Part II: Choice and Demand

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Chapter 5
Income and Substitution Effects
Part I

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Demand Functions

Changes in Income

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A Mathematical Development of Response to Price Changes

Demand Functions

- The optimal levels of $x_1, x_2, \cdots, x_n$ (and λ, the Lagrange multiplier) can be expressed as functions of all prices and income.
- These can be expressed as n demand functions of the form

$$\begin{aligned} x_1^* &= x_1(p_1, p_2, \cdots, p_n, I) \\ x_2^* &= x_2(p_1, p_2, \cdots, p_n, I) \\ &\vdots \\ x_n^* &= x_n(p_1, p_2, \cdots, p_n, I) \end{aligned}$$

- For the case of only two goods, x and y,

$$\begin{aligned} x^* &= x(p_x, p_y, I) \\ y^* &= y(p_x, p_y, I) \end{aligned}$$

- Prices and income are "exogenous." The individual has no control over these parameters.
- Changes in the parameters will shift the budget constraint and cause this person to make different choices.
- In this chapter we will be looking at the partial derivative $\partial x / \partial I$ and $\partial x / \partial p_x$ for any arbitrary good x.

Homogeneity

- If we were to double all prices and income, then the optimal quantities demanded would remain unchanged.

$$x_i^* = x_i(p_1, p_2, \cdots, p_n, I) = x_i(tp_1, tp_2, \cdots, tp_n, tI)$$

  for any $t > 0$.
- Individual demand functions are homogeneous of degree zero in all prices and income.

Example 5.1 Homogeneity

- Homogeneity of demand is a direct result of the utility-maximization assumption.
- For example, if an individual's utility function for food (x) and housing (y) is given by the Cobb-Douglas function

$$U(x, y) = x^{0.3} y^{0.7},$$

the demand functions can be derived as

$$\begin{aligned} x^* &= \frac{0.3I}{p_x}, \\ y^* &= \frac{0.7I}{p_x}. \end{aligned}$$

- These functions exhibit homogeneity because a doubling of all prices and income leave x^* and y^* unaffected.

- If the individual's preferences were reflected by the CES function

$$U(x, y) = x^{0.5} + y^{0.5},$$

then (as shown in Example 4.2) the demand functions are given by

$$\begin{aligned} x^* &= \left(\frac{1}{1 + p_x/p_y}\right) \cdot \frac{I}{p_x}, \\ y^* &= \left(\frac{1}{1 + p_y/p_x}\right) \cdot \frac{I}{p_y}. \end{aligned}$$

- Both of these demand functions are homogeneous of degree 0. A doubling of p_x, p_y, and I would leave x^* and y^* unaffected.

Changes in Income

- An increase in income will cause the budget constraint shift out in a parallel fashion.
- Relative prices p_x/p_y does not change, the MRS will stay constant as the individual moves to higher levels of satisfaction.

Normal and inferior goods

- Normal good: Over some range of income, a good x_i for which $\frac{\partial x_i}{\partial I} \geq 0$ over that range of income.
- Inferior good:Over some range of income, a good x_i for which $\frac{\partial x_i}{\partial I} < 0$ over that range of income.

Figure 5.1 Effect of an Increase in Income on x and y Chosen

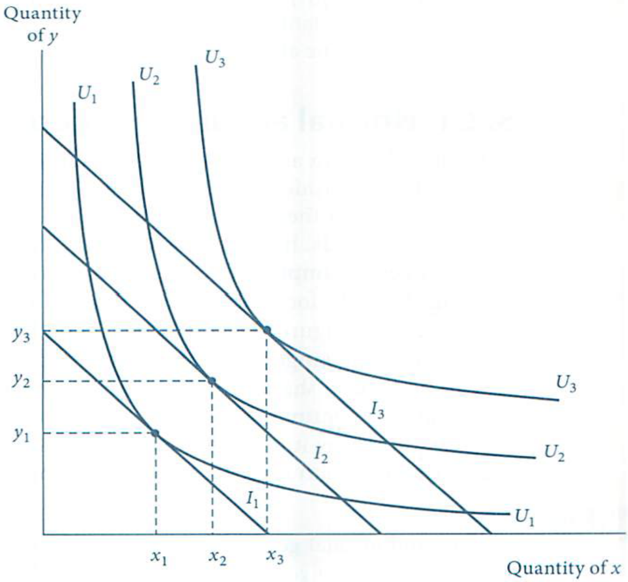
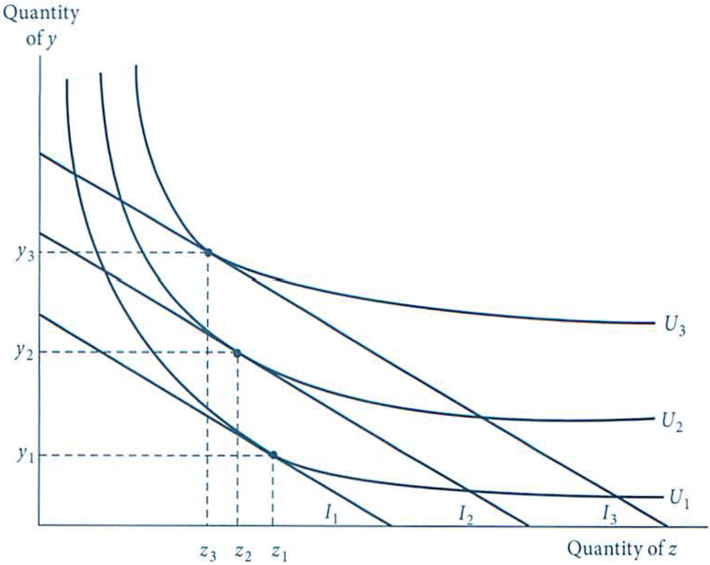


Figure 5.2 An Indifference Curve Map Exhibiting Inferiority



Changes in a Good's Price

- A change in the price of a good alters the slope of the budget constraint, and thus changes the MRS at the consumer's utility-maximizing choices.
- Therefore, when a price changes, two analytically different effects come into play.
- Substitution effect: Even if the individual were to stay on the same indifference curve, consumption patterns would be allocated so as to equate the MRS to the new price ratio.
- Income effect: Arises because a price change necessarily changes an individual's "real" income. The individual cannot stay on the initial indifference curve and must move to a new one.

Figure 5.3 Demonstration of Income and Substitution Effects of a Decrease in the Price of x

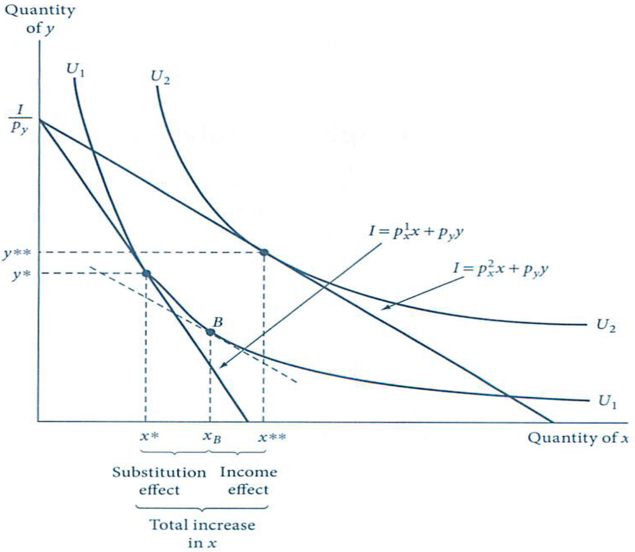
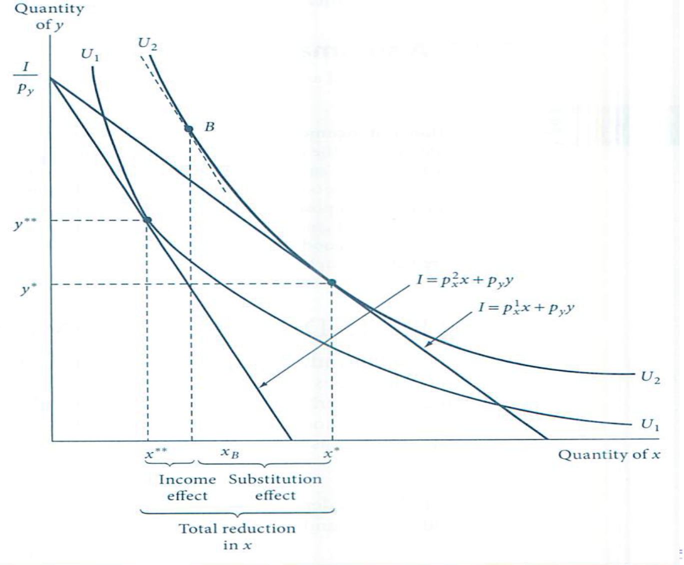


Figure 5.4 Demonstration of Income and Substitution Effects of an Increase in the Price of x



- If a good is normal, substitution and income effects reinforce one another.

When $p \downarrow$:

- Substitution effect $\Rightarrow$ quantity demanded $\uparrow$,
  the individual moves along an indifference curve.

- Income effect $\Rightarrow$ quantity demanded $\uparrow$,
  the resulting rise in purchasing power allows the individual to move to a higher indifference curve.

- Overall, the result of the price decrease is to cause more x to be demanded.

- If a good is inferior, substitution and income effects move in opposite directions.
- When $p \downarrow$:
  - Substitution effect $\Rightarrow$ quantity demanded $\uparrow$,
  - Income effect $\Rightarrow$ quantity demanded $\downarrow$,
- When $p \uparrow$:
  - Substitution effect $\Rightarrow$ quantity demanded $\downarrow$
  - Income effect $\Rightarrow$ quantity demanded $\uparrow$,
- The substitution effect and income effect move in opposite directions, no definite prediction can be made for changes in price.
- If the income effect outweighs the substitution effect, we have a case of Giffen's paradox.

Giffen's paradox

- An increase in price leads to a drop in real income. If the good is inferior, a drop in income causes quantity demanded to rise.
- If the income effect of a price change is strong enough, there could be a positive relationship between price and quantity demanded.
- English economist Robert Giffen observed this paradox in nineteenth-century Ireland: When the price of potatoes rose, people reportedly consumed more of them.
- Potatoes were not only inferior goods but they also used up a large portion of the Irish people's income. An increase in the price of potatoes forced the Irish to cut back on other luxury food consumption to buy more potatoes.

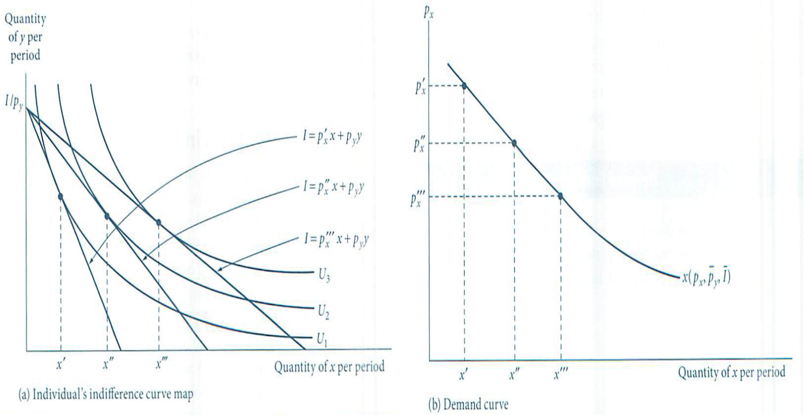
The Individual's Demand Curve

- An individual's demand for x depends on preferences, all prices, and income:

$$x^* = x(p_x, p_y, I)$$

- It may be convenient to graph it assuming that income (I) and the price of y (p_y) are held constant.
- An individual demand curve shows the relationship between the price of a good and the quantity of that good purchased by an individual, assuming that all other determinants of demand are held constant.

Figure 5.5 Construction of an Individual's Demand Curve



Shifts in the demand curve

- Three factors are held constant when a demand curve is derived
  - Income
  - Prices of other goods (p_y)
  - The individual's preferences
- If any of these factors change, the demand curve will shift to a new position.
- *A change in demand*
  - Shift in the demand curve, caused by changes in income, prices of other goods, or preferences
- *A change in quantity demanded*
  - Movement along a given demand curve, caused by a change in the price of the good

Example 5.2 Demand Functions and Demand Curves

- For the Cobb-Douglas utility function from Example 5.1, the demand functions are

$$x = \frac{0.3I}{p_x}, y = \frac{0.7I}{p_y}.$$

- If the individual's income is \$100, these functions became

$$x = \frac{30}{p_x}, y = \frac{70}{p_y}.$$

- The demand curves for these two goods, $p_x x = 30$, $p_y y = 70$, are simple hyperbolas.
- An increase in income would shift both of the demand curves outward.
- The demand curve for x is not shifted by changes in p_y and vice versa.

- For the CES utility function in Example 5.1, the demand function for x is

$$x = \left(\frac{1}{1 + p_x / p_y} \right) \cdot \frac{I}{p_x}$$

- Assume $I = 100$ and $p_y = 1$, the demand function becomes

$$x = \frac{100}{p_x^2 + p_x}$$

The demand curve would be relatively flatter because substitution effects are larger than in the Cobb-Douglas case.

- From the demand function, we also know that

$$\frac{\partial x}{\partial I} = \left(\frac{1}{1 + p_x/p_y}\right) \cdot \frac{1}{p_x} > 0$$

and

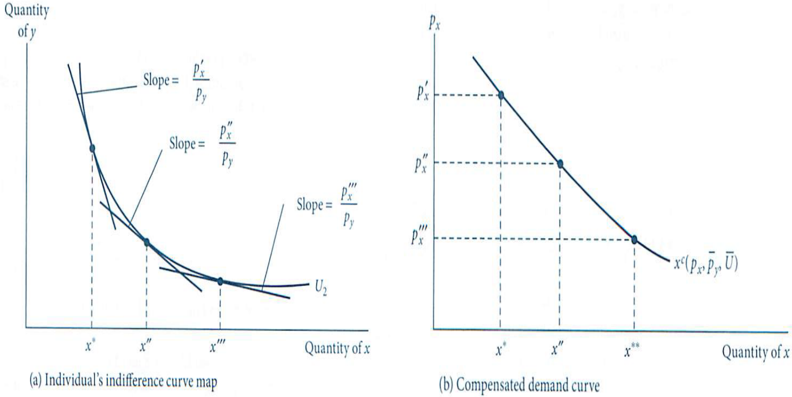
$$\frac{\partial x}{\partial p_y} = \frac{I}{(p_x + p_y)^2} > 0,$$

thus increases in I and p_y would shift the demand curve for good x outward.

Compensated (Hicksian) Demand Curves and Functions

- In Figure 5.5, the level of utility varies along the demand curve. As p_x falls, the individual moves to higher indifference curves.
- Nominal income is held constant as the demand curve is derived. "Real" income rises as the price of x falls.
- An alternative approach holds real income (or utility) constant while examining reactions to changes in p_x.
- The effects of the price change are "compensated" so as to force the individual to remain on the same indifference curve. Thus, reactions to price changes include only substitution effects.

Figure 5.6 Construction of an Compensated Demand Curve



- A *Compensated (Hicksian) demand curve* shows the relationship between the price of a good and the quantity purchased, assuming that other prices and utility are held constant.
- Therefore, the curve (which is termed a *Hicksian demand curve* after the British economist John Hicks) illustrates only substitution effects.
- The curve is a two-dimensional representation of the *compensated demand function*

$$x^c = x^c(p_x, p_y, U)$$

- The major difference between compensated and uncompensated demand curves is whether utility or income is held constant in constructing the curves.

Shephard's lemma

- One remarkable result from duality theory called *Shephard's lemma* was pioneered by R.W. Shephard for the use of duality theory in production and cost functions. (See Chapters 9 and 10).

- Lagrangian expression for the dual expenditure minimization problem is

$$\mathcal{L} = p_x x + p_y y + \lambda[U(x, y) - \overline{U}]$$

- The solution to this problem yields the expenditure function

$$E(p_x, p_y, U)$$

- Applying envelope theorem yields

$$\frac{dE(p_x, p_y, U)}{dp_x} = \frac{\partial \mathcal{L}}{\partial p_x} = x^c(p_x, p_y, U)$$

- That is, compensated demand function for a good can always be found from the expenditure function by differentiation with respect to the good's price.
- It has been shown in chapter 4 that the expenditure function must be concave in prices. In other words,

$$\frac{\partial^2 E(p_x, p_y, V)}{\partial p_x^2} < 0$$

Since

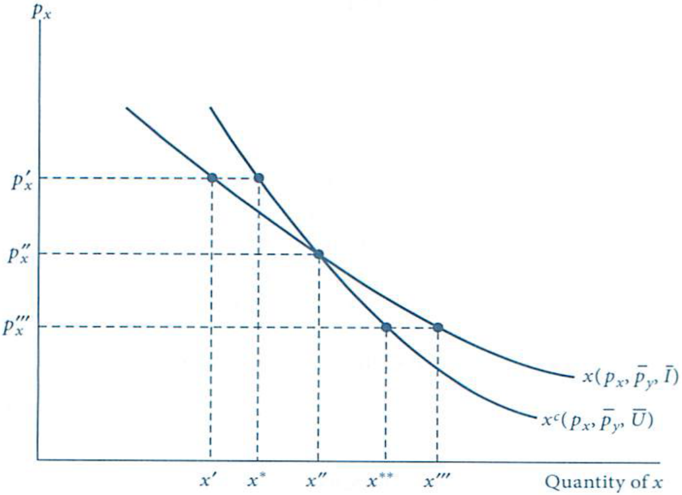
$$\frac{\partial^2 E(p_x, p_y, U)}{\partial p_x^2} = \frac{\partial[\partial E(p_x, p_y, V)/\partial p_x]}{\partial p_x} = \frac{\partial x^c(p_x, p_y, V)}{\partial p_x} < 0$$

Hence the compensated demand curve must have a negative slope.

Relationship between compensated and uncompensated demand curves

- For a normal good, compensated demand curve is less responsive to price changes than is the uncompensated demand curve.
- Uncompensated demand curve reflects both income and substitution effects, whereas compensated demand curve reflects only substitution effects.
- In most empirical work, uncompensated demand curves are used because the data on prices and nominal incomes needed to estimate them are readily available.
- However, for some theoretical purposes, compensated demand curves are a more appropriate concept because the ability to hold utility constant offers some advantages.

Figure 5.7 Comparison of Compensated and Uncompensated Demand Curves



Example 5.3 Compensated Demand Functions

- Suppose the utility function for hamburgers (y) and soft drinks (x) is

$$U(x, y) = x^{0.5} y^{0.5},$$

then the Marshallian demand functions are

$$\begin{aligned} x(p_x, p_y, I) &= \frac{0.5I}{p_x} \\ y(p_x, p_y, I) &= \frac{0.5I}{p_y} \end{aligned}$$

- The indirect utility function is

$$V(p_x, p_y, I) = \frac{0.5I}{p_x^{0.5} p_y^{0.5}}$$

- Therefore, the expenditure functions is

$$\mathrm{E}(p_x, p_y U) = 2p_x^{0.5} p_y^{0.5} U.$$

- We can use Shephard's lemma to calculate the compensated demand functions as

$$\begin{aligned} x^c(p_x, p_y, U) &= \frac{\partial E(p_x, p_y, U)}{\partial p_x} = p_x^{-0.5} p_y^{0.5} U \\ y^c(p_x, p_y, U) &= \frac{\partial E(p_x, p_y, U)}{\partial p_y} = p_x^{0.5} p_y^{-0.5} U. \end{aligned}$$

A Mathematical Development of Response to Price Changes

- Our goal is to examine the partial derivative $\partial x/\partial p_x$— that is, how a change in price of a good affects its purchase.

Direct approach

- Comparative static methods by differentiating the first order conditions for a maximum with respect to p_x.
- This approach is cumbersome and provides little economic insights.

Indirect approach

- Assume there are only two goods, x and y and focus on the compensated demand function, $x^c(p_x, p_y, U)$, and its relationship to the ordinary demand function, $x(p_x, p_y, I)$.
- By definition,

$$x^c(p_x, p_y, U) \quad = \quad x[p_x, p_y, E(p_x, p_y, U)]$$

thus

$$\frac{\partial x^c}{\partial p_x} \quad = \quad \frac{\partial x}{\partial p_x} + \frac{\partial x}{\partial E} \cdot \frac{\partial E}{\partial p_x},$$

or

$$\frac{\partial x}{\partial p_x} \quad = \quad \frac{\partial x^c}{\partial p_x} - \frac{\partial x}{\partial E} \cdot \frac{\partial E}{\partial p_x}.$$

$$\frac{\partial x}{\partial p_x} = \frac{\partial x^c}{\partial p_x} - \frac{\partial x}{\partial E} \cdot \frac{\partial E}{\partial p_x}$$

- The substitution effect.
  The first term $\frac{\partial x^c}{\partial p_x}$ is the slope of the compensated demand curve.
- The income effect.
  The second term measures the way in which changes in p_x affect the demand for x through changes in purchasing power.

The Slutsky equation

- The substitution effect can be written as

$$\text{substitution effect} = \frac{\partial x^c}{\partial p_x} = \left.\frac{\partial x}{\partial p_x}\right|_{U=constant}$$

- The income effect is

$$\text{income effect} = -\frac{\partial x}{\partial E} \cdot \frac{\partial E}{\partial p_x} = -\frac{\partial x}{\partial I} \cdot \frac{\partial E}{\partial p_x} = -x^c \frac{\partial x}{\partial I}$$

- The Slutsky equation is

$$\frac{\partial x(p_x, p_y, I)}{\partial p_x} = \left.\frac{\partial x}{\partial p_x}\right|_{U=constant} - x\frac{\partial x}{\partial I}$$

since $x^c(p_x, p_y, U) = x(p_x, p_y, I)$ at the utility maximizing point.

- The substitution effect is always negative as long as MRS is diminishing. The slope of the compensated demand curve must be negative.
- If x is a normal good, then $\partial x/\partial I > 0$, the entire income effect is negative. If x is an inferior good, then $\partial x/\partial I < 0$, the entire income effect is positive.
- The overall impact of a change in the price of a good is ambiguous. It all depends on the relative size of the effects.
- It is possible that, in the inferior good case, the second term could dominate the first, leading to Giffen's paradox. $(\partial x/\partial p_x > 0)$.

Example 5.4 A Slutsky Decomposition

- For the Cobb-Douglas utility functio $U(x, y) = x^{0.5}y^{0.5}$, the Marshallian demand function for good x is

$$x(p_x, p_y, I) = \frac{0.5I}{p_x}$$

The indirect utility function is

$$V(p_x, p_y, I) = 0.5p_x^{-0.5}p_y^{-0.5}I$$

and the compensated demand function is

$$x^c(p_x, p_y, U) = p_x^{-0.5}p_y^{0.5}U$$

- Hence, the total effect of a price change on Marshallian demand is

$$\frac{\partial x(p_x, p_y, I)}{\partial p_x} = \frac{-0.5I}{p_x^2}$$

- The substitution effect is

$$\frac{\partial x^c(p_x, p_y, U)}{\partial p_x} = -0.5 p_x^{-1.5} p_y^{0.5} U$$
$$= -0.5 p_x^{-1.5} p_y^{0.5} V = -0.25 p_x^{-2} I$$

- The income effect is

$$-x\frac{\partial x}{\partial I} = -\left[\frac{0.5I}{p_x}\right] \cdot \frac{0.5}{p_x} = -\frac{0.25I}{p_x^2}$$

- Therefore,

$$\frac{\partial x^c(p_x, p_y, U)}{\partial p_x} - x\frac{\partial x}{\partial I} = -\frac{0.5I}{p_x^2} = \frac{\partial x(p_x, p_y, I)}{\partial p_x}$$

- Note that in the case of Cobb-Douglas utility function, the substitution and income effect are of the same size.