Part III: Uncertainty and Strategy

7. Uncertainty
8. Game Theory

Chapter 7
Uncertainty
Part II

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Methods for Reducing Uncertainty and Risk

- Risk-averse people will avoid gambles and other risky situations if possible. Often it is impossible to avoid risk entirely.
- If the next four sections, we will study each of four different methods that individuals can take to mitigate the problem of risk and uncertainty:
  - Insurance
  - Diversification
  - Flexibility
  - Information

Insurance

- Risk-averse people would pay a premium to have the insurance company cover the risk of loss.
- A risk-averse person would always want to buy fair insurance to cover any risk he or she faces.
- Insurance company cannot stay in business if it offered fair insurance in the sense that the premium exactly equals the expected payout for claims.
- An insurance customer can always expect to pay more than an actuarially fair premium.
- If people are sufficiently risk averse, they will even buy unfair insurance. The more risk averse they are, the higher the premium they would be willing to pay.

Several factors make insurance difficult or impossible to provide.

- Large-scale disasters
- Rare and unpredictable events offers no reliable track records for insurance companies to establish premiums.
- Informational disadvantage the company may have relative to the customer, causing *adverse selection problem*.
- Having insurance may make customers less willing to take steps to avoid losses. This is called *moral hazard problem*.
- Adverse selection and moral hazard will be discussed in more detail in Chapter 18.

Diversification

- A second way for risk-avese individuals to reduce risk is by diversification. This is the economic principle behind the adage, “Don't put all your eggs in one basket."
- By suitably spreading risk around, it may be possible to reduce the variability of an outcome without lowering the expected payoff.
- Consider an example in which a person has wealth W to invest. This money can be invested in two independent risky assets, 1 and 2, which have equal expected values ($\mu_1 = \mu_2$) and equal variances ($\sigma_1^2 = \sigma_2^2$).

- A person whose undiversified portfolio, UP, includes just one of the assets would earn an expected return of $\mu_{UP} = \mu_1 = \mu_2$ and would face a variance of $\sigma^2_{UP} = \sigma^2_1 = \sigma^2_2$.
- Suppose instead the individual chooses a diversified portfolio, DP. Let α_1 be the fraction invested in the first asset and $\alpha_2 = 1 - \alpha_1$ in the second. Then the expected return on the diversified portfolio is

$$\mu_{DP} = \alpha_1\mu_1 + (1 - \alpha_1)\mu_2 = \mu_1 = \mu_2.$$

  And the variance will depend on the allocation between the two assets:

$$\sigma^2_{DP} = \alpha_1^2\sigma_1^2 + (1 - \alpha_1)^2\sigma_2^2 = (1 - 2\alpha_1 + 2\alpha_1^2)\sigma_1^2.$$

- It can be shown that σ^2_{DP} is minimized when $\alpha_1 = \frac{1}{2}$, and $\sigma^2_{DP} = \frac{\sigma_1^2}{2}$.

Flexibility

- Contrary to diversification, in some situations, a decision can not be divided; it is all or nothing. For example, in shopping for a car, a consumer can not combine the attributes from one model with those of another by buying half of each; cars are sold as a unit.
- The decision-maker can obtain some of the benefit of diversification by making flexible decisions.
- Flexibility allows the person to adjust the initial decision, depending on how the future unfolds. The more uncertain the future, the more valuable this flexibility.
- Flexibility keeps the decision-maker from being tied to one course of action and instead provides a number of options.

- **An example** of the value of flexibility comes from considering the fuels on which cars are designed to run.
- Until now, most cars were limited in how much biofuel (such as ethanol) could be combined with petroleum products (such as gasoline) in the fuel mix.
- A purchaser of such a car would have difficulties if governments passed new regulations increasing the ratio of ethnol in car fuels or banning petroleum products entirely.
- New cars have been designed that can burn ethnol exclusively, but such cars are not useful if current continue to prevail because most filling stations do not sell fuel with high concentrations of ethanol.
- A third type of car can handle a variety of types of fuel, both petroleum-based and ethanol, and any proportions of the two.

Types of options

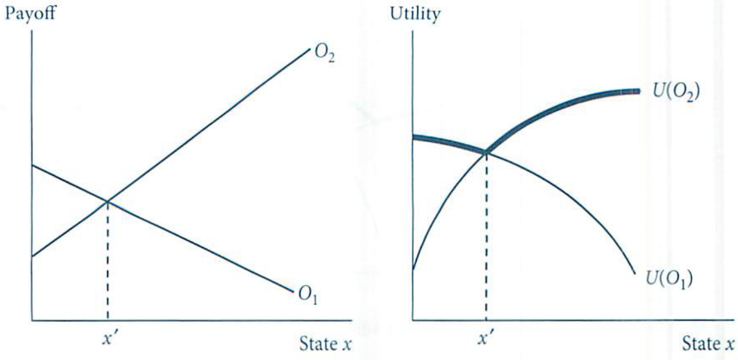
- The "flexible-fuel" cars is valuable because it provides the owner with more options relative to a car that can run on only one type of fuel.
- **Financial option contract**. A *financial option contract* offers the right, but not the obligation, to buy or sell an asset (e.g. a share of stock) during some future period at a certain price.
- **Real option**. A *real option* is an option arising in a setting outside of financial markets.
- All options share three fundamental attributes.
  1. They specify the underlying transaction.
  2. They specify a period over which the option may be exercised.
  3. They specify a price.

Model of real options

- Let x embody all the uncertainty in the economic environment. x is a random variable.
- The individual has some number, $i = 1, \cdots, n$, of choices currently available.
- Let $O_i(x)$ be the payoffs provided by choice i, where the argument (x) allows each choice to provide a different pattern of returns depending on how the future turns out.
- Panel (a) in Figure 7.3 shows the payoffs and panel (b) shows the utilities provided by two alternatives across states of the world (x).

Figure 7.3 The Nature of a Real Option

Panel (a) shows the payoffs and panel (b) shows the utilities provided by two alternatives across states of the world (x). If the decision has to be made upfront, the individual chooses the single curve having the highest expected utility. If the real option to make either decision can be preserved until later, the individual can obtain the expected utility of the upper envelope of the curves, shown in bold.



(a) Payoffs from alternatives

(b) Utilities from alternatives

- If the person does not have the flexibility provided by a real option, he or she must make the choice before observing how the state x turns out. The individual should choose the single alternative that is best on average. The expected utility from this choice is

$$\max\{E[U(O_1)], \cdots, E[U(O_n)]\}$$

- On the other hand, if the real option can be preserved to make a choice that responds to which state of the world x has occurred, the person will be better off. The expected utility is

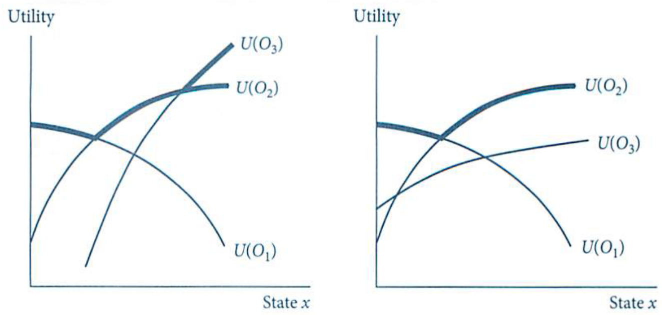
$$\max E\{\max[U(O_1), \cdots, U(O_n)]\}$$

More options are better (typically)

- Adding more options can never harm an individual decision-maker because the extra options can always be ignored. This is the essence of option. (Figure 7.4)
- This insight may no longer hold in a strategic setting with multiple decision-makers.
- In a strategic setting, economic actors may benefit from having some of their options cut off. This may allow a player to commit to a narrower course of action, and this commitment may affect the actions of other parties.
- A famous illustration of this point is provided in one of the earliest treaties on military strategy, by Sun Tzu, a Chinese general writing in 400 BC.
- If the second army observes that the first can not retreat and will fight to death, it may retreat itself before engaging the first.

Figure 7.4 More Options Cannot Make the Individual Decision-Maker Worse Off

The addition of a third alternative to the two drawn in Figure 7.3 is valuable in (a) because it shifts the upper envelope (shown in bold) of utilities up. The new alternative is worthless in (b) because it does not shift the upper envelope, but the individual is not worse off for having it.



(a) Additional valuable option

(b) Additional worthless option

Computing option value

- Let F be the fee that has to be paid for the ability to choose the best alternative after x has been realized instead of before. The individual would be willing to pay the fee as long as
$$E\{\max[U(O_1(x)-F),\cdots,U(O_n(x)-F)]\}$$
$$\geq \max\{E[U(O_1(x))],\cdots,E[U(O_n(x))]\}$$
- The right side is the expected utility from making the choice beforehand. The left side allows for the choice to be made after x has occurred, but subtracts the fee for option from every payoff.
- The real option's value is the highest F for which the equation above is satisfied.

Example 7.5 Value of a Flexible-Fuel Car

- Let $O_1(x) = 1 - x$ be the payoff from a fossil-fuel–only car and $O_2(x) = x$ be the payoff from a biofuel-fuel–only car.
- The state of the world, x, reflects the relative importance of biofuels compared with fossil fuels over the car's lifespan.
- Assume x is a random variable that is uniformly distributed between 0 and 1 with probability density function $f(x) = 1$.
- Suppose first that the car buyer is risk neutral, then the utility level equals to the payoff level.
- Suppose the buyer is forced to choose a biofuel car, this provides an expected utility of

$$E(O_2) = \int_0^1 O_2(x) f(x) dx = \int_0^1 x dx = \left. \frac{x^2}{2} \right|_{x=0}^{x=1} = \frac{1}{2}.$$

- Similar calculations show that the expected utility from buying a fossil-fuel car is also $\frac{1}{2}$.
- Now suppose that a flexible-fuel car is available, which allows the buyer to obtain either $O_1(x)$ or $O_2(x)$, whichever is higher under the latter circumstances. The buyer's expected utility function is

$$\begin{aligned} E[\max(O_1, O_2)] &= \int_0^1 \max(1-x, x) f(x) dx \\ &= \int_0^{\frac{1}{2}} (1-x) dx + \int_{\frac{1}{2}}^1 x dx \\ &= 2\int_{\frac{1}{2}}^1 x dx = x^2 \Big|_{x=\frac{1}{2}}^{x=1} = \frac{3}{4}. \end{aligned}$$

The option value of the flexible-fuel car is $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$, or 25%.

- We next investigate whether risk aversion makes options more or less valuable.
- Now suppose the buyer is risk averse, having von Neumann-Morgenstern utility function $U(x) = \sqrt{x}$. The buyer's expected utility function from a biofuel car is

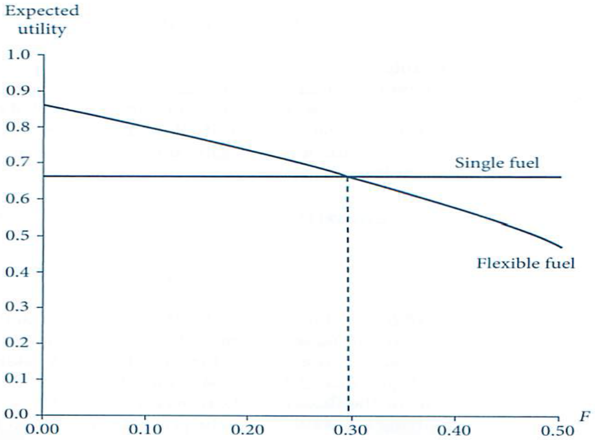
$$E[U(O_2)] = \int_0^1 \sqrt{O_2(x)} f(x)dx = \int_0^1 x^{\frac{1}{2}} dx = \frac{2}{3} x^{\frac{3}{2}} \Big|_{x=0}^{x=1} = \frac{2}{3},$$

which is the same as from a fossil-fuel car, as similar calculations show.

- The expected utility from a flexible-fuel car that costs F more than a single-fuel car is

$$
\begin{aligned}
& E\{\max[U(O_1(x)-F), U(O_2(x)-F)]\} \\
= \; & \int_0^1 \max(\sqrt{1-x-F}, \sqrt{x-F}) f(x)dx \\
= \; & \int_0^{\frac{1}{2}} \sqrt{1-x-F}dx + \int_{\frac{1}{2}}^1 \sqrt{x-F}dx \\
= \; & 2\int_{\frac{1}{2}}^1 \sqrt{x-F}dx, \text{let u = x - F} \\
= \; & 2\int_{\frac{1}{2}-F}^{1-F} u^{\frac{1}{2}}du = \frac{4}{3}u^{\frac{3}{2}}\Big|_{u=\frac{1}{2}-F}^{u=1-F} \\
= \; & \frac{4}{3}\left\{(1-F)^{\frac{3}{2}} - \left(\frac{1}{2}-F\right)^{\frac{3}{2}}\right\}
\end{aligned}
$$

Figure 7.5 Graphical Method for Computing the Premium for a Flexible-Fuel Car



- The two curves intersect at $F = 0.294$, or 29.4%, which is higher than the option value for the risk neutral buyer.

Option value of delay

- "Do not put off to tomorrow what you can do today" is a familiar maxim. Yet the existence of real options suggests a possible value in procrastination. There may be a value in delaying big decisions that are not easily reversed later.
- Delay preserves options. If circumstances continue to be favorable or become even more so, the action can still be taken later.
- But if the future changes and the action is unsuitable, the decision-maker have saved a lot of trouble by not making it.
- Even if circumstances start to favor the biofuel car, the buyer may want to hold off buying a car until he is more sure.
- The value of delay hinges on the irreversibility of the underlying decision.

Implications for cost-benefit analysis

- The *cost-benefit rule* says that an action should be taken if anticipated costs are less than benefits.
- This is a correct course of action in simple settings without uncertainty.
- One must be more careful in applying the rule in setting involving uncertainty.
- The correct decision rule is more complicated because it should account for risk preferences and for the option value of delay.
- Failure to apply the simple cost-benefit rule in settings with uncertainty may indicate sophistication rather than irrationality.

Information

- The fourth method of reducing the uncertainty is to acquire better information about the likely outcome that will arise.
- Delay discussed earlier involves some costs, which can be thought of as a "purchase price" for the information acquired.
- We will consider information as a good that can be purchased directly and analyze why and how much individuals are willing to pay for it.

Information as a good

- Information is a valuable economic resource. For example, a buyer can make a better decision about which type of car to buy if he has better information about the sort of fuels that will be readily available during the life of the car.
- However, unlike the consumer goods, information is difficult to quantify.
- Information has some technical properties that make it an unusual sort of good. Most information is durable and retains value after it has been used.
- Information has the characteristic of a pure *public good* (see Chapter 19). It is both *nonrival* and *nonexclusive*.
- Standard models of supply and demand may be of relatively limited use in understanding such activities.

Quantifying the value of information

- Suppose that an individual is uncertain about what the state of the world (x) will be in the future.
- He or she needs to make one of n choices today. $O_i(x)$ represents the payoffs provided by choice i.
- Interpret F as the fee charged to be told the exact value that x will take on in the future.
- Just as F was the value of the real option in previous sections, it is the value of information.
- The more uncertainty resolved by the new information, the more valuable it is.
- The degree of risk aversion has ambiguous effects on the value of information.

The State-Preference Approach to Choice Under Uncertainty

States of the world and contingent commodities

- We start by thinking about an uncertain future in terms of *states of the world*.
- Assume that it is possible to categorize all the possible things that might happen into a fixed number of well-defined states.
- For example, the world will be in only one of two possible states tomorrow: either "good times" or "bad times,".
- *Contingent commodities* are goods delivered only if a particular state of the world occurs.

- For example, “$1 in good times" is a contingent commodity that promises the individual $1 in good times but nothing should tomorrow trun out to be bad times.
- It is conceivable that an individual could purchase a contingent commodity that someone will pay you $1 if tomorrow turns out to be good times.
- Because tomorrow could be bad, this good will probably sell for less than $1.
- If someone were also willing to to sell me the contingent commodity “$1 in bad times," then I could assure myself of having $1 tomorrow by buying the two contingent commodities “$1 in good times" and “$1 in bad times."

Utility analysis

-

- Denote two contingent goods by W_g (wealth in good times) and W_b (wealth in bad times).

- Assuming that utility is independent of which state occurs and that this individual believes that bad times will occur with probability π, the expected utility is

$$E[U(W)] = (1 - \pi)U(W_g) + \pi U(W_b).$$

This is the value that the individual wants to maximize given his or her initial wealth, W_0.

Prices of contingent commodities

- Assuming that this person can purchase \$1 of wealth in good times for p_g and \$1 of wealth in bad times for p_b, his or her budget constraint is

$$W_0 = p_g W_g + p_b W_b$$

- The price ratio p_g/p_b shows how this person can trade dollars of wealth in good times for dollars in bad times, W_b/W_b.
- For example, if $p_g = 0.8$ and $p_b = 0.2$, the sacrifice of \$1 of wealth in good times would permit this person to buy contingent claims yielding \$4 of wealth in bad times.

Fair markets for contingent goods

- If markets for contingent wealth claims are well-developed and there is general agreement about the likelihood of bad times (π), then prices for these claims will be actuarially fair, that is, they will equal to the underlying probabilities:

$$p_g = 1 - \pi, \; p_b = \pi$$

Hence, the price ratio p_g/p_b will reflect the odds in favor of good times:

$$\frac{p_g}{p_b} = \frac{1-\pi}{\pi}$$

Risk Aversion

- It can be shown that if contingent claims markets are fair, then a utility-maximizing individual will opt for a situation in which $W_g = W_b$; that is, he or she will arrange matters so that the wealth obtained is the same no matter what state occurs.
- Maximization of utility subject to a budget constraint:

$$MRS = \frac{\partial E[U(W)]/\partial W_g}{\partial E[U(W)]/\partial W_b} = \frac{(1-\pi)U'(W_g)}{\pi U'(W_b)} = \frac{p_g}{p_b}.$$

- If markets for contingent claims are fair, the first-order condition is

$$\frac{U'(W_g)}{U'(W_b)} = \frac{\pi}{1-\pi} \cdot \frac{p_g}{p_b} = 1$$

or

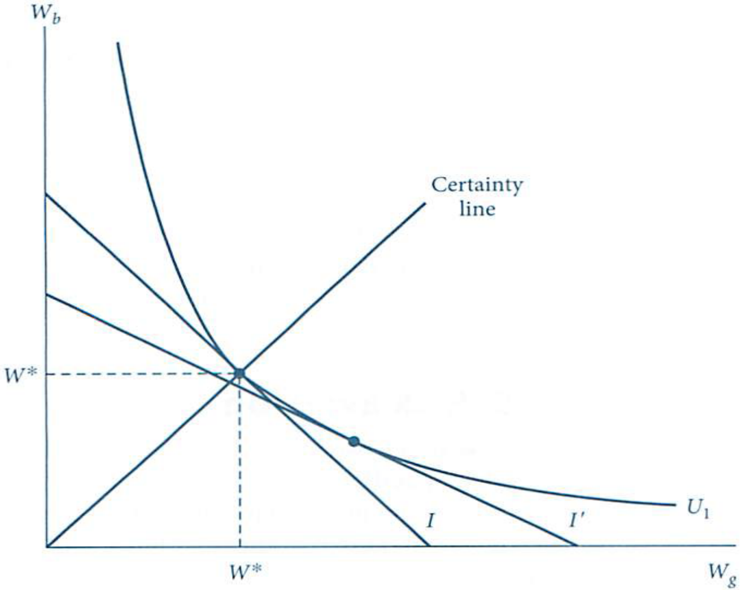
$$W_g = W_b$$

- When faced with fair markets in contingent claims on wealth, the individual will be risk averse and will choose to ensure that he or she has the same level of wealth regardless of which state occurs.

A graphic analysis

- Figure 7.6 illustrates risk aversion with a graph. This individual's budget constraint (I) is tangent to the U_1 indifference curve where $W_g = W_b$— a point on the "certainty line" where wealth (W^*) is independent of which state of the world occurs.
- At W^* the slope of the indifference curve $[(1-\pi)/\pi]$ is equal to the price ratio p_g/p_b.
- If the market for contingent wealth claims were not fair, utility maximization might not occur on the certainty line.
- In the case where $(1-\pi)/\pi = 4$ but $p_g/p_b = 2$ because ensuring wealth in bad times proves costly, the budget constraint may resemble line I' and utility maximization would occur below the certainty line where $W_g > W_b$.

Figure 7.6 Risk Aversions in the State-Preference Model



Example 7.6 Insurance in the State-Preference Model

- Using auto insurance as an example. The buyer's problem involves two contingent commodities 'wealth with no theft" (W_g) and "wealth with a theft" (W_b).
- Assume logarithmic utility and the probability of a theft is $\pi = 0.25$, then the expected utility is

$$E[U(W)] = 0.75U(W_g) + 0.25U(W_b) = 0.75 \ln W_g + 0.25 \ln W_b$$

- If the individual takes no action, the utility is determined by the initial wealth endowment $W_{0g} = 100,000$ and $W_{0b} = 80,000$, so

$$E_{no}[U(W)] = 0.75 \ln 100,000 + 0.25 \ln 80,000 = 11.45714.$$

- Write the budget constraint in terms of the prices of contingent commodities, p_g and p_b:

$$p_g W_{og} + p_b W_{ob} = p_g W_g + p_b W_b.$$

- Assume that these prices equal the probability of the two state, $p_g = 1 - \pi = 0.75$, $p_b = 0.25$, this constraint is

$$0.75 \cdot 100,000 + 0.25 \cdot 80,000 = 95,000 = 0.75 W_g + 0.25 W_b$$

- Maximize expected utility with respect to this budget constraint yields $W_g = W_b = 95,000$. The individual will move to the certainty line and receive an expected utility of

$$E_A[U(W)] = \ln 95,000 = 11.46163 > E_{no}[U(W)] = 11.45714,$$

a clear improvement over doing nothing.

- To obtain this improvement, the individual must be able to transfer $5,000 in extra wealth in good times into $15,000 of extra wealth in bad times.

- A fair insurance contract will allow this because the wealth changes promised by insurance is

$$\frac{dW_b}{dW_g} = \frac{95,000 - 80,000}{95,000 - 100,000} = \frac{15,000}{-5,000} = -3$$

which is exactly equal the negative of the odds ratio

$$-\frac{1-\pi}{\pi} = -\frac{0.75}{0.25} = -3$$

- **A policy with a deductible provision.** For example, a policy B that cost \$5,200 and returned \$20,000 in case of theft would permit this person to reach the certainty line with $W_g = W_b = 94,800$ and expected utility

$$E_B[U(W)] = \ln 94,800 = 11.45953,$$

which exceeds the utility from the initial endowment.

- A policy that costs \$4,900 and requires the individual to incur the first \$1,000 of a loss from theft would yield

$$\begin{aligned} W_g &= 100,000 - 4,900 = 95,100, \\ W_b &= 80,000 - 4,900 + (20,000 - 1,000) = 94,100 \end{aligned}$$

the expected utility from this policy, label it C, equals

$$\mathrm{E}_C[U(W)] = 0.75 \ln 95,100 + 0.25 \ln 94,100 = 11.46004.$$

The policy still provided higher utility than doing nothing.

Risk aversion and risk premiums

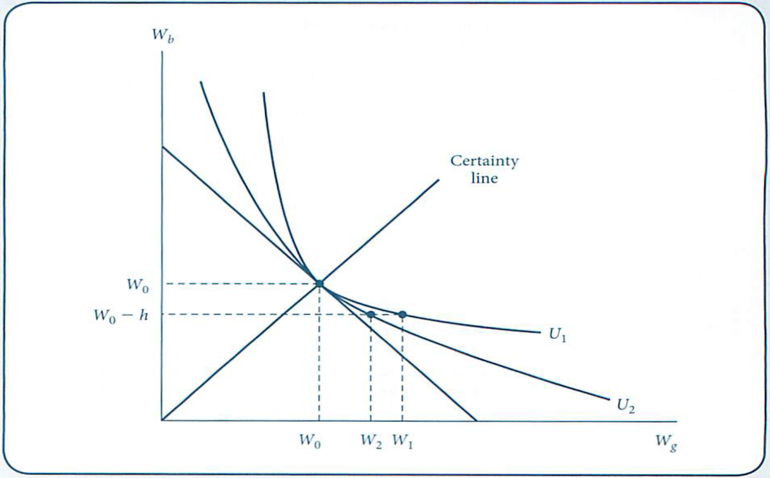
- Consider two people, each starts with a certain wealth, W_0. Each person seeks to maximize an expected utility function of the form

$$E[U(W)] = (1-\pi)\frac{W_g^R}{R} + \pi\frac{W_b^R}{R}$$

- The utility function exhibits constant relative risk aversion. The parameter R determines both the degree of risk aversion and the degree of curvature of indifference curves implied by the function.
- A risk-averse individual will have a large negative value for R, such as U_1 shown in Figure 7.7.

Figure 7.7 Risk Aversion and Risk Premiums

Indifference curve U_1 represents the preferences of a risk-averse person, whereas the person with preferences represented by U_2 is willing to assume more risk. When faced with the risk of losing h in bad times, person 2 will require compensation of $W_2 - W_0$ in good times, whereas person 1 will require a larger amount given by $W_1 - W_0$.



- A person with more tolerance for risk will have a higher value of R and the flatter indifference curves such as U_2.
- When faced with the risk of losing h in bad times, person 2 will require compensation of $W_2 - W_0$ in good times, whereas person 1 will require a larger amount given by $W_1 - W_0$.
- Therefore, the difference between W_1 and W_2 indicates the effect of risk aversion on willingness to assume risk.

Asymmetry of Information

- The level of information that a person buys will depend on the per-unit price of information messages.
- Information costs may differ significantly among individuals because individuals
  - may possess specific skills for acquiring information,
  - may have experiences that yield valuable information,
  - may have invested in some types of information services.
- When information costs are high and variable across individuals, we would expect them to find it advantageous to acquire different amounts of information. (Chapter 18)

Extensions: The Portfolio Problem

- One of the classic problems in the theory of behavior under uncertainty is the issue of how much of his or her wealth a risk-averse investor should invest in a risky asset.
- Intuitively, it seems that the fraction invested in risky assets should be smaller for more risk-averse investors, and one goal of our analysis in these Extensions will be to show that formally.
- We then generalize the model to consider portfolios with many such assets, and finally working up to the Capital Asset Pricing model (CAPM), a staple of financial economics courses.

E7.1 Basic model with one risky asset

- Assume an individual has wealth, (W_0), to invest in one of two assets. The first asset yields a certain return of r_f, whereas the other asset's return is a random variable, r.

- Denote the amount invested in the risky asset by k, then this person's wealth at the end of one period is

$$
\begin{aligned}
W &= (W_0 - k)(1 + r_f) + k(1 + r) \\
&= W_0(1 + r_f) + k(r - r_f).
\end{aligned}
$$

where W is a random variable; k can be positive (buy) or negative (sell); k can be greater than W_0.

- The investor will choose k to maximize $E[U(W)]$. The first-order conditions is

$$\frac{\partial E[U(W)]}{\partial k} = \frac{\partial E[U(W_0(1+r_f)+k(r-r_f))]}{\partial k}$$
$$= E[U' \cdot (r-r_f)] = 0.$$

- If the investment does well, W will be large and U' will be relatively low because of diminishing marginal utility.
- If the investment does poorly, W will be relatively low and U' will be relatively high.
- In the expected value calculation of the first-order condition, negative outcomes for $r - r_f$ will be weighted more heavily than positive outcomes.

Two conclusions can be drawn from the first-order conditions.

- First, as long as $E(r - r_f) > 0$, an investor will choose positive amount of the risky asset.

  Notice that from the first-order condition, fairly large values of U' is required to be attached to situations where $r - r_f$ turns out to be negative. This can only happen if the investor owns positive amounts of the risky asset so that end-of-period wealth is low in such situations.

- The second conclusion is that investors who are more risk averse will hold smaller amounts of the risky asset.

  Because for risk-averse investors, U' rises rapidly as wealth falls. They need relatively little exposure to potential negative outcomes from holding the risky asset.

E7.2 CARA utility

- Suppose the investor's utility function is given by the CARA form: $U(W) = -e^{-AW}$. Then the marginal utility function is

$$
\begin{aligned}
U'(W) &= A\exp(-AW) \\
&= A\exp[-A(W_0(1+r_f) + k(r-r_f))] \\
&= A\exp[-AW_0(1+r_f)]\exp[-Ak(r-r_f)]
\end{aligned}
$$

- The optimality condition can be written as

$$
\begin{aligned}
E[U'\cdot(r-r_f)] &= A\exp[-AW_0(1+r_f)]\cdot \\
&\quad E[\exp(-Ak(r-r_f))\cdot(r-r_f)] = 0 \\
\text{or} &\quad E[\exp(-Ak(r-r_f))\cdot(r-r_f)] = 0.
\end{aligned}
$$

The solution for k will be independent of W_0.

- The CARA function implies that the fraction of wealth that an investor holds in risky assets (k/W_0)should decrease as wealth increases. Empirical data tend to show the fraction of wealth held in risky assets increasing with wealth.
- If the utility takes the CRRA rather than CARA form, it can be shown that all individuals with the same risk tolerance will hold the same fraction of wealth in risky assets, regardless of their absolute levels of wealth.
- However, the CRRA form utility function still can not explain why the fraction of wealth held in risky assets tends to increase with wealth.

E7.3 Portfolios of many risky assets

- Let the return on each of n risky assets be the random variable $r_i (i = 1, \cdots, n)$., with $E(r_i) = \mu_i$ and $Var(r_i) = \sigma_i^2$.
- An investor who invests a portion of the wealth in a portfolio of these assets will obtain a random return (r_p) given by

$$r_p = \sum_{i=1}^{n} \alpha_i r_i,$$

where $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$.
- The expected return is

$$E(r_p) = \mu_p = \sum_{i=1}^{n} \alpha_i \mu_i$$

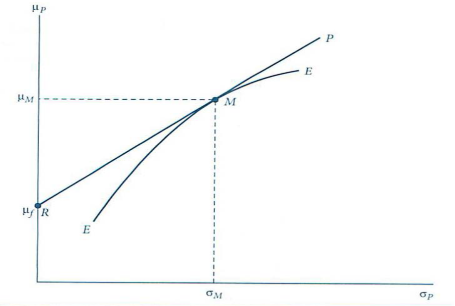
If the returns of each asset are independent, the variance of return is

$$Var(r_p) = \sigma_p^2 = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2.$$

E7.4 Optimal portfolios

- To solve for the optimal portfolio of just the risky assets, we choose a general set of weightings (the α_i) to minimize the variance of the portfolio for each potential expected return.
- The solution to this problem yields an "efficiency frontier" for risky asset portfolios such as line EE in Figure E7.1.

Figure E7.1 Efficient Portfolios

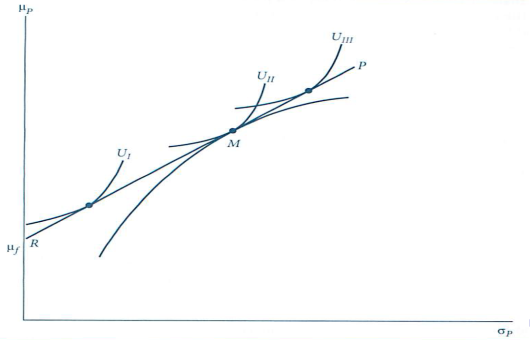


- Next, add a risk-free asset with expected return μ_f and $\sigma_f = 0$, shown as point R in Figure E7.1. Optimal portfolios will consist of mixture of this asset with risky ones.
- In equilibrium, M will be the "market portfolio" consisting of all capital assets held in proportion to their market valuations, with mean μ_M and variance σ_M^2.
- The equation for the line RP that represented any mixed portfolio is
$$\mu_p = \mu_f + \frac{\mu_M - \mu_f}{\sigma_M} \cdot \sigma_p.$$
- The market line RP permits individual investors to "purchase" returns in excess of the risk-free return $\mu_M - \mu_f$ by taking on proportionally more risk (σ_p/σ_M).
- For points to the left of the market point$M$, $\sigma_p/\sigma_M < 1$ and $\mu_F < \mu_p < \mu_M$. High-risk points to the right of M, $\sigma_p/\sigma_M > 1$ and $\mu_F < \mu_p > \mu_M$.

E7.5 Individual choices

- Individuals with low tolerance for risk (I) will opt for portfolios heavily weighted toward the risk-free asset.
- Investors willing to assume a modest degree of risk (II) will opt for portfolios close to the market portfolio.
- High-risk investors (III) may opt for leveraged portfolios

Figure E7.2 Investor Behavior and Risk Aversion



Mutual funds

- Mutual funds pool the funds of many individuals, they are able to achieve economies of scale in transactions and management costs.
- This permits fund owners to share in the fortunes of a much wider variety of equities.
- But mutual funds managers have incentives of their own, the portfolios they hold may not always be perfect representations of the risk attitudes of their clients.
- The classic investigation by Jensen(1968) finds that mutual funds managers are seldom able to attain extra returns large enough to offset the expenses they charge investors.
- This has led many mutual buyers to favor “index" funds that seek simply to duplicate the market average.

E7.6 Capital asset pricing model (CAPM)

- Consider a portfolio that combines a small amount (α) of an asset with a random return of x with the market portfolio, which has a random return of M.
- The return on this portfolio (z) would be

$$z = \alpha x + (1 - \alpha)M.$$

The expected return is

$$\mu_z = \alpha\mu_x + (1 - \alpha)\mu_M$$

with variance

$$\sigma_z^2 = \alpha^2\sigma_x^2 + (1 - \alpha)^2\sigma_M^2 + 2\alpha(1 - \alpha)\sigma_{x,M}$$

- Previous analysis also shows

$$\mu_z = \mu_f + (\mu_M - \mu_f) \cdot \frac{\sigma_z}{\sigma_M}.$$

Therefore,

$$\frac{\partial \mu_z}{\partial \alpha} = \mu_x - \mu_M = \frac{\mu_M - \mu_f}{\sigma_M} \cdot \frac{\partial \sigma_z}{\partial \alpha}$$

Since $\sigma_z = \left(\alpha^2\sigma_x^2 + (1-\alpha)^2\sigma_M^2 + 2\alpha(1-\alpha)\sigma_{x,M}\right)^{1/2}$,

$$\begin{aligned} \left.\frac{\partial \sigma_z}{\partial \alpha}\right|_{\alpha\to 0} &= \left.\frac{1}{2}\sigma_z^{-1/2}\left(2\alpha\sigma_x^2 - 2(1-\alpha)\sigma_M^2 + 2(1-2\alpha)\sigma_{x,M}\right)\right|_{\alpha\to 0} \\ &= \frac{1}{2}{\sigma_M^2}^{-1/2}\left(-2\sigma_M^2 + 2\sigma_{x,M}\right) = \frac{\sigma_{x,M} - \sigma_M^2}{\sigma_M} \end{aligned}$$

- Therefore

$$\mu_x - \mu_M = \frac{\mu_M - \mu_f}{\sigma_M}\left(\frac{\sigma_{x,M} - \sigma_M^2}{\sigma_M}\right) = (\mu_M - \mu_f)\left(\frac{\sigma_{x,M}}{\sigma_M^2} - 1\right)$$

$$\mu_x = \mu_f + (\mu_M - \mu_f) \cdot \frac{\sigma_{x,M}}{\sigma_M^2}$$

$$\mu_x = \mu_f + (\mu_M - \mu_f) \cdot beta$$

- The risk has a reward of $\mu_M - \mu_f$, and the quantity of risk is measured by $\frac{\sigma_{x,M}}{\sigma_M^2}$.
- This ratio of the covariance between the return x and the market to the variance of the market return is referred to as the *beta* coefficient for the asset.
- Estimated beta coefficients for financial assets are reported in many publications.