Part II: Choice and Demand

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Chapter 6
Demand Relationship among Goods

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2020.11.6.

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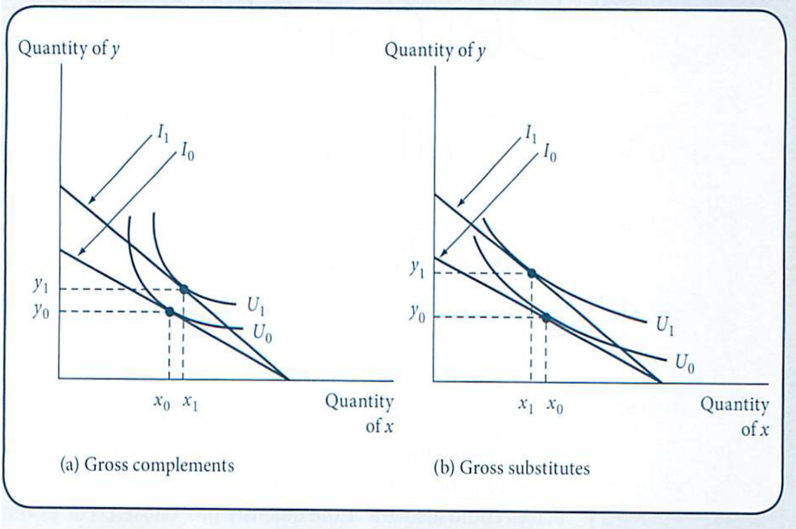
The Two-Good Case

Figure 6.1 shows how the quantity of x chosen might be affected by a decrease in the price of y?

- Shifting the budget constraint outward
- The quantity of good y chosen has increased
- In Figure 6-1(a), small substitution effect, x and p_y move in opposite directions. $\frac{\partial x}{\partial p_y} < 0$.
- In Figure 6-1(b), large substitution effect, x and p_y move in the same direction, $\frac{\partial x}{\partial p_y} > 0$.

Figure 6.1 Differing Directions of Cross-Price Effects



A mathematical treatment

- A Slutsky-type equation can be shown that

$$\begin{aligned}\frac{\partial x(p_x, p_y, I)}{\partial p_y} &= \text{substitution effect} + \text{income effect} \\ &= \left.\frac{\partial x}{\partial p_y}\right|_{U=constant} - y \cdot \frac{\partial x}{\partial I},\end{aligned}$$

or, in elasticity terms,

$$e_{x,p_y} = e_{x^c,p_y} - s_y e_{x,I}$$

where $\left.\frac{\partial x}{\partial p_y}\right|_{U=constant}$ and $y \cdot \frac{\partial x}{\partial I}$ are both positive if x is a normal good.

Example 6.1 Another Slutsky Decomposition for Cross-Price Effects

- In Example 5.4, the uncompensated and compensated demand functions for x are given by

$$\begin{aligned} x(p_x, p_y, I) &= \frac{0.5I}{p_x}, \\ x^c(p_x, p_y, V) &= V p_y^{0.5} p_x^{-0.5}. \end{aligned}$$

- The Marshallian demand function yields

$$\frac{\partial x}{\partial p_y} = 0$$

This is because the substitution and income effects of a price change are precisely counterbalancing.

- The substitution effect is

$$\left.\frac{\partial x}{\partial p_y}\right|_{U=constant} = \frac{\partial x^c}{\partial p_y} = 0.5Vp_y^{-0.5}p_x^{-0.5}$$
$$= 0.25Ip_y^{-1}p_x^{-1},$$

because $V = 0.5Ip_y^{-0.5}p_x^{-0.5}$

- Since $y = 0.5Ip_y^{-1}$, the income effect is

$$-y\frac{\partial x}{\partial I} = -\left[0.5Ip_y^{-1}\right] \cdot \left[0.5p_x^{-1}\right] = -0.25Ip_y^{-1}p_x^{-1}$$

- Thus, the total effect of the price change is

$$\frac{\partial x}{\partial p_y} = 0.25Ip_y^{-1}p_x^{-1} - 0.25Ip_y^{-1}p_x^{-1} = 0$$

Substitutes and Complements

- With many goods, the Slutsky equation for any two goods x_i, x_j is

$$\frac{\partial x_i(p_1, \cdots, p_n, I)}{\partial p_j} = \left.\frac{\partial x_i}{\partial p_j}\right|_{U=constant} - x_j \frac{\partial x_i}{\partial I},$$

Or, in elasticity form,

$$e_{i,j} = e^c_{i,j} - s_j e_{i,I}.$$

- Two goods are *substitutes* if one good may, as a result of changed conditions, replace the other in use. “Substitutes" substitute for one another in the utility function.
- *Complements* are goods that “go together." “Complements" complement each other in the utility function.

Gross (Marshallian) substitutes and complements

- Two goods, x_i and x_j, are said to be gross substitutes if

$$\frac{\partial x_i}{\partial p_j} > 0.$$

  An increase in the price of one good causes more of the other good to be bought.

- Two goods, x_i and x_j, are said to be gross complements if

$$\frac{\partial x_i}{\partial p_j} < 0.$$

  An increase in the price of one good causes less of the other good to be purchased.

Asymmetry of the gross definitions

- This definition is a "gross" definition in that it includes both income and substitution effects that arises from price changes.
- However, the gross definitions of substitutes and complements are not symmetric.
- It is possible, by the definitions, for x_1 to be a substitute for x_2 and at the same time, for x_2 to be a complement of x_1.
- The presence of income effects can produce paradoxical results such as the case in Example 6.2.

Example 6.2 Asymmetry in Cross-Price Effects

- Suppose a quasi-linear utility function $U(x, y) = \ln x + y$. The Lagrangian expression is

$$\mathcal{L} = \ln x + y + \lambda(I - p_x x - p_y y)$$

and the first-order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{1}{x} - \lambda p_x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= 1 - \lambda p_y = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= I - p_x x - p_y y = 0 \end{aligned}$$

Thus from the first two equations, $p_x x = p_y$.

- Substitution into the budget constraint to solve for the Marshallian demand function for y

$$
\begin{aligned}
I &= p_x x + p_y y = p_y + p_y y \\
y &= \frac{I - p_y}{p_y}
\end{aligned}
$$

and $x = \frac{p_y}{p_x}$. Therefore,

$$\frac{\partial x}{\partial p_y} > 0$$

x and y are gross substitutes.

- On the other hand, $\frac{\partial y}{\partial p_x} = 0$, thus x and y are independent.
- Relying on gross responses to price changes to define the relationship between x and y would therefore run into ambiguity.

Net (Hicksian) Substitutes and Complements

- Because of the possible asymmetries involved in the definition of gross substitutes and complements, an alternative definition that focuses only on substitution effects is often used.

- Goods x_i and x_j are said to be net substitutes if

$$\left.\frac{\partial x_i}{\partial p_j}\right|_{U=constant} > 0$$

and net complements if

$$\left.\frac{\partial x_i}{\partial p_j}\right|_{U=constant} < 0$$

- This definition looks only at the shape of the indifference curve. It is unambiguous because the definitions are perfectly symmetric.

$$\left.\frac{\partial x_i}{\partial p_j}\right|_{U=constant} = \left.\frac{\partial x_j}{\partial p_i}\right|_{U=constant}$$

- This symmetry can be shown using Shephard's lemma. Since

$$x_i^c(p_1, \cdots, p_n, V) = \frac{\partial E(p_1, \cdots, p_n, V)}{\partial p_i},$$

the substitution effect is

$$\left.\frac{\partial x_i}{\partial p_j}\right|_{U=constant} = \frac{\partial x_i^c}{\partial p_j} = \frac{\partial^2 E}{\partial p_j \partial p_i} = E_{ij}$$

- Apply Young's theorem to the expenditure fucntion

$$E_{ij} = E_{ji} = \frac{\partial x_j^c}{\partial p_i} = \frac{\partial x_j}{\partial p_i}\bigg|_{U=constant}$$

- The difference between Hicks' and Marshall's definitions of substitutes and complements is apparent in Figure 6.1.
- The two goods are always Hicks substitutes because of the convexity of the indifference curves so that $\frac{\partial x^c}{\partial p_y} > 0$.
- However, Marshall's definition involves income effects.
  - Figure 6.1(a): negative income effect dominates so that $\frac{\partial x}{\partial p_y} < 0$ and the two goods are "Marshall complements."
  - Figure 6.1(b): positive substitution effect dominates so that $\frac{\partial x}{\partial p_y} > 0$ and the goods are "Marshall substitutes."

Substitutability with Many Goods

- Once the utility-maximizing model is extended to many goods, a wide variety of demand patterns become possible.
- A major theoretical question that has concerned economists is whether substitutability or complementarity is more prevalent.
- *Hicks' second law of demand* states that "most" goods must be substitutes.

- The compensated demand function for a particular good, $x_i^c(p_1, \cdots, p_n, V)$, is homogeneous of degree 0 in all prices.
- Apply Euler's theorem yields

$$p_1 \cdot \frac{\partial x_i^c}{\partial p_1} + p_2 \cdot \frac{\partial x_i^c}{\partial p_2} + \cdots + p_n \cdot \frac{\partial x_i^c}{\partial p_n} = 0$$

- Dividing each term by x_i yields

$$e_{i1}^c + e_{i2}^c + \cdots + e_{in}^c = 0$$

- Since $e^c_{ii} \leq 0$ because of the negativity of the own-substitution effect. Hence it must be the case that

$$\sum_{j \neq i} e^c_{ij} \geq 0$$

- The sum of all the compensated cross-price elasticities for a particular good must be positive (or zero). This is the sense that “most" goods are substitutes.
- Empirical evidence seems generally consistent with this theoretical finding.

Composite Commodities

- In the most general case, an individual who consumes n goods will have demand functions that reflect $n(n+1)/2$ different substitution effects.
- It is often far more convenient to group goods into larger aggregates such as food, clothing, and so forth.
- We might wish to examine one specific good x and its relationship to "all other goods" y.
- This section shows the conditions under which this procedure can be defended.

Composite commodity theorem

- Suppose consumers choose among n goods, the demand for x_1 will depend on the prices of the other $n - 1$ commodities.
- Assuming that the prices of all of these other $n - 1$ goods move proportionally together, the "composite commodity theorem" states that we can define a single composite of all of these goods, y, so that the utility maximization problem can be compressed into a simpler problem of choosing only between x_1 and y.
- Let $p_2^0, p_3^0, \cdots, p_n^0$ represent the initial prices, assume that they can only vary together so that the relative price of $x_2, \cdots, x_n$ would not change.

- Define the composite commodity y to be total expenditures on $x_2, \cdots, x_n$ with the initial prices,

$$y = p_2^0 x_2 + p_3^0 x_3 + \cdots + p_n^0 x_n$$

then the initial budget constraint is

$$I = p_1 x_1 + p_2^0 x_2 + \cdots + p_n^0 x_n = p_1 x_1 + y$$

- Assume all prices change by a factor t ($t > 0$), the budget constraint becomes

$$I = p_1 x_1 + t p_2^0 x_2 + \cdots + t p_n^0 x_n = p_1 x_1 + t y$$

- Changes in p_1 or t induce the same kinds of substitution effects we have been analyzing.

Generalizations and limitations

- A **composite commodity** is a group of goods for which all prices move together.
- These goods can be treated as a single commodity.
- The individual behaves as if he is choosing between other goods and spending on this entire composite group.
- One must be rather careful in applying the theorem to the real world because its conditions are stringent. Finding a set of commodities whose prices move together is rare.

Example 6.3 Housing Costs as a Composite Commodity

- Suppose that an individual receives utility from three goods:
  - Food (x)
  - Housing services (y), measured in hundreds of square feet
  - Household operations (z), measured by electricity use
- Suppose the three-good CES utility function is

$$U(x, y, z) = -\frac{1}{x} - \frac{1}{y} - \frac{1}{z}$$

the Lagrangian expression is

$$\mathcal{L} = -\frac{1}{x} - \frac{1}{y} - \frac{1}{z} + \lambda(I - p_x x - p_y y - p_z z)$$

- The first-order conditions are

$$\begin{aligned} \frac{1}{x^2} - \lambda p_x &= 0 \\ \frac{1}{y^2} - \lambda p_y &= 0 \\ \frac{1}{z^2} - \lambda p_z &= 0 \\ I - p_x x - p_y y - p_z z &= 0 \end{aligned}$$

The first three equations yield

$$y = \sqrt{\frac{p_x}{p_y}} x, \qquad z = \sqrt{\frac{p_x}{p_z}} x,$$
$$p_y y = \sqrt{p_x p_y} x, \qquad p_z z = \sqrt{p_x p_z} x,$$

- The Marshallian demand functions are

$$
\begin{aligned}
x &= \frac{I}{p_x + \sqrt{p_x p_y} + \sqrt{p_x p_z}} \\
y &= \frac{I}{p_y + \sqrt{p_y p_x} + \sqrt{p_y p_z}} \\
z &= \frac{I}{p_z + \sqrt{p_z p_x} + \sqrt{p_z p_y}}
\end{aligned}
$$

- Suppose initially $I = 100$, $p_x = 1$, $p_y = 4$, $p_z = 1$, then the quantity demanded are $x^* = 25$, $y^* = 12.5$, $z^* = 25$.

- \$25 is spent on food and a total of \$75 is spent on housing-related (y and z) needs.

- If we assume that p_y and p_z always move together, then we can define the "composite commodity" housing (h) as
$$h = 4y + 1z,$$
and arbitrarily define the initial price of housing p_h to be 1. The initial quantity of housing is
$$h = 4 \cdot 12.5 + 1 \cdot 25 = 75$$
- Because p_y and p_x always move together, p_h will always be related to these prices by
$$p_h = p_z = 0.25 p_y$$
- The demand function for x becomes
$$x = \frac{I}{p_x + \sqrt{4 p_x p_h} + \sqrt{p_x p_h}} = \frac{I}{p_x + 3\sqrt{p_x p_h}}$$
- As before, $I = 100$, $p_x = 1$, $p_h = 1$, thus $x^* = 25$. Total spending on housing is $p_h h^* = I - p_x x^* = 75$, $h^* = 75$.

- **An increase in housing costs.** If p_y and p_z increase proportionally to $p_y = 16$ and $p_z = 4$, then p_h would also increase to 4. The demand for x would decrease to

$$x^* = \frac{100}{1 + 3\sqrt{4}} = \frac{100}{7}$$

and the housing purchases would be

$$p_h h^* = 100 - \frac{100}{7} = \frac{600}{7}$$

or, because $p_h = 4$,

$$h^* = \frac{150}{7}.$$

- With $I = 100$, $p_x = 1$, $p_y = 16$, $p_z = 4$, we can solve for the original model and the demand for these goods would be

$$x^* = \frac{100}{7}$$
$$y^* = \frac{100}{28}$$
$$z^* = \frac{100}{14},$$

so the total amount of the composite good housing consumed is

$$h^* = 4y^* + 1z^* = \frac{150}{7}$$

This is precisely the level of housing purchases under the composite good model.

Home Production, Attributes of Goods, and Implicit Prices

- Thus far in this chapter we have focused on what economists can learn about the relationships among goods by observing individuals' changing consumption of these goods in reaction to changes in market prices.
- To develop a deeper understanding of such questions, economist have sought to explore activities such as parental child care or meal preparation within individuals' households.

Household production model

- Suppose that individuals do not receive utility directly from goods they purchase in the market.
- Utility is received when the individual produces goods by combining market goods with time inputs.
- In formal terms, assume that there are three goods that a person might purchase in the market: x, y, and z. Purchasing these goods provides no direct utility.
- The goods can be combined by the individual to produce either of two home-produced goods: a_1 or a_2.

- The technology of this household production can be represented by the production functions f_1 and f_2. Therefore,

$$a_1 = f_1(x, y, z)$$
$$a_2 = f_2(x, y, z)$$

and

$$\text{utility} = U(a_1, a_2)$$

- The individual's goal is to choose x, y, z so as to maximize utility subject to the production constraints and to a financial budget constraint

$$p_x x + p_y y + p_z z = I$$

Two insights can be drawn from the model.

- The production functions are measurable using detailed data on household operations, Households can be treated as "multi-product" firms and studied using many of the techniques economist used to study production,
- Consuming more a_1 requires more use of x, y, and z, this activity has an opportunity cost measured in terms of the quantity of a_2 that can be produced. This is an implicit price.

The linear attributes model

- A particularly simple form of the household production model was first developed by K, J, Lancaster in 1966 to examine the underlying "attributes" of goods.
- In this model, it is the attributes of goods that provide utility to individuals, and each specific good contains a fixed set of attributes.
- For example, if we focus only on the calories (a_1) and vitamins (a_2) that various foods provide. And assumes that the "production" equations for a_1 and a_2 have the simple form

$$a_1 = a_x^1 x + a_y^1 y + a_z^1 z$$
$$a_2 = a_x^2 x + a_y^2 y + a_z^2 z$$

- If the individual spends all of his or her income on good x:

$$x^* = \frac{I}{p_x},$$

and that will yield

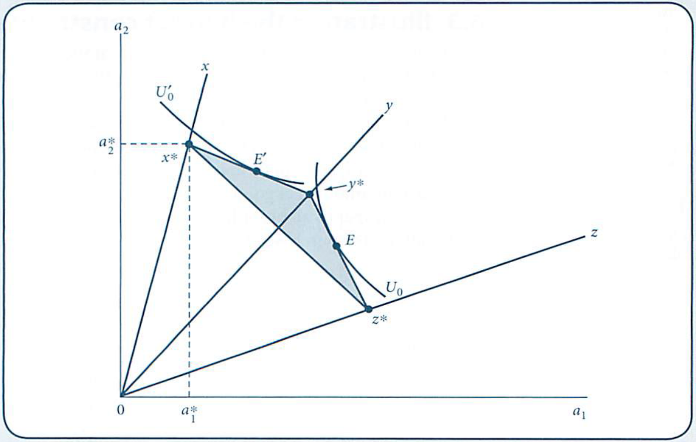
$$a_1^* = a_x^1 x^* = \frac{a_x^1 I}{p_x}$$
$$a_2^* = a_x^2 x^* = \frac{a_x^2 I}{p_x}$$

This point is recorded as point x^* on the Ox ray in Figure 6.2. Similarly for y^* and z^* in the figure.

- Bundles of a_1 and a_2 that are obtainable by purchasing both x and y are represented by the line joining x^* and y^* in Figure 6.2. Similarly for line x^*z^* and line y^*z^*.

Figure 6.2 Utility Maximization in the Attributes Model

The points x^*, y^*, and z^* show the amounts of attributes a_1 and a_2 that can be purchased by buying only x, y, or z, respectively. The shaded area shows all combinations that can be bought with mixed bundles. Some individuals may maximize utility at E, others at E'.



Corner solutions

- From Figure 6.2, a utility-maximizing individual would never consume positive quantities of all three of these goods.
- The attribute model predicts that corner solutions will be relatively common, especially in cases where individuals attach value to fewer attributes than there are market goods to choose from.
- Consumption patterns may change abruptly if income, prices, or preferences change.
- This is a direct result of the linear assumptions inherent in the production functions assumed here.

Extension: Simplifying Demand and Two-Stage Budgeting

- There are two general ways to simplify the problem of utility maximization.
- The first uses the composite commodity theorem to aggregate goods into categories within which relative prices moves together.
- For situations where relative prices changes within a category of spending, however, this process will not do.

- An alternative is to assume that consumers engage in a two-stage process.
  - First they allocate income to various broad groupings (e.g. food, clothing) of goods.
  - Then, given these expenditure constraints, they maximize utility within each of the subcategories of goods using only information about those goods' relative prices.
  - This process is called *two-stage budgeting*.

E6.1 Theory of two-stage budgeting

- Issues that arises in two-stage budgeting can be stated as: Does there exist a partition of goods into m non-overlapping groups ($r = 1, m$) and a separate budget (I_r) devoted to each category such that demand functions for the goods within any one category depend only on the prices of goods within the category and on the category's budget allocation.

- That is, can we partition goods so that demand is given by

$$x_i(p_1, \cdots, p_n, I) = x_{i \in r}(p_{i \in r, I_r})$$

for $r = 1, m$?

E6.2 Relation to the composite commodity theorem

- Unfortunately neither of the two available approaches to demand simplification is completely satisfying.
- Economists have tried to devise even more elaborate, hybrid methods of aggregation among goods.
- Lewbel (1996) shows how the composite theorem might be generalized to cases where within-group relative price exhibit considerable variability.
- He uses this generalization for aggregating U.S. consumer expenditures into six large groups (i.e. food, clothing, household operation, medical care, transportation, and recreation).
- He concludes that his procedure is much more accurate than assuming two-stage budgeting among these expenditure categories.

E6.3 Homothetic functions and energy demand

- One way to simplify the study of demand when there are many commodities is to assume that utility for certain sub-categories of goods is homothetic and may be separated from the demand for other commodities.
- Joegenson, Slednick and Stoker (1997) assumed that demand functions for specific types of energy are proportional to total spending on energy.
- The authors were able to estimate the price elasticities of demand for various type of energy.
- They conclude that most types of energy (i.e. electricity, natural gas, gasoline) have fairly elastic demand functions. Demand appears to be most responsive to price for electricity.