
- But some production functions have $f_{kl} < 0$ over some input ranges.
- Assuming diminishing RTS means that we are assuming that MP_l and MP_k diminish rapidly enough to compensate for any possible negative cross-productivity effects.

Example 9.2 A Diminishing RTS

- Production function in Example 9.1 is

$$q = f(k, l) = 600k^2l^2 - k^3l^3.$$

- Marginal productivity functions are

$$\begin{aligned} MP_l &= f_l = \frac{\partial q}{\partial l} = 1200k^2l - 3k^3l^2 \\ MP_k &= f_k = \frac{\partial q}{\partial k} = 1200kl^2 - 3k^2l^3 \end{aligned}$$

  Both will be positive for values of k and l for which $kl < 400$.

- Because

$$\begin{aligned} f_{ll} &= 1200k^2 - 6k^3l \\ f_{kk} &= 1200l^2 - 6kl^3, \end{aligned}$$

  Thus function exhibits diminishing marginal productivities for sufficiently large values of k and l. $f_{ll}, f_{kk} < 0$ if $kl > 200$.

- However, even within the range $200 < kl < 400$ where the marginal productivity behave “normally," this production function may not necessarily have a diminishing RTS.
- Cross differentiation of either of the marginal productivity functions yields

$$f_{kl} = f_{lk} = 2400kl - 9k^2l^2$$

which is positive only for $kl < 2400/9 \doteq 266$

Returns to Scale

- How does output respond to increases in all inputs together? Suppose that all inputs are doubled, would output double?
- This is a question of the *returns to scale* exhibited by the production function that has been of interest to economists ever since Adam Smith intensively studies the production of pins.
- A doubling of scale permits a greater division of labor and specialization of function.
- Doubling of the inputs also entails some loss in efficiency because managerial overseeing may become more difficult.

- **Returns to scale.** If the production function is given by $q = f(k, l)$ and if all inputs are multiplied by the same positive constant t (where $t > 1$), then we classify the *returns to scale* of the production function by

Effect on Output	Returns to Scale
$f(tk, tl) = tf(k, l) = tq$	Constant
$f(tk, tl) < tf(k, l) = tq$	Decreasing
$f(tk, tl) = tf(k, l) = tq$	Increasing

- It is possible for a function to exhibit constant returns to scale for some levels of input usage and increasing or decreasing returns for other levels.
- The degree of returns to scale is generally defined within a fairly narrow range of variation in input usage.

Constant returns to scale

- Constant returns-to-scale production functions are homogeneous of degree one in inputs because

$$f(tk, tl) = t^1 f(k, l) = tq$$

- If a function is homogeneous of degree k, its derivatives are homogeneous of degree $k - 1$.
- The marginal productivity functions derived from a constant returns to scale production are homogeneous of degree zero.

- That is,

$$
\begin{aligned}
MP_k &= \frac{\partial f(k,l)}{\partial k} = \frac{\partial f(tk,tl)}{\partial k}, \\
MP_l &= \frac{\partial f(k,l)}{\partial l} = \frac{\partial f(tk,tl)}{\partial l},
\end{aligned}
$$

for any $t > 1$. Let $t = 1/l$, then

$$
\begin{aligned}
MP_k &= \frac{\partial f(k/l,1)}{\partial k}, \\
MP_l &= \frac{\partial f(k/l,1)}{\partial l},
\end{aligned}
$$

- The marginal productivity of any input depends on the ratio of capital and labor, not on the absolute levels of these inputs.

Homothetic production functions

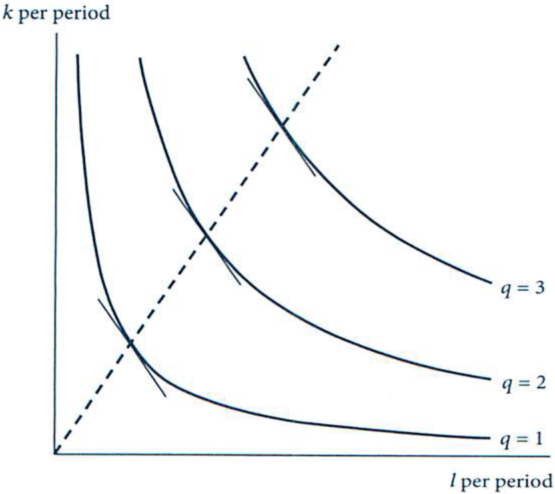
- The $RTS(= MP_l/MP_k)$ for any constant-returns-to scale production will depend only on the ratio of k and l since

$$RTS = \frac{MP_l}{MP_k} = \frac{\frac{\partial f(k/l,1)}{\partial l}}{\frac{\partial f(k/l,1)}{\partial k}},$$

not on the absolute level of k and l.

- This is a homothetic function, and its isoquants will be radial expansion of one another.
- However, a production function can have a homothetic isoquant map even if it does not exhibit constant returns to scale.
- As shown in Chapter 2, this property of homotheticity is retained by any monotonic transformation of a homogeneous function.

Figure 9.2 Isoquant Map for a Constant Returns-to-Scale Production Function



- For example, if $f(k,l)$ is a constant returns-to-scale production function, let
$$F(k,l) = f(k,l)^{\gamma},$$
where γ is any positive exponent. If $\gamma > 1$ then
$$F(tk,tl) = f(tk,tl)^{\gamma} = t^{\gamma} f(k,l)^{\gamma} = t^{\gamma} F(k,l) > tF(k,l)$$
for any $t > 1$. F exhibits increasing returns to scale and γ captures the *degree* of the increasing returns to scale.
- An identical proof can show that the function F exhibits decreasing returns to scale for $\gamma < 1$.
- In these cases, changes in the returns to scale will just change the labels on the isoquants rather than their shapes.

The n-input case

- The definition of returns to scale can be generalized to a production function with n inputs,

$$q = f(x_1, x_2, \cdot, x_n)$$

If all inputs are multiplied by $t > 1$, we have

$$f(tx_1, tx_2, \cdot, tx_n) = t^k f(x_1, x_2, \cdot, x_n) = t^k q$$

- If $k = 1$, the production function exhibits constant returns to scale.
- Decreasing and increasing returns to scale correspond to the cases $k < 1$ and $k > 1$, respectively.

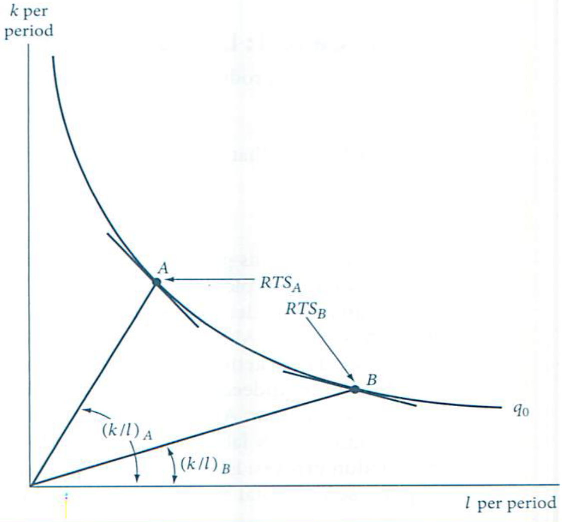
The Elasticity of Substitution

- An important characteristic of the production function is how "easy" it is to substitute one input for another.
- This is a question about the shape of a single isoquant rather than about the whole isoquant map.
- The *elasticity of substitution* measures the proportionate change in k/l relative to the proportionate change in the RTS along an isoquant.
- For the production function $q = f(k, l)$,

$$\sigma = \frac{\%\Delta(k/l)}{\%\Delta RTS} = \frac{d(k/l)}{dRTS} \times \frac{RTS}{k/l} = \frac{d\ln(k/l)}{d\ln RTS} = \frac{d\ln(k/l)}{d\ln(f_l/f_k)}$$

- The value of σ will always be positive because k/l and RTS move in the same direction.

Figure 9.3 Graphic Description of the Elasticity of Substitution

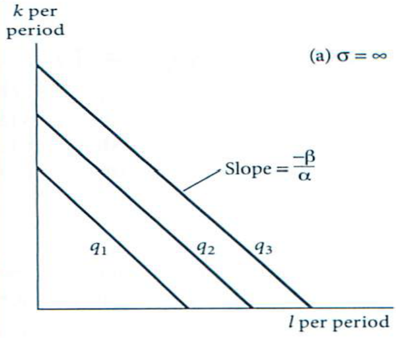


- If σ is high, the RTS will not change much relative to k/l, the isoquant will be close to linear.
- If σ is low, the RTS will change by a substantial amount as k/l changes, the isoquant will be sharply curved.
- In general, it is possible that σ will vary as one moves along an isoquant and as the scale of production changes.
- Often, it is convenient to assume that σ is constant along an isoquant.

Four Simple Production Functions

Case 1: Linear ($\sigma = \infty$)

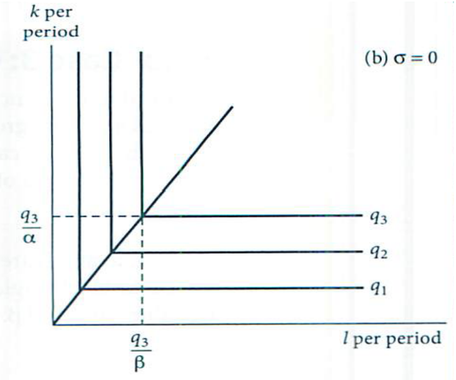
$$q = f(k, l) = \alpha k + \beta l$$



- All isoquants are straight lines with slope $-\beta/\alpha$.
- Constant returns to scale.
- RTS is constant.

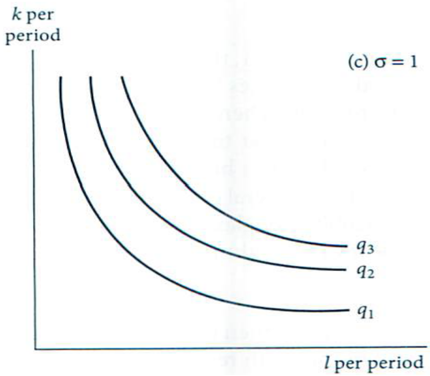
Case 2: Fixed proportions ($\sigma = 0$)

$$q = \min(\alpha k, \beta l), \alpha, \beta > 0$$



- Capital and labor must always be used in a fixed ratio.
- The firm will always operate along a ray where k/l is constant.
- Because k/l is constant, $\sigma = 0$.

Case 3: Cobb-Douglas ($\sigma = 1$)



$$q = f(k, l) = Ak^{\alpha} l^{\beta}$$

$$f(tk, tl) = A(tk)^{\alpha}(tl)^{\beta} = t^{\alpha+\beta} f(k, l)$$

- $$\begin{aligned} RTS = \frac{f_l}{f_k} &= = \frac{\beta A k^{\alpha} l^{\beta-1}}{\alpha A k^{\alpha-1} l^{\beta}} = \frac{\beta}{\alpha} \cdot \frac{k}{l} \\ \ln(RTS) &= \ln\left(\frac{\beta}{\alpha}\right) + \ln\left(\frac{k}{l}\right) \\ \sigma &\equiv \frac{\partial \ln(k/l)}{\partial \ln(RTS)} = 1 \end{aligned}$$

- This production function can exhibit any returns to scale, depending on whether $\alpha + \beta \gtrless 1$.
- The Cobb-Douglas production function is useful in many applications because it is linear in logarithms:

$$\ln q = \ln A + \alpha \ln k + \beta \ln l$$

α is the elasticity of output with respect to k while β is the elasticity of output with respect to l.

Case 4: CES production function

- The constant elasticity of substitution (CES) production was first introduced by Arrow et al. in 1961. It is given by

$$q = f(k, l) = (k^\rho + l^\rho)^{\gamma/\rho}$$

for $\rho \leq 1, \rho \neq 0$ and $\gamma > 0$. Since

$$f(tk, tl) = [(tk)^\rho + (tl)^\rho]^{\gamma/\rho} = t^\gamma f(k, l)$$

For $\gamma > 1$ the function exhibits increasing returns to scale, whereas for $\gamma < 1$ it exhibits decreasing returns.

- From the production function and marginal product of k and l,

$$\begin{aligned} q &= f(k,l) = (k^\rho + l^\rho)^{\gamma/\rho} \\ MP_l &= \frac{\partial f}{\partial l} = (k^\rho + l^\rho)^{\gamma/\rho - 1} \cdot \rho l^{\rho-1} \\ MP_k &= \frac{\partial f}{\partial l} = (k^\rho + l^\rho)^{\gamma/\rho - 1} \cdot \rho k^{\rho-1}, \end{aligned}$$

we have

$$\begin{aligned} RTS &= \frac{MP_l}{MP_k} = \left(\frac{k}{l}\right)^{1-\rho} \\ \ln RTS &= (1-\rho)\ln\left(\frac{k}{l}\right) \end{aligned}$$

Therefore,

$$\sigma = \frac{\partial \ln(k/l)}{\partial \ln RTS} = \frac{1}{1-\rho}$$

- The linear, fixed-proportions, and Cobb-Douglas cases correspond to $\rho = 1$, $\rho = -\infty$ and $\rho = -0$, respectively.

- Often the CES function is used with a distributional weight, $\alpha (0 \leq \alpha \leq 1)$, to indicate the relative significance of the inputs:

$$q = f(k, l) = [\alpha k^{\rho} + (1 - \alpha) l^{\rho}]^{\gamma/\rho}$$

- With constant returns to scale and $\rho = 0$, this function converges to the Cobb-Douglas form

$$q = f(k, l) = k^{\alpha} l^{1-\alpha}.$$

Example 9.3 A Generalized Leontief Production Function

- Suppose that the production function is given by

$$q = f(k, l) = k + l + 2\sqrt{kl}.$$

  This is a special case of a class of functions named for the Russian-American economist Wassily Leontief.
- This function clearly exhibits constant returns to scale.
- Marginal productivities are

$$\begin{aligned} f_k &= 1 + (k/l)^{-0.5}, \\ f_l &= 1 + (k/l)^{0.5}. \end{aligned}$$

- $RTS = f_l/f_k$ diminishes as k/l falls, so the isoquants have the usual convex shape.

- Two ways to find the elasticity of substitution for this production function. First, the function can be factored as

$$q = f(k,l) = k + l + 2\sqrt{kl} = (\sqrt{k} + \sqrt{l})^2 = (k^{0.5} + l^{0.5})^2$$

which makes clear that this function has a CES form with $\rho = 0.5$ and $\gamma = 1$. Hence $\sigma = 1/(1-\rho) = 2$.

- Another more exhaustive approach is to apply the definition in footnote 6 directly.

$$\begin{aligned} \sigma &= \frac{f_k f_l}{f \cdot f_{kl}} = \frac{[1 + (k/l)^{-0.5}][1 + (k/l)^{0.5}]}{q \cdot 0.5(kl)^{-0.5}} \\ &= \frac{2 + (k/l)^{-0.5} + (k/l)^{0.5}}{0.5(k/l)^{0.5} + 0.5(k/l)^{-0.5} + 1} = 2 \end{aligned}$$

- For the production function $q = f(k, l)$, it can be shown that the elasticity of substitution $\sigma = \frac{d\ln(k/l)}{d\ln(f_l/f_k)}$ can be derived to be

$$\sigma = \frac{(kf_k + lf_l)f_k f_l}{kl(-f_{kk}f_l^2 + 2f_k f_l f_{kl} - f_{ll}f_k^2)}.$$

- If the production function exhibits constant returns to scale, then the elasticity of substitution can be reduced to

$$\sigma = \frac{f_k \cdot f_l}{f \cdot f_{kl}}$$

Proof:

$$\sigma \equiv \frac{d\ln(k/l)}{d\ln(f_l/f_k)} = \frac{f_l/f_k}{k/l} \cdot \frac{d(k/l)}{d(f_l/f_k)}$$

Total differentiating $d(k/l)$ and $d(f_l/f_k)$, along with the fact that $-\frac{dk}{dl} = \frac{f_l}{f_k}$ and thus $dl = -\frac{f_k}{f_l}dk$ gives

$$\begin{aligned}
d(k/l) &= \frac{1}{l^2}(ldk - kdl) \\
&= \frac{1}{l^2}\left(l + k\frac{f_k}{f_l}\right)dk \\
&= \frac{1}{l^2}(kf_k + lf_l)\frac{dk}{f_l} \\
\frac{d(k/l)}{k/l} &= \frac{1}{kl}(kf_k + lf_l)\frac{dk}{f_l}
\end{aligned}$$

$$
\begin{aligned}
d\left(\frac{f_l}{f_k}\right) &= \frac{\partial(f_l/f_k)}{\partial k}\cdot dk + \frac{\partial(f_l/f_k)}{\partial l}\cdot dl \\
&= \left(\frac{\partial(f_l/f_k)}{\partial k} + \frac{\partial(f_l/f_k)}{\partial l}\cdot\left(-\frac{f_k}{f_l}\right)\right) dk \\
&= \left(f_l\frac{\partial(f_l/f_k)}{\partial k} - f_k\frac{\partial(f_l/f_k)}{\partial l}\right)\frac{dk}{f_l} \\
\frac{d\left(\frac{f_l}{f_k}\right)}{f_l/f_k} &= \frac{f_k}{f_l}\left(f_l\frac{\partial(f_l/f_k)}{\partial k} - f_k\frac{\partial(f_l/f_k)}{\partial l}\right)\frac{dk}{f_l}
\end{aligned}
$$

Therefore,

$$\sigma = \frac{\frac{d(k/l)}{k/l}}{\frac{d(f_l/f_k)}{f_l/f_k}} = \frac{f_l(kf_k + lf_l)}{f_k kl\left(f_l \frac{\partial(f_l/f_k)}{\partial k} - f_k \frac{\partial(f_l/f_k)}{\partial l}\right)}$$

Since

$$\frac{\partial(f_l/f_k)}{\partial k} = \frac{1}{f_k^2}(f_k f_{lk} - f_l f_{kk})$$

$$\frac{\partial(f_l/f_k)}{\partial l} = \frac{1}{f_k^2}(f_k f_{ll} - f_l f_{kl})$$

$$f_l \frac{\partial(f_l/f_k)}{\partial k} - f_k \frac{\partial(f_l/f_k)}{\partial l} = \frac{1}{f_k^2}\left(f_l f_k f_{lk} - f_l^2 f_{kk} - f_k^2 f_{ll} + f_k f_l f_{kl}\right)$$

$$= \frac{1}{f_k^2}\left(-f_l^2 f_{kk} - f_k^2 f_{ll} + 2 f_k f_l f_{kl}\right)$$

Finally,

$$\sigma = \frac{f_k f_l(kf_k + lf_l)}{kl\left(-f_l^2 f_{kk} - f_k^2 f_{ll} + 2 f_k f_l f_{kl}\right)}$$

- For a general production function $q = f(k,l)$, the elasticity of substitution is

$$\sigma = \frac{f_k f_l (k f_k + l f_l)}{kl\left(-f_l^2 f_{kk} - f_k^2 f_{ll} + 2 f_k f_l f_{kl}\right)}$$

- If $f(k,l)$ exhibits constant returns to scale, or, $f(k,l)$ is homogeneous of degree one, the the marginal products of the inputs, f_k, and f_l, are homogeneous of degree zero. Thus, according to the Euler's theorem,

$$\begin{aligned} f_k k + f_l l &= f \\ f_{kk} k + f_{kl} l &= 0 \Rightarrow f_{kk} = -\frac{l}{k} f_{kl} \\ f_{lk} k + f_{ll} l &= 0 \Rightarrow f_{ll} = -\frac{k}{l} f_{lk} \end{aligned}$$

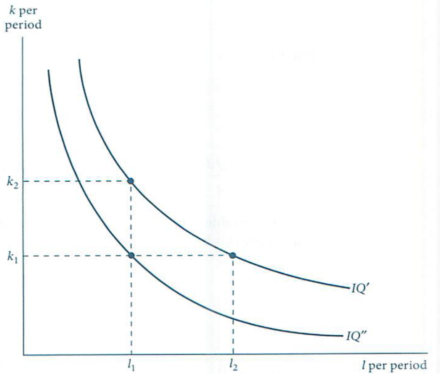
- Thus,

$$\begin{aligned}
\sigma &= \frac{f_k f_l f}{kl\left(-f_l^2\left(-\frac{l}{k}f_{kl}\right) - f_k^2\left(\frac{k}{l}f_{lk}\right) + +2f_k f_l f_{kl}\right)} \\
&= \frac{f_k f_l f}{\left(l^2 f_l^2 + k^2 f_k^2 + 2kl f_k f_l\right) f_{kl}} \\
&= \frac{f_k f_l f}{(kf_k + lf_l)^2 f_{kl}} \\
&= \frac{f_k f_l}{f \cdot f_{kl}}
\end{aligned}$$

Technical Progress

- Following the development of superior production techniques, the same level of output can be produced with fewer inputs. The isoquant shifts inward.

Figure 9.5 Technical Progress



Measuring technical progress

- The first observation to be made about technical progress is that historically the rate of growth of output overtime has exceeded the growth rate that can be attributed to the growth in conventionally defined inputs.
- Suppose that we let

$$q = A(t)f(k, l)$$

be the production function for some good, where $A(t)$ represents all influences that go into determining q other than k and l.

- Changes in A over time represent technical progress. A is shown as a function of time (t), $dA/dt > 0$.

Growth accounting

- Differentiating the production function with respect to time,

$$\begin{aligned}\frac{dq}{dt} &= \frac{dA}{dt}\cdot f(k,l) + A\cdot\frac{df(k,l)}{dt}\\ &= \frac{dA}{dt}\cdot\frac{q}{A} + \frac{q}{f(k,l)}\left(\frac{\partial f}{\partial k}\cdot\frac{dk}{dt} + \frac{\partial f}{\partial l}\cdot\frac{dl}{dt}\right)\end{aligned}$$

- Dividing by q gives

$$\begin{aligned}\frac{dq/dt}{q} &= \frac{dA/dt}{A} + \frac{\partial f/\partial k}{f(k,l)}\cdot\frac{dk}{dt} + \frac{\partial f/\partial l}{f(k,l)}\cdot\frac{dl}{dt}\\ &= \frac{dA/dt}{A} + \frac{\partial f}{\partial k}\cdot\frac{k}{f(k,l)}\cdot\frac{dk/dt}{k} + \frac{\partial f}{\partial l}\cdot\frac{l}{f(k,l)}\cdot\frac{dl/dt}{k}\end{aligned}$$

- Let G_x denote the proportional rate of growth of variable x per unit of time, $(dx/dt)/x$, then the previous equation can be rewritten as

$$G_q = G_A + e_{q,k}G_k + e_{q,l}G_l,$$

where

- $e_{q,k} = \frac{\partial f}{\partial k} \cdot \frac{k}{f(k,l)}$ is the elasticity of output with respect to capital.
- $e_{q,l} = \frac{\partial f}{\partial l} \cdot \frac{l}{f(k,l)}$ is the elasticity of output with respect to labor.

- In a pioneer study of U.S. economy between the years 1909 and 1949, R. M. Solow recorded the following values:

$$\begin{aligned} G_q &= 2.75\% \text{ per year} \\ G_l &= 1.00\% \\ G_k &= 1.75\% \\ e_{q,l} &= 0.65 \\ e_{q,k} &= 035 \end{aligned}$$

Consequently,

$$\begin{aligned} G_A &= G_q - e_{q,l} G_l - e_{q,k} G_k \\ &= 275 - 0.65 \cdot 1.00 - 0.35 \cdot 1.75 = 1.50 \end{aligned}$$

- More than half ($1.50/2.75 \doteq 55\%$) of the growth in real output could be attributed to technical change (A).

Example 9.4 Technical Progress in the Cobb-Douglas Production Function

- The Cobb-Douglas production function with technical progress provides an especially easy avenue for illustrating technical progress.
- Assuming constant returns to scale, the production function is

$$q = A(t)f(k,l) = A(t)k^{\alpha}l^{1-\alpha}.$$

- Also assume that technical progress occurs at a constant exponential (θ), $A(t) = Ae^{\theta t}$, then the production function becomes

$$q = Ae^{\theta t}k^{\alpha}l^{1-\alpha}.$$

- Taking logarithm and differentiate with respect to t gives

$$\begin{aligned}
\ln q &= \ln A + \theta t + \alpha \ln k + (1-\alpha)\ln l \\
\frac{\partial \ln q}{\partial t} &= \frac{\partial q/\partial t}{q} = G_q = \theta + \alpha \cdot \frac{\partial \ln k}{\partial t} + (1-\alpha) \cdot \frac{\partial \ln l}{\partial t} \\
&= \theta + \alpha G_k + (1-\alpha) G_l.
\end{aligned}$$

- Suppose $A = 10$, $\theta = 0.03$, $\alpha = 0.5$ and that a firm uses an input mix of $k = l = 4$.
- At $t = 0$, output is 40 (=$10 \cdot 4^{0.5} \cdot 4^{0.5}$).
- After 20 years ($t = 20$), the production function becomes

$$q = 10e^{0.03 \cdot 20} k^{0.5} l^{0.5} = 10 \cdot 1.82 \cdot k^{0.5} l^{0.5} = 18.2 \cdot k^{0.5} l^{0.5}$$

With $k = l = 4$, $q = 72.8$.

- **Input-augmenting technical progress.** A plausible approach to modeling improvements in labor and capital separately is to assume that the production function is

$$q = A(e^{\psi t}k)^{\alpha}(e^{\varepsilon t}l)^{1-\alpha},$$

where ψ represents the annual rate of improvement in capital input and ε represents the annual rate of improvement in labor input.

- However, because of the exponential nature of the Cobb-Douglas function, this would be indistinguishable from our original example:

$$q = Ae^{[\alpha\psi+(1-\alpha)\varepsilon]t}k^{\alpha}l^{1-\alpha} = Ae^{\theta t}k^{\alpha}l^{1-\alpha}$$

where $\theta = \alpha\psi + (1-\alpha)\varepsilon$.

Many-Input Production Functions

E9.1 Cobb-Douglas

The many-input Cobb-Douglas production function is given by

$$q = \prod_{i=1}^{n} x_i^{\alpha_i}$$

a. This function exhibits constant returns to scale if $\sum_{i=1}^{n} \alpha_i = 1$.

b. α_i is the elasticity e_{q,x_i}. Since $0 \leq \alpha_i \leq 1$, each input exhibits diminishing marginal productivity.

c. Any degree of increasing returns to scale can be cooperated into this function, depending on

$$\varepsilon = \sum_{i=1}^{n} \alpha_i .$$

d. The elasticity of substitution between any two inputs in this production function is 1.

$$\sigma_{ij} = \frac{\partial \ln(x_i/x_j)}{\partial \ln(f_j/f_i)}$$

$$\frac{f_j}{f_i} = \frac{\alpha_j x_j^{\alpha_j - 1} \Pi_{i \neq j} x_i^{\alpha_i}}{\alpha_i x_j^{\alpha_i - 1} \Pi_{j \neq i} x_j^{\alpha_j}} = \frac{\alpha_j}{\alpha_i} \cdot \frac{x_i}{x_j}$$

Hence,

$$\ln\left(\frac{f_j}{f_i}\right) = \ln\left(\frac{\alpha_j}{\alpha_i}\right) + \ln\left(\frac{x_i}{x_j}\right)$$

and $\sigma_{ij} = 1$.

- Because the parameter is so constrained, the function is generally not used in econometric analyses of microeconomic data on firms. However, the function has a variety of general uses in macroeconomics.

The Solow growth model

- The Solow model of equilibrium growth can be derived using a two-input constant returns-to-scale Cobb-Douglas function of the form

$$q = Ak^{\alpha}l^{1-\alpha},$$

where A is a technical change factor that can can be represented by exponential growth of the form

$$A = e^{at}$$

Dividing both sides by l yields

$$\hat{q} = e^{at}\hat{k}^{\alpha}$$

where $\hat{q} = q/l, \hat{k} = k/l.$

- Solow shows that economies will evolve toward an equilibrium value of $\hat{k}$. Hence cross-country differences in growth rates can be accounted for only by differences in the technical change factor a.
- However, the equation is incapable of explaining the large differences in per capita output $(\hat{q})$ observed around the world.
- A second shortcoming is that it offers no explanation of the technical change parameter a. By adding additional factors, it becomes easier to understand how the parameter a may respond to is easier to economic incentives. This is the key insight of literature on "endogenous" growth theory. (Romer 1996)

E9.2 CES

The many-input constant elasticity of substitution (CES) production function is given by

$$q = \left(\sum \alpha_i x_i^{\rho}\right)^{\gamma/\rho}, \rho \leq 1.$$

a. This function exhibits constant returns to scale for $\gamma = 1$. For $\gamma > 1$, the function exhibits increasing returns to scale.
b. This function exhibits diminishing marginal productivities for each inout when $\gamma \leq 1$.
c. The elasticity of substitution is given by

$$\sigma = \frac{1}{1-\rho},$$

and this applies to substitution between any two of the inputs.

E9.3 Nested production functions

- In some applications, Cobb-Douglas and CES production functions are combined into a “nested" single function.
- For example, there are three primary inputs, x_1, x_2, x_3. Suppose that x_1 and x_2 are relatively closely related in their use (e.g. capital and energy), whereas the third input (labor) is relatively distinct.
- One can use a CES aggregator function to construct a composite input for capital services of the form
$$x_4 = [\gamma x_1^{\rho} + (1-\gamma)x_2^{\rho}]^{1/\rho}.$$
- Then the final production might take a Cobb-Douglas form
$$q = x_3^{\alpha} x_4^{\beta}$$
- Nested production functions have been used in studies that seek to measure the precise nature of the substitutability between capital and energy inputs.

E9.4 Generalized Leontief

$$q = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} \sqrt{x_i x_j},$$

where $\alpha_{ij} = \alpha_{ji}$,

- The function exhibits constant returns to scale.
- Because each input appears both linearly and under the radical, the function exhibits diminishing marginal productivities to all inputs.
- The restriction $\alpha_{ij} = \alpha_{ji}$ is used to ensure symmetry of the second-order partial derivatives.

E9.5 Translog

$$\ln q = \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln x_i + 0.5 \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} \ln x_i \ln x_j, \alpha_{ij} = \alpha_{ji}$$

- Cobb-Douglas function is a special case of this function where $\alpha_0 = \alpha_{ij} = 0$ for all i, j.
- This function may assume any degree of returns to scale. If

$$\sum_{i=1}^{n} \alpha_i = 1, \sum_{j=1}^{n} \alpha_{ij} = 0$$

  for all i, then this function exhibits constant returns to scale.
- The condition $\alpha_{ij} = \alpha_{ji}$ is required to ensure equality of the cross-partial derivatives.
- Translog production function has been used to study the ways in which newly arrived workers may substitute for existing workers.