Part III: Uncertainty and Strategy

7. Uncertainty
8. Game Theory

Chapter 8
Game Theory
Part I

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- This chapter provides an introduction to noncooperative game theory, a tool used to understand the strategic interactions among two or more agents.
- The range of game theory has been growing constantly, including all areas of economics (from labor economics to macroeconomics) and other fields such as political science and biology.
- Game theory is particularly useful in understanding the interaction between firms in an oligopoly. (Chapter 15)
- We begin with the central concept of Nash equilibrium and study its application in simple games.
- We then go on to study refinements of Nash equilibrium that are used in games with more complicated timing and information structures.

Basic Concepts

- In a strategic setting, what is best for one decision-maker may depend on the actions of other people.
- There are two major tasks involved when using game theory to analyze an economic situation.
- The first is to distill the situation into a simple game.
- The second task is to “solve" the game, which results in a prediction about what will happen.
- A *game* is an abstract model of a strategic situation, which have three essential elements: players, strategies, and payoffs.

Players

- Each decision-maker in a game is called a *player*. These players may be individuals, firms, or countries.
- A player is characterized as having the ability to choose from among a set of possible actions.

Strategies

- Each course of action open to a player during the game is called a *strategy*. A strategy may be a simple action or a complex plan of action.
- Let S_i be the set of strategies open to player i, $s_i \in S_i$ is the strategy chosen by player i.
- A strategy *profile* will refer to a listing of particular strategies chosen by each of the a group of players.

Payoffs

- The final return to each player at the conclusion of a game is called a *payoff*. Payoffs are measured in levels of utility obtained by the players.
- In a two-player game, $U_1(s_1, s_2)$ denotes player 1's payoff assuming she follows s_1 and player 2 follows s_2. $U_2(s_2, s_1)$ would be player 2's payoff under the same circumstances.
- In an n-player game, we can write the payoff of player i as $U_i(s_i, s_{-i})$, which depends on player i's own strategy s_i and the profile $s_{-i} = (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n)$ of the strategies of all players other than i.

Prisoner's Dilemma

- The Prisoners' Dilemma, introduced by A. W. Tucker in the 1940s, is one of the most famous studied in game theory.
- Two suspects are arrested for a crime. The district attorney want to extract a confession so he offers each the following deal.
  - If you fink on your companion, but your companion doesn't fink on you, you get a one-year sentence and your companion gets a four-year sentence.
  - If you both fink on each other, you will each get a three-year sentence.
  - If neither finks, you will get tried for a lesser crime and each get a two-year sentence.

Figure 8.1 Normal Form for the Prisoner's Dilemma

		Suspect 2	
		Fink	Silent
Suspect 1	Fink	$U_1 = 1, U_2 = 1$	$U_1 = 3, U_2 = 0$
	Silent	$U_1 = 0, U_2 = 3$	$U_1 = 2, U_2 = 2$

- The Prisoner's Dilemma can be summarized by the matrix shown in Figure 8.1, called the *normal form* of the game.
- To avoid negative numbers we specify payoffs as the years of freedom over the next 4 years. For example, $U_1(fink, silent) = 3$ and $U_2(fink, silent) = 0$.

Thinking strategically about the Prisoners' Dilemma

- On first thought one might predict that both will be silent because this gives the most total years of freedom for both compared with any other outcome.
- Regardless of what the other player does, finking is better than being silent because it results in an extra year of freedom.
- Because players are symmetric, the same reasoning holds for the other player.
- The best prediction is that both will fink.
- This prediction reveals a central insight from the game theory that putting player against each other in strategic situations sometimes lead to outcomes that are inefficient for the players.

Nash Equilibrium

- In the economic theory of markets, the concept of equilibrium is developed to indicate a situation in which both suppliers and demanders are content with the market outcome.
- In the strategic setting of game theory, *Nash equilibrium*, formalized by John Nash in the 1950s, involves strategic choices that, once made, provide no incentives for players to alter their behavior.
- A Nash equilibrium is a strategy for each player that is best choice for each player given the others' equilibrium strategies.

A formal definition

- **Best response.** s_i is a best response for player i to rivals' strategies s_{-i}, denoted $s_i \in BR_i(s_{-i})$, if

$$U_i(s_i, s_{-i}) \geq U_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i.$$

- **Nash equilibrium.** A Nash equilibrium is a strategy profile $(s_1^*, s_2^*, \cdots, s_n^*)$ such that for each player $i = 1, 2, \cdots, n$, s_i^* is a best response to other players' equilibrium strategies, s_{-i}^*. That is, $s_i^* \in BR_i(s_{-i}^*)$.
- In a two-player game, (s_1^*, s_2^*) is a Nash equilibrium if s_1^* and s_2^* are mutual best responses against each other:

$$U_1(s_1^*, s_2^*) \geq U_1(s_1, s_2^*) \text{ for all } s_1 \in S_1$$
$$U_2(s_2^*, s_1^*) \geq U_2(s_2, s_1^*) \text{ for all } s_2 \in S_2$$

- A Nash equilibrium is stable in that, even if all players revealed their strategies to each other, no play would have an incentive to deviate from his or her equilibrium strategy.
- Another reason Nash equilibrium is widely used is that it is guaranteed to exist for all games we will study (allowing for mixed strategies, to be defined later; Nash equilibrium in pure strategies do not have to exist). This existence result will be discussed in the Extensions to this chapter.
- Nash equilibrium has some drawbacks.
  - There may be multiple Nash equilibria, making it hard to have a unique prediction.
  - It is unclear how a player can choose a best-response strategy before knowing how rivals will play.

Nash equilibrium in the Prisoners' Dilemma

- Finking is player 1's best response to player 2's finking.
- Because players are symmetric, the same logic implies that player 2's finking is a best response to player 1's finking.
- Therefore, both finking is a Nash equilibrium.

Figure 8.2 Underlining Procedure in the Prisoners' Dilemma

Suspect 1 \ Suspect 2	Fink	Silent
Fink	$\underline{U_1 = 1}$, $\underline{U_2 = 1}$	$\underline{U_1 = 3}$, $U_2 = 0$
Silent	$U_1 = 0$, $\underline{U_2 = 3}$	$U_1 = 2$, $U_2 = 2$

Dominant strategies

- A strategy that is a best response to any strategy the other players might choose is called a *dominant strategy*.
- A dominant strategy is a strategy s_i^* for player i that is a best response to all strategy profile of other players. That is, $s_i^* \in BR_i(s_{-i})$ for all s_{-i}.
- If all players in a game have a dominant strategy, then we say the game has a *dominant strategy equilibrium*.
- It is generally true for all games that a dominant strategy equilibrium, if it exists, is also a Nash equilibrium and is the unique Nash equilibrium.

Battle of the Sexes

- Battle of the Sexes game is another example that illustrates the concepts of best response and Nash equilibrium.
- A wife and husband may either go to the ballet or to a boxing match. They both prefer spending time together. The wife prefers ballet and the husband prefers boxing match.

Figure 8.3 Normal Form for the Battle of the Sexes

		Player 2 (Husband)	
		Ballet	Boxing
Player 1 (Wife)	Ballet	2, 1	0, 0
	Boxing	0, 0	1, 2

- There are two Nash equilibria, (ballet, ballet) and (boxing, boxing).
- There is no dominant strategy.

Figure 8.4 Underlining Procedure in the Battle of the Sexes

		Player 2 (Husband)	
		Ballet	Boxing
Player 1 (Wife)	Ballet	2, 1	0, 0
	Boxing	0, 0	1, 2

Example 8.1 Rock, Paper, Scissors

- Two players display one of three hand signals, rock breaks scissors, scissors cut paper, paper covers rock.
- None of the nine boxes represents a Nash equilibrium. Any strategy pair is unstable because it offers at least one of the players an incentive to deviate.

Figure 8.5 Rock, Paper, Scissors

Player 1 \ Player 2	Rock	Paper	Scissors
Rock	0, 0	−1, 1	1, −1
Paper	1, −1	0, 0	−1, 1
Scissors	−1, 1	1, −1	0, 0

Figure 8.5 Rock, Paper, Scissors

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0, 0	−1, 1	1, −1
	Paper	1, −1	0, 0	−1, 1
	Scissors	−1, 1	1, −1	0, 0

Mixed Strategies

- Players' strategies can be more complicated than just **pure strategies**, where a player chooses an action with certainty.
- *Mixed strategies* have the player randomly select from several possible actions.
- Reasons for studying mixed strategies:
    - Some games have no Nash equilibria in pure strategies, but will have one in mixed strategies.
    - Strategies involving randomization are familiar and natural in certain settings such as class exams and sports.

Formal definitions

- Suppose that player i has a set of M possible actions,

$$A_i = \{a_i^1, \cdots, a_i^m, \cdots, a_i^M\},$$

where the subscript, i, refers to the player and the superscript, m, to the different choices.

- A mixed strategy is a probability distribution over the M actions,

$$s_i = (\sigma_i^1, \cdots, \sigma_i^m, \cdots, \sigma_i^M),$$

where σ_i^m indicates the probability of player i playing action a_i^m, with

$$0 \le \sigma_i^m \le 1$$

and

$$\sum_{m=1}^{M} \sigma_i^m = 1.$$

- In the Battles of the Sexes, both player have the same two actions of ballet and boxing, so $A_1 = A_2 = \{ballet, boxing\}$.
- A mixed strategy $(\sigma, 1-\sigma)$ indicates the probability that the player chooses ballet is σ. For example, mixed strategy $(1/3, 2/3)$ means that the player plays ballet with probability 1/3 and boxing with probability 2/3.
- (1,0) means that the player chooses ballet with certainty, a pure strategy. A *pure strategy* is a special case of a mixed strategy.
- Mixed strategies that involve two or more actions being played with positive probability are called *strictly mixed strategies*.

Example 8.2 Expected Payoffs in the Battle of the Sexes

- Suppose the wife chooses mixed strategy $(w, 1-w)$ and the husband chooses $(h, 1-h)$. The wife plays ballet with probability w and the husband with probability h, then her expected payoff is

$$\begin{aligned} & U_1((w,1-w),(h,1-h)) \\ = \; & whU_1(ballet, ballet) + w(1-h)U_1(ballet, boxing) \\ + \; & (1-w)hU_1(boxing, ballet) \\ + \; & (1-w)(1-h)wU_1(boxing, boxing) \\ = \; & wh \cdot 2 + w(1-h) \cdot 0 + (1-w)h \cdot 0 + (1-w)(1-h) \cdot 1 \\ = \; & 1 - w - h + 3wh \end{aligned}$$

Computing mixed-strategy equilibria

- The key to guessing whether a game has a Nash equilibrium in strictly mixed strategy is the surprising result that

  almost all games have an odd number of Nash equilibrium.

- We found an odd number (one) of pure-strategy Nash equilibrium in the Prisoner's Dilemma, suggesting we need not look further for one in strictly mixed strategies.
- In the Battle of Sexes, we found an even number (two) of pure-strategy Nash equilibria, suggesting the existence of a third one in strictly mixed strategies.
- Rock, Paper, Scissors has no pure-strategy Nash equilibria, we would expect to find one Nash equilibrium in strictly strategies.

Example 8.3 Mixed-Strategy Nash Equilibrium in Battle of the Sexes

- A general mixed strategy: the wife chooses $(w, 1-w)$ and the husband chooses]red$(h, 1-h)$, where w and h are the probabilities of playing ballet for the wife and husband.
- From Example 8.2, the wife's expected payoff is

$$U_1((w, 1-w), (h, 1-h)) = 1 - w - h + 3wh$$

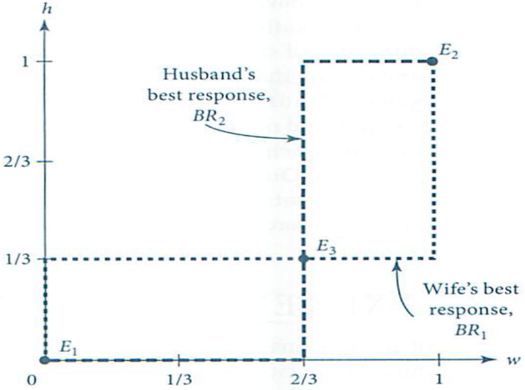
- The wife's best response depends on h. Note that $\frac{\partial U_1}{\partial w} = -1 + 3h$.
  - If $h < 1/3$, she should set $w = 0$.
  - If $h > 1/3$, she should set $w = 1$.
  - if $h = 1/3$, her expected payoff is $2/3$ no matter what value of w she chooses.

- Similarly, the expected payoff for the husband is

$$\begin{aligned} & U_2((h, 1-h), (w, 1-w)) \\ = \; & hw \cdot 1 + h(1-w) \cdot 0 + (1-h)w \cdot 0 + (1-h)(1-w) \cdot 2 \\ = \; & 2 - 2h - 2w + 3hw \end{aligned}$$

- The husband's best response depends on w. Note that $\frac{\partial U_2}{\partial h} = -2 + 3w$.
    - When $w < 2/3$, he should set $h = 0$.
    - When $w > 2/3$, he should set $h = 1$.
    - when $w = 2/3$, his expected payoff is $2/3$ no matter what value of h he chooses.

Figure 8.6 Nash Equilibria in Mixed Strategies in Battle of Sexes



- The three intersection points E_1, E_2, and E_3 are Nash equilibria. The Nash equilibrium in strictly mixed strategy, E_3, is $w = 2/3, h = 1/3$.

- Note that a player will be willing to randomize between two actions in equilibrium only if he or she gets the same expected payoff from playing either action or, in other words, is indifferent between the two actions in equilibrium.
- Suppose the husband is playing mixed strategy $(h, 1-h)$, the wife's expected payoff from playing ballet is

$$U_1(ballet, (h, 1-h)) = h \cdot 2 + (1-h) \cdot 0 = 2h.$$

Her expected payoff from playing boxing is

$$U_1(boxing, (h, 1-h)) = h \cdot 0 + (1-h) \cdot 1 = 1-h$$

- For the wife to be indifferent between ballet and boxing in equilibrium, $2h = 1 - h$, and $h^* = 1/3$.

- Similarly, suppose the wife is playing mixed strategy $(w, 1-w)$, the husband's expected payoff from playing ballet is

$$U_2(ballet, (w, 1-w)) = w \cdot 1 + (1-w) \cdot 0 = w.$$

His expected payoff from playing boxing is

$$U_1(boxing, (w, 1-w)) = w \cdot 0 + (1-w) \cdot 2 = 2 - 2w$$

- For the husband to be indifferent between ballet and boxing in equilibrium, $w = 2 - 2w$, and $w^* = 2/3$.

- Notice that the wife's indifference condition does not "pin down" her equilibrium mixed strategy. Rather, the wife's indifference condition pins down the other player's mixed strategy.

Existence of Equilibrium

- Nash equilibrium is widely used because a Nash equilibrium is guaranteed to exist in a wide class of games.
- This is not true for some other equilibrium concepts such as the concept of dominant strategy equilibrium.
- The Extensions section at the end of the chapter will provide the technical details behind John Nash's proof of the existence of his equilibrium in all finite games.
- The existence theorem does not guarantee the existence of a pure-strategy Nash equilibrium. It does guarantee that, if a pure-strategy Nash equilibrium does not exist, a mixed-strategy Nash equilibrium does exist.

Continuum of Actions

- Some settings are more realistically modeled via a continuous range of actions.
- It is natural to allow firms to choose any non-negative price or quantity rather than artificially restricting them to just two prices (say, $2 and $5) or two quantities (say, 100 or 1,000 units).
- The familiar methods from calculus can often be used to solve for Nash equilibria.
- It is also possible to analyze how the equilibrium actions vary with changes in underlying parameters.

Tragedy of the Commons

How to solve for the Nash equilibrium when the game involves a continuum of actions .

- Write down the payoff for each player as a function of all players' actions.
- Compute the first-order condition associated with each player's payoff maximum.
- This will give an equation that can be rearranged into the best response of each player as a function of all other players' actions.
- There will be one equation for each player.
- Solve the system of n equations for the n unknown equilibrium actions.

Example 8.4 Tragedy of the Commons

- The term *Tragedy of the Commons* describes the overuse problem that arises when scarce resources are treated as common property.
- Assume that two herders decide how many sheep to graze on the village commons. The problem is that the commons is small and can rapidly succumb to overgrazing.
- Let q_i be the number of sheep chosen by herder $i = 1, 2$, and the per-sheep value of grazing on the commons is

$$v(q_1, q_2) = 120 - (q_1 + q_2)$$

- The normal form is a listing of payoff functions

$$\begin{aligned} U_1(q_1, q_2) &= q_1 v(q_1, q_2) = q_1(120 - q_1 - q_2) \\ U_2(q_1, q_2) &= q_2 v(q_1, q_2) = q_2(120 - q_1 - q_2) \end{aligned}$$

- To find the Nash equilibrium, we solve herder 1's value-maximization problem:

$$\max_{q_1} \quad \{q_1(120 - q_1 - q_2)\}.$$

and get his best-response function

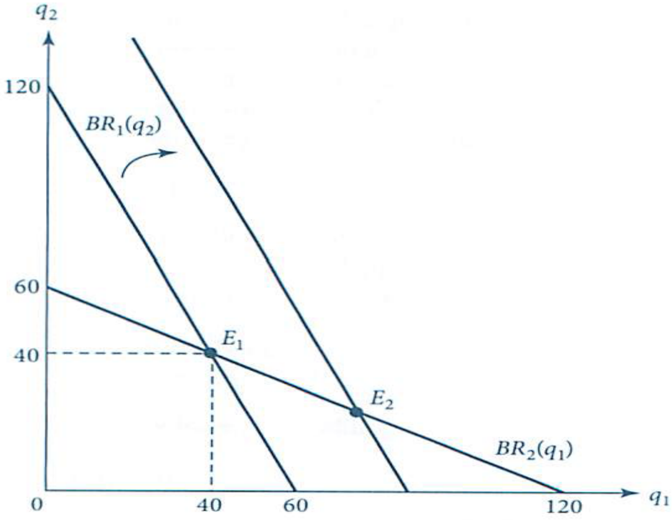
$$q_1 = 60 - \frac{q_2}{2} = BR_1(q_2)$$

- Similar steps show that herder 2's best-response function is

$$q_2 = 60 - \frac{q_1}{2} = BR_2(q_1)$$

- The Nash equilibrium is given by the pair (q_1^*, q_2^*) that satisfied these two best-response functions, which gives $q_1^* = q_2^* = 40$, each earns a payoff of 1,600.

Figure 8.7 Best-Response Diagram for the Tragedy of the Commons



- The Nash equilibrium is not the best use of the commons.
- The "joint payoff maximization"problem

$$\max_{q_1,q_2}\{(q_1+q_2)v(q_1+q_2)\} = \max_{q_1,q_2}\{(q_1+q_2)(120-q_1-q_2)\}$$

is solved by

$$q_1 = q_2 = 30,$$

or, by any q_1 and q_2 that sum to 60.

Sequential Games

- In some games, the order of moves matters. A player who moves later in the game can see how others have played up to that moment.
- The player can use this information to form more sophisticated strategies than simply choosing an action.
- The player's strategy can be a contingent plan with the action played depending on what the other players have done.

Sequential Battle of the Sexes

- Suppose the wife chooses first, and the husband observes her choice before making his. Her possible strategies haven't changed. His possible strategies have expanded.
- For each of his wife's actions, he can choose one of two actions. Therefore, he has four possible strategies.

Table 8.1 Husband's Contingent Strategies

Contingent Strategy	Written in Conditional Format
Always go to the ballet	(ballet | ballet, ballet | boxing)
Follow his wife	(ballet | ballet, boxing | boxing)
Do the opposite	(boxing | ballet, ballet | boxing)
Always go to boxing	(boxing | ballet, boxing | boxing)

- "boxing | ballet" should be read as "the husband chooses boxing conditional on the wife's choosing ballet."

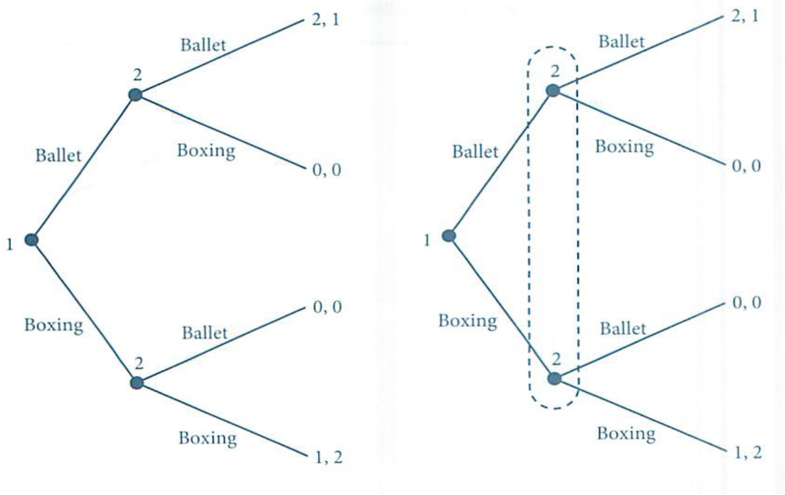
Figure 8.8 Normal Form for the Sequential Battle of the Sexes

		Husband			
		(Ballet | Ballet Ballet | Boxing)	(Ballet | Ballet Boxing | Boxing)	(Boxing | Ballet Ballet | Boxing)	(Boxing | Ballet Boxing | Boxing)
Wife	Ballet	2, 1	2, 1	0, 0	0, 0
	Boxing	0, 0	1, 2	0, 0	1, 2

- The normal form is twice as complicated as that for the simultaneous version of the game in Figure 8.2.
- This motivate a new way to represent games, called the *extensive form*, which is especially convenient for sequential games.

Extensive form

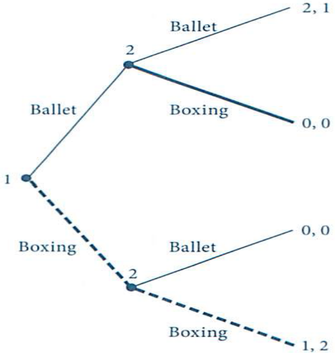
Figure 8.9 Extensive Form of the Battle of Sexes



(a) Sequential version (b) Simultaneous version

- From the normal form in Figure 8.8, there are three pure-strategy Nash equilibria.
    1. Wife plays ballet, husband plays (ballet | ballet, ballet | boxing)
    2. Wife plays ballet, husband plays (ballet | ballet, boxing | boxing)
    3. Wife plays boxing, husband plays (boxing | ballet, boxing | boxing)
- Consider the third Nash equilibrium. (boxing | ballet) is not a credible threat (empty threat) for the husband, while (boxing| boxing) is a Nash equilibrium.
- Similarly, in the first Nash equilibrium, (ballet| boxing) is an empty threat while (ballet| ballet) is a Nash equilibrium.

Figure 8.10 Equilibrium Path

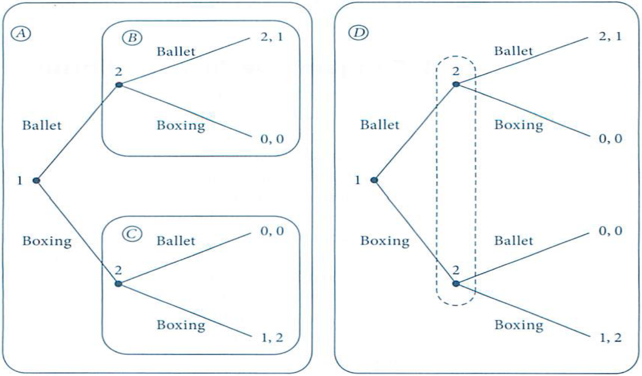


- The third outcome is a Nash equilibrium because the strategies are rational along the equilibrium path.
- Following the wife's choosing ballet, the husband's strategy is irrational.

Subgame-perfect equilibrium

- Subgame-perfect equlibrium is a refinement that rules out empty threats by requiring strategies to be rational even for contingencies that do not arise in equilibrium.
- A *subgame* is a part of the extensive form beginning with a decision node and including everything that branches out to the right of it.
- A *proper subgame* is a subgame that starts at a decision node not connected to another in an information set.

Figure 8.11 Proper Subgames in the Battle of the Sexes



(a) Sequential (b) Simultaneous

- The sequential version in (a) has three proper subgames, labeled A, B, C.
- The simultaneous version in (b) has only one proper subgame, the whole game itself, labeled D.

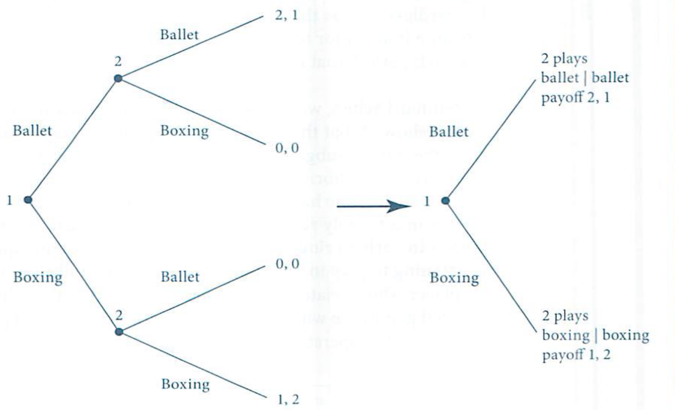
- A **subgame-perfect equilibrium** is a strategy profile $(s_1^*, s_2^*, \cdots, s_n^*)$ that constitutes a Nash equilibrium for every proper subgame.
- A subgame-perfect equilibrium must be a Nash equilibrium for the whole game.
- The husband has only one strategy that can be part of a subgame-perfect equilibrium: (ballet| ballet, boxing| boxing).
- Generally, subgame-perfect equilibrium rules out any sort of empty threat in a sequential game.
- Subgame-perfect equilibrium requires rational behavior both on and off the equilibrium path. Threats to play irrationally are ruled out as being empty.

Backward induction

A shortcut for finding the perfect-subgame equilibrium directly is to use *backward induction*, the process of solving for equilibrium by working backwards from the end of the game to the beginning as follows.

- Identify all the subgames at the bottom of extensive form.
- Find the Nash equilibrium on these subgames.
- Replace the subgames with the actions and payoffs resulting from Nash equilibrium play on these subgames.
- Then move up to the next level of subgames and repeat the procedure

Figure 8.12 Apply Backward Induction



- The last subgames are replaced by the Nash equilibria on these subgames.
- The simple game that results at right can be solved for player 1's equilibrium action.