Part II: Choice and Demand

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Chapter 4
Utility Maximization and Choice

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- This chapter examines the basic model of choice that economists use to explain individuals' behavior.
- Individuals are assumed to behave as though they maximize utility subject to a budget constraint.
- To maximize utility, individuals will choose bundles of commodities for which the rate of trade-off between any two goods (the MRS) is equal to the ratio of the goods' market prices.
- Market prices convey information about opportunity costs to individuals, and this information plays an important role in affecting the choices actually made.

Two complaints non-economists often make about the economic approach.

1. No real person can make the kinds of "lightning calculations" required for utility maximization?

- The pool player also can not make the lightning calculations required to plan a shot according to the laws of physics, but the laws still predict the player's behavior.
- The utility-maximization model predicts many aspects of behavior.
- Economists assume that people behave as if they made such calculations; thus, the complaint that the calculations can not possibly be made is largely irrelevant.

2. The economic model of choice is extremely selfish. No one has such solely self-centered goals.

- Nothing in the utility-maximization model prevents individuals from deriving satisfaction from philanthropy or generally "doing good."
- Economists have used the utility-maximization model to study such issues as donating time and money to charity, leaving bequests to children, or even giving blood.
- One need not take a position on whether such activities are selfish or selfless because economists doubt people would undertake them if they were against their own best interests, broadly conceived.

An Initial Survey

- **Utility maximization**: To maximize utility, given a fixed amount of income to spend, an individual will buy those quantities of goods that exhaust his or her total income, and for which the psychic rate of trade-off between any two goods (the MRS) is equal to the rate at which the goods can be traded one for the other in the marketplace.

$$\begin{aligned} MRS_{xy} = \frac{MU_x}{MU_y} &= \frac{p_x}{p_y}, \\ \text{or } \frac{MU_x}{p_x} &= \frac{MU_y}{p_y} \end{aligned}$$

The Two-Good Case: A Graphical Analysis

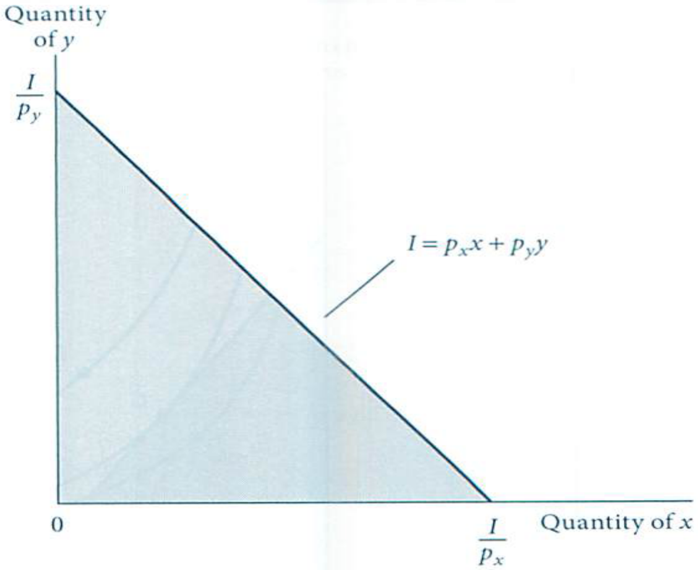
Budget constraint

- Assume that the individual has I dollars to allocate between good x and good y.
- If p_x is the price of x and p_y is the price of y, then the individual is constrained by

$$p_x x + p_y y \leq I$$

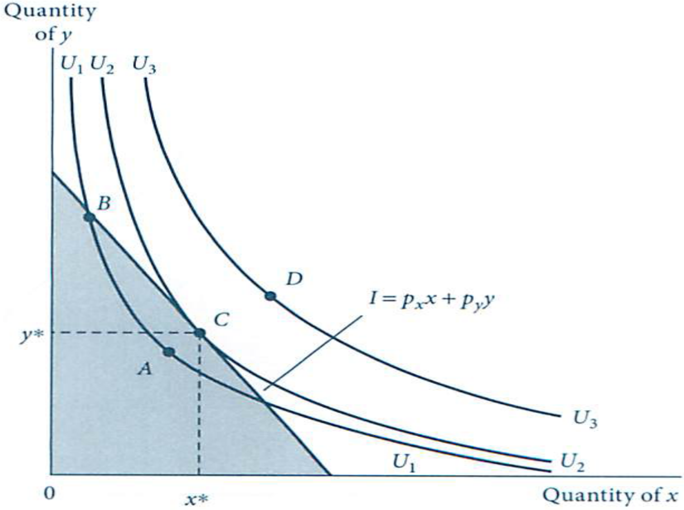
- The slope of the constraint is $-\frac{p_x}{p_y}$. This slope shows how y can be traded for x in the market.

Figure 4.1 The Individual's Budget Constraints for Two Goods



First-order conditions for a maximum

Figure 4.2 A Graphical Demonstration of Utility Maximization



- C is a point of tangency between the budget constraint and the indifference curve. Therefore, at C we have

$$\begin{aligned} \text{slope of budget constraint} &= -\frac{p_x}{p_y} = \\ \text{slope of indifference curve} &= \left.\frac{dy}{dx}\right|_{U=\text{constant}} \end{aligned}$$

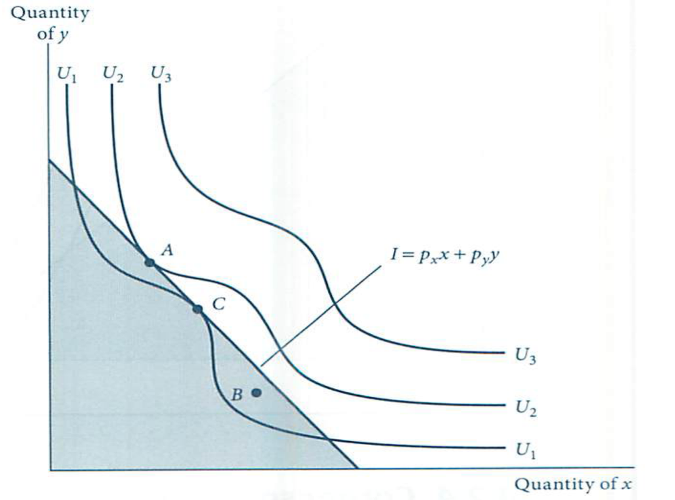
or

$$\frac{p_x}{p_y} = -\left.\frac{dy}{dx}\right|_{U=\text{constant}} = MRS \text{ (of } x \text{ for } y\text{)}$$

Second-order conditions for a maximum

- The tangency rule is necessary but not sufficient unless we assume that MRS is diminishing.
- If MRS is diminishing, then indifference curves are strictly convex. The condition of tangency is both a necessary and sufficient condition for a maximum.
- If MRS is not diminishing, we must check second-order conditions to ensure that we are at a maximum.

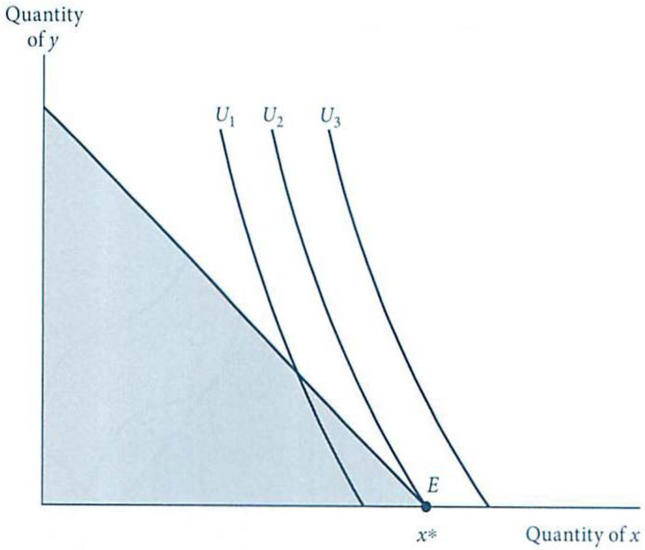
Figure 4.3 Example of an Indifference Curve Map for Which the Tangency Rule Does Not Ensure a Maximum



Corner solutions

- Individuals may maximize utility by choosing to consume only one of the goods.
- At the optimal point, E, in Figure 4.4, the budget constraint is flatter than the indifference curve.
- The rate at which x can be traded for y in the market is lower than the MRS.

Figure 4.4 Corner Solution for Utility Maximization



The n-Good Case

- The individual's objective is to maximize

$$\text{utility} = U(x_1, x_2, \cdots, x_n)$$

subject to the budget constraint

$$I = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n.$$

The Lagrangian expression is

$$\mathcal{L} = U(x_1, x_2, \cdots, x_n) + \lambda(I - p_1 x_1 - p_2 x_2 - \cdots - p_n x_n)$$

First-order conditions

- First-order conditions for an interior maximum ($n+1$ equations)

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{\partial U}{\partial x_1} - \lambda p_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= \frac{\partial U}{\partial x_2} - \lambda p_2 = 0 \\
\dots \\
\frac{\partial \mathcal{L}}{\partial x_n} &= \frac{\partial U}{\partial x_n} - \lambda p_n = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= I - p_1 x_1 - p_2 x_2 - \cdots - p_n x_n = 0
\end{aligned}$$

Implications of first-order conditions

- For any two goods, x_i and x_j, we have

$$\frac{\partial U/\partial x_i}{\partial U/\partial x_j} = \frac{p_i}{p_j}$$

- It has been shown that the ratio of the marginal utilities of two goods along an indifference curve is equal to the marginal rate of substitution between them, the conditions for an optimal allocation of income become

$$MRS(x_i \text{ for } x_j) = \frac{p_i}{p_j}.$$

This is exactly the result derived earlier.

Interpreting the Lagrange multiplier

- $$\lambda = \frac{\partial U/\partial x_1}{p_1} = \frac{\partial U/\partial x_2}{p_2} = \cdots = \frac{\partial U/\partial x_n}{p_n}$$

  λ is the marginal utility of an extra dollar of consumption expenditure. Or, the marginal utility of “income."
- Another way to rewrite the necessary condition

  $$p_i = \frac{\partial U/\partial x_i}{\lambda}$$

  for every i. At the margin, the price of a good represents the consumer’s evaluation of the utility of the last unit consumed.
- The price of a goods also represents how much the consumer is willing to pay for the last unit.

Corner solutions

- When corner solutions arise, the first-order conditions must be modified as

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial U}{\partial x_i} - \lambda p_i \leq 0 \quad (i = 1, \cdots, n),$$

$$\text{and if } \frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial U}{\partial x_i} - \lambda p_i < 0, \text{ then } x_i = 0$$

- This means that

$$p_i > \frac{\partial U / \partial x_i}{\lambda}.$$

In other words, any goods whose price(p_i) exceeds its marginal value to the consumer will not be purchased ($x_i = 0$).

Example 4.1 Cobb-Douglas Demand Functions

- The Cobb-Douglas utility function is given by

$$U(x, y) = x^{\alpha} y^{\beta}$$

where, for simplicity, we assume $\alpha + \beta = 1$.

- The Lagrangian expression

$$\mathcal{L} = x^{\alpha} y^{\beta} + \lambda(I - p_x x - p_y y)$$

yields the first-order conditios

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= \alpha x^{\alpha-1} y^{\beta} - \lambda p_x = 0 \\
\frac{\partial \mathcal{L}}{\partial y} &= \beta x^{\alpha} y^{\beta-1} - \lambda p_y = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= I - p_x x - p_y y = 0
\end{aligned}$$

- Taking the ratio of the first two terms shows that

$$\begin{aligned} \frac{\alpha y}{\beta x} &= \frac{p_x}{p_y} \\ \text{or } p_y y = \frac{\beta}{\alpha} p_x x &= \frac{1-\alpha}{\alpha} p_x x \end{aligned}$$

- Substituting into the budget constraint gives

$$\begin{aligned} I = p_x x + p_y y &= \frac{p_x x}{\alpha} \\ p_x x^* &= \alpha I \\ p_y y^* &= (1-\alpha) I = \beta I \\ x^* &= \frac{\alpha I}{p_x} \\ y^* &= \frac{\beta I}{p_y} \end{aligned}$$

- With Cobb-Douglas utility function, the individual will allocate α proportion of his or her income to good x and β proportion of his or her income to good y.
- Although this feature of the Cobb-Douglas function often makes it easy to work out simple problems, it does suggest that the function has limits in its ability to explain actual consumption behavior.
- Because the share of income devoted to particular goods often changes significantly in response to changing economic conditions, a more general functional form may provide insights not provided by the Cobb-Douglas function.

Numerical example. Suppose $p_x = 1$, $p_y = 4$, $I = 8$. Suppose also that $\alpha = \beta = 0.5$, then

$$\begin{aligned} x^* &= \frac{\alpha I}{p_x} = \frac{0.5I}{p_x} = \frac{0.5(8)}{1} = 4, \\ y^* &= \frac{\beta I}{p_y} = \frac{0.5I}{p_y} = \frac{0.5(8)}{4} = 1, \end{aligned}$$

and at these optimal choices,

$$\begin{aligned} U &= x^{0.5}y^{0.5} = (4)^{0.5}(1)^{0.5} = 2, \\ \lambda &= \frac{\alpha x^{\alpha-1}y^{\beta}}{p_x} = \frac{0.5(4)^{-0.5}(1)^{0.5}}{1} = 0.25 \end{aligned}$$

Example 4.2 CES Demand

- Three specific examples of the CES function to illustrate cases in which budget shares are responsive to relative prices.

Case 1: δ=0.5. In this case, $\sigma = 1/(1-\delta) = 2$, utility function is

$$U(x,y) = x^{0.5} + y^{0.5}$$

Setting up the Lagrangian expression

$$\mathcal{L} = x^{0.5} + y^{0.5} + \lambda(I - p_x x - p_y y)$$

yields the following first-order conditions for a maximum:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= 0.5x^{-0.5} - \lambda p_x = 0, \\
\frac{\partial \mathcal{L}}{\partial y} &= 0.5y^{-0.5} - \lambda p_y = 0, \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= I - p_x x - p_y y = 0.
\end{aligned}$$

Division of the first two equations shows that

$$\left(\frac{y}{x}\right)^{0.5} = \frac{p_x}{p_y}$$

$$p_y y = p_y x \left(\frac{p_x}{p_y}\right)^2 = p_x x \left(\frac{p_x}{p_y}\right)$$

Substituting this into the budget constraint, we have

$$p_x x + p_y y = p_x x + p_x x \left(\frac{p_x}{p_y}\right) = I$$

$$\text{and } x^* = \frac{I}{p_x[1 + (p_x/p_y)]}, y^* = \frac{I}{p_y[1 + (p_y/p_x)]}$$

- The share of income spent on good x depends on the price ratio p_x/p_y. The higher is the relative price of x, the smaller will be the share of income spent on x.

Case 2: δ=-1. In this case, $\sigma = 1/(1-\delta) = 0.5$, the utility function is given by

$$U(x, y) = -x^{-1} - y^{-1},$$

It can be show that the first-order conditions require

$$\frac{y}{x} = \left(\frac{p_x}{p_y}\right)^{0.5}$$

Substituting into the budget constraints, we have

$$\begin{aligned} p_x x &+ p_y \left(\frac{p_x}{p_y}\right)^{0.5} x = I \\ x^* &= \frac{I}{p_x + p_y (p_x/p_y)^{0.5}} \\ &= \frac{I}{p_x[1 + (p_y/p_x)^{0.5}]} \end{aligned}$$

$$x^* = \frac{I}{p_x[1 + (p_y/p_x)^{0.5}]}$$
$$y^* = \frac{I}{p_y[1 + (p_x/p_y)^{0.5}]}$$

These demand functions are less price responsive than the Cobb-Douglas function in two ways.

- The share of income spent on good x, $p_x x/I = 1/[1 + (p_y/p_x)^{0.5}]$, responds positively to increases in p_x.
- The demand functions are less price responsive than the Cobb-Douglas is also illustrated by the relatively small implied exponents of each good's own price (-0.5).

Case 3: $\delta = -\infty$. The utility function is, for example,

$$U(x, y) = \min(x, 4y)$$

A utility-maximizing person will choose only combinations of the two goods for which $x = 4y$. Substituting this condition into the budget constraint:

$$I = p_x x + p_y y = p_x x + p_y \frac{y}{4} = (p_x + 0.25 p_y)x.$$

Hence

$$x^* = \frac{I}{p_x + 0.25 p_y}$$

Similarly,

$$y^* = \frac{I}{4p_x + p_y}$$

- In this case, the share of a person's budget devoted to good x rises rapidly as the price of x increases because x and y must be consumed in fixed proportions.

$$\frac{p_x x^*}{I} = \frac{1}{1+0.25(p_y/p_x)}$$
$$\frac{p_y y^*}{I} = \frac{1}{1+4(p_x/p_y)}$$

Indirect Utility Function

- Examples 4.1 and .2 illustrates that it is often possible to manipulate first-order conditions to solve for optimal values of $x_1, x_2, \cdots, x_n$.
- These optimal values in general will depend on the prices of all the goods and on the individual's income. That is,

$$
\begin{aligned}
x_1^* &= x_1(p_1, p_2, \cdots, p_n, I), \\
x_2^* &= x_2(p_1, p_2, \cdots, p_n, I), \\
&\vdots \\
x_n^* &= x_n(p_1, p_2, \cdots, p_n, I).
\end{aligned}
$$

- We can use the optimal values of the x's to find the indirect utility function.

$$\begin{aligned} & \text{maximum utility} \\ = \quad & U[x_1^*(p_1, \cdots, p_n, I), x_2^*(p_1, \cdots, p_n, I), \cdots, x_n^*(p_1, \cdots, p_n, I)] \\ = \quad & V(p_1, p_2, \cdots, p_n, I). \end{aligned}$$

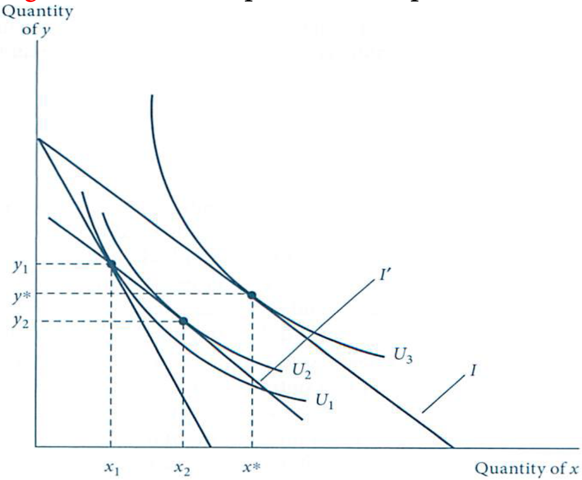
- The indirect utility function is an example of a value function.
- The optimal level of utility will depend indirectly on prices and income.

The Lump Sum Principle

- Many economic insights stem from the recognition that utility ultimately depends on the income of individuals and on the prices they face.
- One of the most important of these insights is the so-called lump sum principles that illustrates the superiority of taxes on an individual's general purchasing power to taxes on specific goods.
- A related insight is that general income grants to low-income people will raise utility more than will a similar amount of money spent subsidizing specific goods.

- The intuition behind these results derives directly from the utility-maximization hypothesis; an income tax or subsidy leaves the individual free to decide how to allocate whatever final income he or she has.
- Taxes or subsidies on specific goods both change a person's purchasing power and distort his or her choices because of the artificial prices incorporated in such schemes.
- General income taxes and subsidies are to be preferred if efficiency is an important criterion in social policy.
- The lump sum principle as it applies to taxation is illustrated in Figure 4.5.

Figure 4.5 The Lump Sum Principle of Taxation



- initial choice: (x^*, y^*)
- choice under a tax on x: (x_1, y_1)
- choice under income tax (x_2, y_2), $U_2 > U_1$

Example 4.3 Indirect Utility and the Lump Sum Principle

Case 1: Cobb-Douglas.

For the Cobb-Douglas utility function $U(x, y) = x^{\alpha} y^{\beta}$ with $\alpha = \beta = 0.5$, optimal purchases are

$$x^* = \frac{I}{2p_x}, y^* = \frac{I}{2p_y}$$

Thus the indirect utility function is

$$V(p_x, p_y, I) = U(x^*, y^*) = (x^*)^{0.5}(y^*)^{0.5} = \frac{I}{2p_x^{0.5} p_y^{0.5}}$$

With $p_x = 1, p_y = 4, I = 8$, $V = \frac{8}{2 \cdot 1 \cdot 2} = 2$.

The lump sum principle

- For the case of Cobb-Douglas utility function, $V = \frac{I}{2p_x^{0.5}p_y^{0.5}}$, with $p_x = 1, p_y = 4, I = 8, V = 2$.
- Suppose that a tax of \$1 were imposed on good x, then p_x increases from \$1 to \$2. Therefore $V(p_x, p_y, I)$ becomes $V(2, 4, 8) = \frac{8}{2 \cdot 2^{0.5} \cdot 2} = 1.41$.
- Since $x^* = \frac{8}{2p_x} = 2$, when $p_x = 2$, total tax collections will be \$2 . Therefore, an equal-revenue income tax would reduce net income to \$8-\$2=\$6, and the indirect utility would be

$$V(p_x, p_y, I) = V(1, 4, 6) = \frac{6}{2 \cdot 1 \cdot 2} = 1.5$$

- Thus, the income tax is a clear improvement in utility.

Case 2: Fixed proportions.

For the fixed proportions utility function $U(x, y) = \min(x, 4y)$, optimal purchases are

$$x^* = \frac{I}{p_x + 0.25p_y}, y^* = \frac{I}{4p_x + p_y}$$

Thus the indirect utility function is

$$V(p_x, p_y, I) = \min(x^*, 4y^*) = x^* = 4y^* = \frac{I}{p_x + 0.25p_y}$$

With $p_x = 1$, $p_y = 4$, $I = 8$, $V = \frac{8}{1+0.25\cdot 4} = 4$.

The lump sum principle

- A \$1 tax on good x would reduce indirect utility from 4 to

$$V(p_x, p_y, I) = V(2, 4, 8) = \frac{8}{2 + 0.25 \cdot 4} = \frac{8}{3}$$

- In this case, $x^* = \frac{8}{2+0.25\cdot 4} = \frac{8}{3}$, and tax collections would be $\frac{8}{3}$.
- An income tax that collected \$ $\frac{8}{3}$ would leave this consumer with \$$\frac{16}{3}$ and yield an indirect utility of

$$V(1, 4, \frac{16}{3}) = \frac{16/3}{1 + 0.25 \cdot 4} = \frac{8}{3}$$

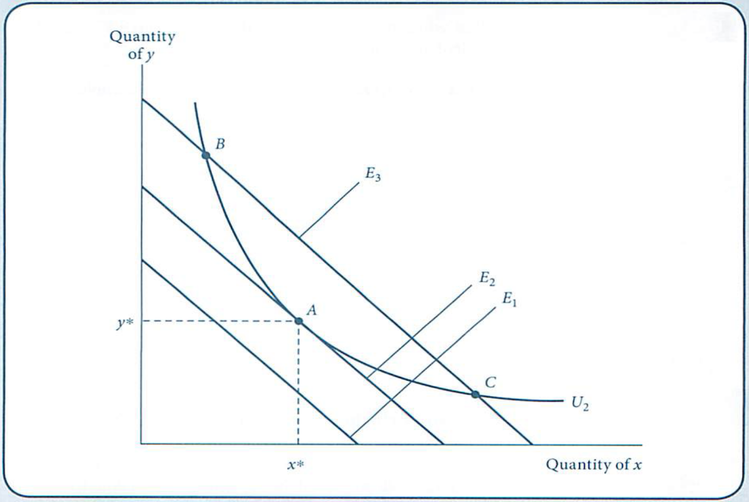
- Hence after-tax utility is the same under both the excise and income taxes. Since preferences are rigid, the tax on x does not distort choices.

Expenditure Minimization

- Many constrained maximum problems have associated "dual" constrained minimum problems.
- For the case of utility maximization, the associated dual minimization problem concerns allocating income to achieve a given level of utility with the minimal expenditure.
- The goal and the constraint have been reversed.
- Often the expenditure-minimization approach is more useful because expenditures are directly observable, whereas utility is not.

Figure 4.6 The Dual Expenditure-Minimization Problem

The dual of the utility-maximization problem is to attain a given utility level (U_2) with minimal expenditures. An expenditure level of E_1 does not permit U_2 to be reached, whereas E_3 provides more spending power than is strictly necessary. With expenditure E_2, this person can just reach U_2 by consuming x^* and y^*.



A mathematical statement

- The individual's dual expenditure-minimization problem is to choose $x_1, x_2, \cdots, x_n$ to minimize

$$\text{total expemditures} = E = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n$$

subject to the constraint

$$\text{utility} = \overline{U} = U(x_1, x_2, \cdots, x_n).$$

- **Expenditure function**:

  The individual's expenditure function shows the minimal expenditures necessary to achieve a given utility level for a particular set of prices. That is

  $$\text{minimal expenditure} = E(p_1, p_2, \cdots, p_n, U)$$

  This is a value function.

- The expenditure function and the indirect utility function are inverse functions of one another. Both depend on market prices, but involve different constraints (income or utility)

Example 4.4 Two Expenditure Functions

Case 1: Cobb-Douglas utility.

- The indirect utility function in the two-good, Cobb-Douglas case is
$$V(p_x, p_y, I) = \frac{I}{2p_x^{0.5}p_y^{0.5}}.$$
Then we have the expenditure function
$$E(p_x, p_y, U) = 2p_x^{0.5}p_y^{0.5}U.$$
- For $P_x = 1$, $p_y = 4$, with a utility target $U = 2$, then the required minimal expenditures are $2 \cdot 1^{0.5} \cdot 4^{0.5} \cdot 2$=\$8.
- Suppose the price of good y were to increase from \$4 to \$5, expenditures would have to be increase to $2 \cdot 1 \cdot 5^{0.5} \cdot 2$=\$8.94 to provide enough extra purchasing power to precisely compensate for this price increase.

Case 2: Fixed proportions.

- The indirect utility function is

$$V(p_x, p_y, I) = \frac{I}{p_x + 0.25p_y}$$

Then the expenditure function is

$$E(p_x, p_y, U) = (p_x + 0.25p_y)U.$$

- For $P_x = 1$, $p_y = 4$, with a utility target $U = 4$, the required minimal expenditures are $(1 + 0.25 \cdot 4) \cdot 4$=\$8.
- Suppose p_y were to increase from \$4 to \$5, expenditures would have to be increase to $(1 + 0.25 \cdot 5) \cdot 4$=\$9 to provide enough extra purchasing power to precisely compensate for this price increase.

Properties of Expenditure Functions

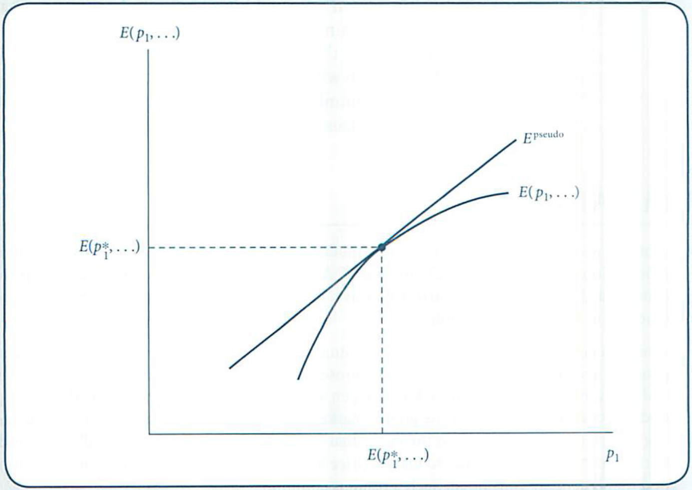
- *Homogeneity.* A doubling of all prices will precisely double the value of required expenditures. That is, it is homogeneous of degree one.
- *Expenditure functions are nondecreasing in prices.*

$$\frac{\partial E}{\partial p_i} \geq 0, \text{ for every good } i.$$

- *Expenditures functions are concave in prices.* Concave functions are Functions that always lie below tangents to them.

Figure 4.7 Expenditure Functions Are Concave in Prices

At p_1^* this person spends $E(p_1^*, \ldots)$. If he or she continues to buy the same set of goods as p_1 changes, then expenditures would be given by E^{pseudo}. Because his or her consumption patterns will likely change as p_1 changes, actual expenditures will be less than this.



Extensions: Budget Shares

- **Engel's law**: Fraction of income spent on food decreases as income increases.
$$\frac{\partial s_i}{\partial I} < 0,$$
where $s_i = \frac{p_i x_i}{I}$ is the budget shares.
- Hayashi (1995) shows that the share of income devoted to foods favored by the elderly is much larger in two-generation households than in one-generation households.
- Findings by Behrman (1989) from less-developed countries shows that people's desires for a more varied diet as their incomes increase may result in reducing the fraction of income spent on particular nutrients.

E4.1 The variability of budget shares

Table E4.1 Budget shares of U.S. households, 2008

TABLE E4.1 BUDGET SHARES OF U.S. HOUSEHOLDS, 2008

	Annual Income		
	\$10,000 – \$14,999	\$40,000 – \$49,999	Over \$70,000
Expenditure Item			
Food	15.7	13.4	11.8
Shelter	23.1	21.2	19.3
Utilities, fuel, and public services	11.2	8.6	5.8
Transportation	14.1	17.8	16.8
Health insurance	5.3	4.0	2.6
Other health-care expenses	2.6	2.8	2.3
Entertainment (including alcohol)	4.6	5.2	5.8
Education	2.3	1.2	2.6
Insurance and pensions	2.2	8.5	14.6
Other (apparel, personal care, other housing expenses, and misc.)	18.9	17.3	18.4

- Engel's law is clearly visible.
- Cobb–Douglas utility function is not useful for detailed empirical studies of household behavior since budget shares are constant for all observed income levels.

E4.2 Linear expenditure system

- A generalization of the Cobb–Douglas function that incorporates the idea that certain minimal amounts of each good must be bought by an individual (x_0, y_0) is the utility function

$$U(x, y) = (x - x_0)^{\alpha}(y - y_0)^{\beta}$$

for $x \geq x_0$ and $y \geq y_0$, where $\alpha + \beta = 1$. This is also called Stone-Geary utility function.

- Let supernumerary income (I^*) be the amount of purchasing power remaining after purchasing the minimum bundle

$$I^* = I - p_x x_0 - p_y y_0$$

- The demand functions are

$$
\begin{aligned}
x &= x_0 + \frac{\alpha I^*}{p_x} = \frac{p_x x_0 + \alpha I^*}{p_x} \\
y &= y_0 + \frac{\beta I^*}{p_y} = \frac{p_y y_0 + \beta I^*}{p_y}
\end{aligned}
$$

Then

$$
\begin{aligned}
s_x &= \frac{p_x x}{I} = \frac{p_x x_0 + \alpha(I - p_x x_0 - p_y y_0)}{I} = \alpha + \frac{\beta p_x x_0 - \alpha p_y y_0}{I} \\
s_y &= \frac{p_y y}{I} = \frac{p_y y_0 + \beta(I - p_x x_0 - p_y y_0)}{I} = \beta + \frac{\alpha p_y y_0 - \beta p_x x_0}{I}
\end{aligned}
$$

- The budget share is positively related to the minimal amount of that good needed and negatively related to the minimal amount of the other good required.

E4.3 CES utility

- The CES utility function

$$U(x,y) = \frac{x^\delta}{\delta} + \frac{y^\delta}{\delta}$$

for $\delta \leq 1, \delta \neq 0$.

- From the first-order conditions, it can be shown that the share equations are

$$\begin{aligned} s_x &= \frac{1}{1 + (p_y/p_x)^K}, \\ s_y &= \frac{1}{1 + (p_x/p_y)^K} \end{aligned}$$

where $K = \delta/(\delta - 1)$

- The homothetic nature of the CES function is shown by the fact the budget shares depend only on the price ratio, p_x/p_y.
- For the Cobb-Douglas case, $\delta = 0$ and so $K = 0$, and $s_x = s_y = 1/2$.
- When $\delta > 0$, substitution possibilities are great and $K < 0$. If p_x/p_y increases, the individual substitutes y for x to such an extent that s_x decreases.
- When $\delta < 0$, substitution possibilities are limited and $K > 0$. An increase in p_x/p_y causes only minor substitution of y for x, and s_x actually increases.

E4.4 The almost ideal demand system (AIDS)

- Starts from a specific expenditure function.

$$\begin{aligned}\frac{\partial \ln E(p_x, p_y, V)}{\partial \ln p_x} &= \frac{1}{E(p_x, p_y, V)} \cdot \frac{\partial E}{\partial p_x} \cdot \frac{\partial p_x}{\partial \ln p_x} \\ &= \frac{x p_x}{E} = s_x\end{aligned}$$

- The expenditure function of the *almost ideal demand system* takes the form

$$\begin{aligned}\ln E(p_x, p_y, V) &= a_0 + a_1 \ln p_x + a_2 \ln p_y \\ &+ 0.5 b_1 (\ln p_x)^2 + b_2 \ln p_x \ln p_y \\ &+ 0.5 b_3 (\ln p_y)^2 + V c_0 p_x^{c_1} p_y^{c_2}\end{aligned}$$

- For the expenditure function to be homogeneous of degree one in prices, the parameters must obey the constraints

$$a_1 + a_2 = 1, b_1 + b_2 = 0, b_2 + b_3 = 0, c_1 + c_2 = 0$$

- It can be shown that, for this function,

$$\begin{aligned} s_x &= a_1 + b_1 \ln p_x + b_2 \ln p_y + c_1 V c_0 p_x^{c_1} p_y^{c_2} \\ s_y &= a_2 + b_2 \ln p_x + b_3 \ln p_y + c_2 V c_0 p_x^{c_1} p_y^{c_2} \end{aligned}$$

and

$$\begin{aligned} s_x &= a_1 + b_1 \ln p_x + b_2 \ln p_y + c_1 \ln(E/p) \\ s_y &= a_2 + b_2 \ln p_x + b_3 \ln p_y + c_2 \ln(E/p) \end{aligned}$$

where p is an index of prices defined by

$$\begin{aligned} \ln p &= a_0 + a_1 \ln P_x + a_2 \ln p_y + 0.5 b_1 (\ln p_x)^2 \\ &= b_2 \ln p_x p_y + 0.5 b_3 (\ln p_y)^2. \end{aligned}$$