Chapter 2 Mathematics for Microeconomics, Part I

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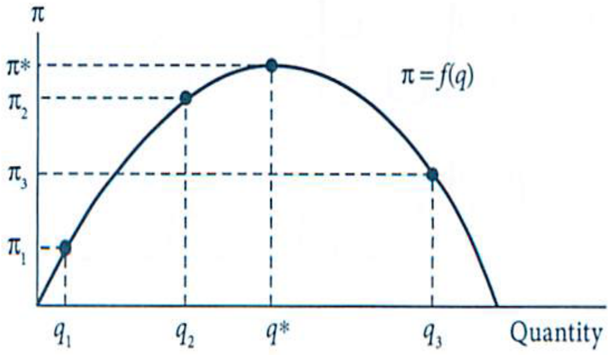
Maximization of a Function of One Variable

- Economic theories assume that an economic agent is seeking to find the optimal value of some function.
  - Consumers seek to maximize utility.
  - Firms seek to maximize profit.

- For example, the manager of a firm wants to maximize profits. Suppose that the profits (π) received depend only on the quantity (q) of the good sold.

$$\pi = f(q)$$

Figure 2.1 Hypothetical Relationship between Quantity Produced and Profits



- If a manager wishes to produce the level of output that maximizes profits, then q^* should be produced. Notice that at q^*, $d\pi/dq = 0$.

The manager may try varying q to see where a maximum profit is obtained.

- An increase from q_1 to q_2 leads to a rise in π.

$$\frac{\pi_2 - \pi_1}{q_2 - q_1} > 0 \text{ or } \frac{\Delta \pi}{\Delta q} > 0$$

- If output is increased beyond q^*, profit will decline.

  An increase from q^* to q_3 leads to a drop in π.

$$\frac{\Delta \pi}{\Delta q} < 0$$

Derivatives

- The derivative of $\pi = f(q)$ is the limit of $\Delta\pi/\Delta q$ for very small changes in q.
- It is the slope of the curve.
- The value depends on the value of q.
- The derivative of $\pi = f(q)$ at the point q_1 is

$$\frac{d\pi}{dq} = \frac{df}{dq} = \lim_{h \to 0} \frac{f(q_1 + h) - f(q_1)}{h}$$

Value of a derivative at a point (the slope)

- The evaluation of the derivative at the point $q = q_1$ can be denoted

$$\left.\frac{d\pi}{dq}\right|_{q=q_1}$$

- In our previous example,

$$\left.\frac{d\pi}{dq}\right|_{q=q_1} > 0, \left.\frac{d\pi}{dq}\right|_{q=q_3} < 0, \left.\frac{d\pi}{dq}\right|_{q=q^*} = 0$$

First-order condition for a maximum

- For a function of one variable to attain its maximum value at some point, the derivative at that point must be zero.

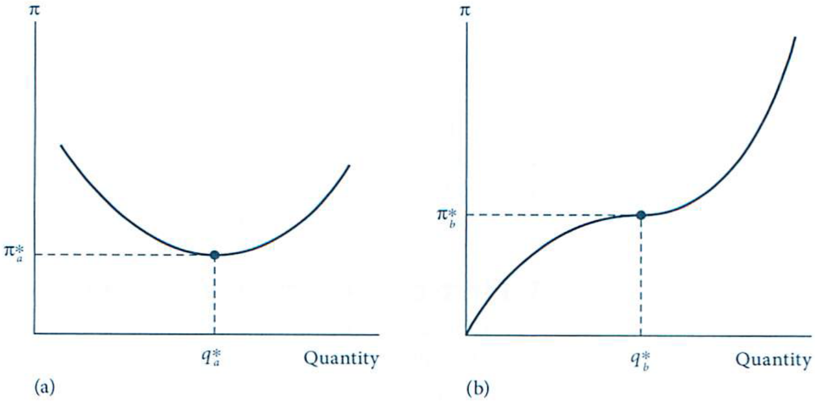
$$\left.\frac{df}{dq}\right|_{q=q^*} = 0$$

- The first order condition ($d\pi/dq$) is a necessary condition for a maximum. But it is not a sufficient condition.

The second order condition

- In order for q^* to be the maximum, $\frac{d\pi}{dq} > 0$ for $q < q^*$ and $\frac{d\pi}{dq} < 0$ for $q > q^*$.
- At q^*, $d\pi/dq$ must be decreasing. The derivative of $d\pi/dq$ must be negative at q^*.

Figure 2.2 Two Profit Functions That Give Misleading Results if the First Derivative Rule is Applied Uncritically



Second derivative

- The derivative of a derivative is called a *second derivative* and is denoted by

$$\frac{d^2\pi}{dq^2} \text{ or } \frac{d^2 f}{dq^2} \text{ or } f''(q)$$

- The second order condition for q^* to represent a (local) maximum is:

$$\left.\frac{d^2\pi}{dq^2}\right|_{q=q^*} = \left.f''(q)\right|_{q=q^*} < 0$$

Rules for finding derivatives

1. If a is a constant, then $\frac{da}{dx} = 0$
2. If a is a constant, then $\frac{d[af(x)]}{dx} = af'(x)$
3. If a is a constant, then $\frac{dx^a}{dx} = ax^{n-1}$
4. $\frac{d \ln x}{dx} = \frac{1}{x}$
5. $\frac{da^x}{dx} = a^x \ln a$ for any constant a

   special case: $\frac{de^x}{dx} = e^x$

Suppose that $f(x)$ and $g(x)$ are two functions of x and $f'(x)$ and $g'(x)$ exist, then:

6. $\frac{d[f(x)+g(x)]}{dx} = f'(x) + g'(x)$
7. $\frac{d[f(x)\cdot g(x)]}{dx} = f(x)g'(x) + f'(x)g(x)$
8. $\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$ provided that $g(x) \neq 0$

If $y = f(x)$ and $x = g(z)$ and if both $f'(x)$ and $g'(x)$ exist, then:

9. $\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz} = \frac{df}{dx}\frac{dg}{dz}$

This is called the *chain rule*. This Allows us to study how one variable (z) affects another variable (y) through its influence on some intermediate variable (x). Some examples of the chain rule include:

10. $\frac{de^{ax}}{dx} = \frac{de^{ax}}{d(ax)} \cdot \frac{d(ax)}{dx} = e^{ax} \cdot a = ae^{ax}$

11. $\frac{d[\ln(ax)]}{dx} = \frac{d[\ln(ax)]}{d(ax)} \cdot \frac{d(ax)}{dx} = \frac{1}{ax} \cdot a = \frac{1}{x}$

12. $\frac{d[\ln(x^2)]}{dx} = \frac{d[\ln(x^2)]}{d(x^2)} \cdot \frac{d(x^2)}{dx} = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$

Example 2.1 Profit Maximization

- Suppose that the relationship between profit and output is

$$\pi = 1,000q - 5q^2$$

- The first order condition for a maximum is

$$\begin{aligned} \frac{d\pi}{dq} &= 1,000 - 10q = 0 \\ q^* &= 100 \end{aligned}$$

Since the second derivative is always -10, then $q = 100$ is a global maximum.

Functions of Several Variables

- Most goals of interest to economic agents depend on several variables, and trade-offs must be made among these variables.
- The dependence of one variable (y) on a series of other variables ($x_1, x_2, \cdots, x_n$) is denoted by

$$y = f(x_1, x_2, \cdots, x_n)$$

Partial derivatives

- The only directional slopes of interest are those that are obtained by increasing one of the x's while holding all the other variables constant.
- These directional slopes are called *partial derivatives*.
- The partial derivative of y with respect to x_1 is dented by

$$\frac{\partial y}{\partial x_1} \text{ or } \frac{\partial f}{\partial x_1} \text{ or } f_{x_1} \text{ or } f_1$$

  All of the other x's are held constant.
- A more formal definition is

$$\left.\frac{\partial f}{\partial x_1}\right|_{\bar{x}_2,\cdots,\bar{x}_n} = \lim_{h\to 0} \frac{f(x_1+h, \bar{x}_2, \cdots, \bar{x}_n) - f(x_1, \bar{x}_2, \cdots, \bar{x}_n)}{h}$$

Calculating Partial Derivatives

The calculation of partial derivatives proceeds as for the usual derivative by *treating* $x_2, \cdots, x_n$ *as constants.*

1. If $y = f(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$, then

$$\frac{\partial f}{\partial x_1} = f_1 = 2ax_1 + bx_2, \quad \frac{\partial f}{\partial x_2} = f_2 = bx_1 + 2cx_2$$

2. If $y = f(x_1, x_2) = e^{ax_1 + bx_2}$, then

$$\frac{\partial f}{\partial x_1} = f_1 = ae^{ax_1 + bx_2}, \quad \frac{\partial f}{\partial x_2} = f_2 = be^{ax_1 + bx_2}$$

3. If $y = f(x_1, x_2) = a \ln x_1 + b \ln x_2$, then

$$\frac{\partial f}{\partial x_1} = f_1 = \frac{a}{x_1}, \quad \frac{\partial f}{\partial x_2} = f_2 = \frac{b}{x_2}$$

Partial derivatives and the ceteris paribus assumption

- Partial derivatives are the mathematical expression of the *ceteris paribus* assumption.
- For example, the fundamental law of demand is reflected by the mathematical statement $\partial q/\partial p < 0$.

Partial derivatives and units of measurement

- The numerical size of partial derivatives on the chosen units of measurement poses problems for economists.
- Making comparisons among studies could prove practically impossible, especially given the wide variety of measuring systems in use around the world.
- Economists have chosen to adapt a different, unit-free way to measure quantitative impacts.

Elasticity— a general definition

- Elasticities measures the proportional effect of a change in one variable on another. They are unit-free.
- Elasticity of y with respect to x is

$$e_{y,x} = \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \frac{\Delta y}{\Delta x} \cdot \frac{x}{y} = \frac{dy(x)}{dx} \cdot \frac{x}{y}$$

- Elasticity is a pure figure with no dimensions.

Example 2.2 Elasticity and Functional Form

- Suppose y is a linear function of x of the form

$$y = a + bx + \text{ other terms}$$

- Then, the elasticity is:

$$e_{y,x} = \frac{dy}{dx} \cdot \frac{x}{y} = b \cdot \frac{x}{y} = b \cdot \frac{x}{a + bx + \cdots}$$

- $e_{y,x}$ is not constant.
- It is important to note the point at which the elasticity is to be computed.

- If the relationship between y and x is of the exponential form
$$y = ax^b,$$
then the elasticity is a constant.
$$e_{y,x} = \frac{dy}{dx} \cdot \frac{x}{y} = abx^{b-1} \cdot \frac{x}{ax^b} = b$$
- The logarithmic transformation of $y = ax^b$ is
$$\ln y = \ln a + b \ln x,$$
The elasticity is also a constant because
$$e_{y,x} = b = \frac{d\ln y}{d\ln x}$$
Elasticities can be calculated through logarithmic differentiation.

Second-order partial derivatives

- The partial derivative of a partial derivative is the second-order partial derivative.

$$\frac{\partial(\partial f/\partial x_i)}{\partial x_j} = \frac{\partial^2 f}{\partial x_j x_i} = f_{ij}$$

1. $y = f(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$, then

$$f_{11} = 2a, f_{12} = b, f_{21} = b, f_{22} = 2c$$

2. $y = f(x_1, x_2) = e^{ax_1+bx_2}$, then

$$f_{11} = a^2 e^{ax_1+bx_2}, f_{12} = abe^{ax_1+bx_2}$$

$$f_{21} = abe^{ax_1+bx_2}, f_{22} = b^2 e^{ax_1+bx_2}$$

3. $y = f(x_1, x_2) = a \ln x_1 + b \ln x_2$, then

$$f_{11} = -ax_1^{-2}, f_{12} = 0, f_{21} = 0, f_{22} = -bx_2^{-2}$$

Young's theorem

- Under general conditions, the order in which partial differentiation is conducted to evaluate second-order partial derivatives does not matter. That is

$$f_{ij} = f_{ji}$$

Uses of second-order partials

- Second-order partials play an important role in many economic theories.
- A variable's own second-order partial, f_{ii} shows how $\partial y/\partial x_i$ changes as the value of x_i increases. $f_{ii} < 0$ indicates diminishing marginal effectiveness.
- The cross-partial f_{ij} indicates how the marginal effectiveness of x_i changes as x_j increases.

The chain rule with many variables

- $y = f(x_1, x_2, x_3)$

  Each of these x's is itself a function of a single parameter, a.

- $y = f[x_1(a), x_2(a), x_3(a)]$

- How a change in a affects the value of y:

$$\frac{dy}{da} = \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{da} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{da} + \frac{\partial f}{\partial x_3} \cdot \frac{dx_3}{da}$$

Special case: if $x_3(a) = a$, then:

$$y = f[x_1(a), x_2(a), a]$$

The effect of a on y:

- A direct effect, which is given by f_a
- An indirect effect that operates only through the ways in which a affects the $x's$

$$\frac{dy}{da} = \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{da} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{da} + \frac{\partial f}{\partial a}$$

Example 2.3 Using the Chain Rule

- Each week, a pizza fanatic consumes three kinds of pizza, denoted by x_1, x_2, and x_3
  - Cost of type 1 pizza is p per pie
  - Cost of type 2 pizza is $2p$
  - Cost of type 3 pizza is $3p$
- Allocates $30 each week to each type of pizza.
- How the total number of pizzas purchased is affected by the underlying price p?

- Quantity purchased of each type:
  $x_1 = 30/p$; $x_2 = 30/2p$; $x3 = 30/3p$.
- Total pizza purchases:

$$y = f[x_1(p), x_2(p), x_3(p)] = x_1(p) + x_2(p) + x_3(p)$$

- Applying the chain rule:

$$\begin{aligned} \frac{dy}{dp} &= f_1 \cdot \frac{dx_1}{dp} + f_2 \cdot \frac{dx_2}{dp} + f_3 \cdot \frac{dx_3}{dp} \\ &= -30p^{-2} - 15p^{-2} - 10p^{-2} = -55p^{-2} \\ & \quad (f_1 = f_2 = f_3 = 1) \end{aligned}$$

Implicit functions

- If the value of a function is held constant, an implicit relationship is created among the independent variables that enter into the function.
- The independent variables can no longer take on any values, but must instead take on only that set of values that result in the function's retaining the required value.
- The most useful result provided by this approach is in the ability to quantify the trade-offs inherent in most economic models.

- Consider a simple case

$$y = f(x_1, x_2)$$

- Holding y constant allows the creation of an *implicit function* of the form $x_2 = g(x_1)$.
- Set the original function equal to a constant (say, zero) and write the function as

$$y = 0 = f(x_1, x_2) = f(x_1, g(x_1))$$

- Differentiate with respect to x_1 yields:

$$0 = f_1 + f_2 \cdot \frac{dg(x_1)}{dx_1}$$

- Rearranging terms gives the final result that

$$\frac{dg(x_1)}{dx_1} = \frac{dx_2}{dx_1} = -\frac{f_1}{f_2}$$

Example 2.4 A Production Possibility Frontier— Again

- A production possibility frontier for two goods of the form

$$x^2 + 0.25y^2 = 200$$

- The implicit function is

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-2x}{0.5y} = \frac{-4x}{y},$$

which is precisely the result we obtained earlier, with considerably less work.

A special case— comparative statics analysis

- One important application of the implicit function theorem is *comparative statics analysis*.
- From $y = 0 = f(x_1, x_2) = f(x_1, g(x_1))$, with exogenous variable, a, the implicit form of the function can be written as

$$f(a, x(a)) = 0$$

- Applying the implicit function theorem would yield

$$\frac{dx(a)}{da} = -\frac{f_1}{f_2} = -\frac{\frac{\partial f}{\partial a}}{\frac{\partial f}{\partial x}}$$

This shows directly how changes in the exogenous variable a affect the endogenous variable x.

Example 2.5 Comparative Statics of a Price-Taking Firm

- The first order condition for a profit firm that takes market price as given is

$$f(q(p), p) = p - C'(q(p)) = 0$$

- Applying the implicit function theorem to this expression yields

$$\frac{dq(p)}{dp} = -\frac{f_p}{f_q} = -\frac{1}{\partial(-C'(q))/\partial q} = \frac{1}{C''(q)} > 0,$$

which is precisely the result we obtained earlier.

2.3 Maximization of Functions of Several Variables

Suppose an agent wishes to maximize

$$y = f(x_1, x_2, \cdots, x_n)$$

- The change in y from a change in x_1 (holding all other x's constant) is equal to the change in x_1 times the slope measured in the x_1 direction.

$$dy = \frac{\partial f}{\partial x_1} dx_1 = f_1 dx_1$$

First-order conditions for a maximum

- A necessary condition for a point to be a local maximum of the function $f(x_1, x_2, \cdots, x_n)$ is that $dy = 0$ for any combination of small changes in the x's.

$$f_1 = f_2 = \cdots f_n = 0$$

- This is called a *critical point* of the function.

Second-order conditions

- However, a second-order condition is needed to ensure that the point found by applying the first-order conditions is a local maximum.
- If we confine our attention only to movements in a single direction, then the condition required for a maximum is $f_{ii} < 0$, —the second partial derivatives must be negative.
- Unfortunately, the conditions that assure the value of f decreases for movements in any arbitrary direction involve all the second partial derivatives. The general case is best discussed with matrix algebra (see the Extensions to this chapter).

Example 2.6 Finding a Maximum

Suppose that y is a function of x_1 and x_2

$$y = -(x_1 - 1)^2 - (x_2 - 2)^2 + 10$$
$$y = -x_1^2 + 2x_1 - x_2^2 + 4x_2 + 5$$

First-order conditions imply that

$$\frac{\partial y}{\partial x_1} = -2x_1 + 2 = 0$$
$$\frac{\partial y}{\partial x_2} = -2x_2 + 4 = 0$$

or $x_1^* = 1, x_2^* = 2$ and $f_{11} = f_{22} = -2$

The Envelope Theorem

- The envelope theorem concerns how an optimized function changes when a parameter of the function changes.

A specific example

Suppose y is a function of a single variable (x) and an exogenous parameter (a) given by

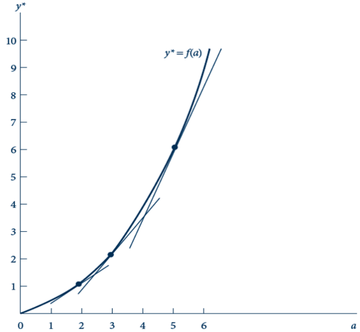
$$y = -x^2 + ax$$

- For different values of a, this function represents a family of inverted parabolas.
- If a is assigned a specific value, it is a function of x only. We can calculate the value of x that maximizes y.

Table 2.1 Optimal values of y and x for alternative values of a in $y = -x^2 + ax$

Value of a	Value of x^*	Value of y^*
0	0	0
1	$\frac{1}{2}$	$\frac{1}{4}$
2	1	1
3	$\frac{3}{2}$	$\frac{9}{4}$
4	2	4
5	$\frac{5}{2}$	$\frac{25}{4}$
6	3	9

Figure 2.3 Illustration of the Envelope Theorem



- The envelope theorem states that the slope of the relationship between y^* and the parameter a can be found by calculating the slope of the auxiliary relationship found by substituting the respective optimal values for x into the objective function and calculating $\partial y / \partial a$.

A direct, time-consuming approach

Calculate the slope of y^* directly

- Solve for the optimal value of x for any value of a:

$$\frac{dy}{dx} = -2x + a = 0, x^* = \frac{a}{2}$$

- Substituting the value of x^* gives

$$\begin{aligned} y^* &= -(x^*)^2 + a(x^*) = -(\frac{a}{2})^2 + a(\frac{a}{2}) \\ &= -\frac{a^2}{4} + \frac{a^2}{2} = \frac{a^2}{4} \end{aligned}$$

Therefore,

$$\frac{dy^*}{da} = \frac{a}{2}$$

The envelope shortcut

- For small changes in a, dy^*/da can be computed by holding x at its optimal value (x^*) and calculating $\partial y/\partial a$ from the objective function directly.

$$\frac{dy^*}{da} = \frac{\partial y}{\partial a}\bigg|_{x=x^*(a)} = \frac{\partial(-x^2 + ax)}{\partial a}\bigg|_{x=x^*(a)} = x^*(a)$$

- Holding $x = x^*$:

$$\frac{dy^*}{da} = x^*(a) = \frac{a}{2}$$

The envelope theorem

- The change in the value of an optimized function with respect to a parameter of that function can be found by partially differentiating the objective function while holding x (or several x's) at its optimal value.

$$\frac{dy^*}{da} = \frac{\partial y}{\partial a}\{x = x^*(a)\}$$

Many-variable case

- Suppose y is a function of a set of x's and a particular parameter of interest a:

$$y = f(x_1, \cdots, x_n, a)$$

- Finding an optimal value for y would consist of solving n first-order equations of the form

$$\frac{\partial y}{\partial x_i} = 0, (i = 1, \cdots, n)$$

- Optimal values for these x's would be a function of a

$$x_1^* = x_1^*(a), x_2^* = x_2^*(a), \cdots, x_n^* = x_n^*(a)$$

- Substituting these functions into the original objective yields an expression in which the optimal value of y (say, y^*) depend on a both directly or indirectly through the effect of a on x^*'s:

$$y^* = f[x_1^*(a), x_2^*(a), \cdots, x_n^*(a), a]$$

This function is called a "value function."

- Total differentiating y^* with respect to a yields

$$\begin{aligned}\frac{dy^*}{da} &= \frac{\partial f}{\partial x_1}\cdot\frac{dx_1}{da} + \frac{\partial f}{\partial x_2}\cdot\frac{dx_2}{da} + \cdots\frac{\partial f}{\partial x_n}\cdot\frac{dx_n}{da} + \frac{\partial f}{\partial a} \\ &= \left.\frac{\partial f}{\partial a}\right|_{x_i = x_i^*(a)} \text{ for all } x_i, \text{ because } \frac{\partial f}{\partial x_i} = 0 \quad \forall i\end{aligned}$$

Example 2.7 A Price-Taking Firm's Supply Function

- Suppose that a price-taking firm has a cost function given by $C(q) = 5q^2$.
- A direct way of finding its supply function is to use the first order condition

$$p = C'(q) = 10q$$

to get $q^* = 0.1p$.

- An alternative way is to calculate the firm's profit function. The optimal value of the firm's profit is

$$\pi^*(p) = pq^* - C(q^*) = p(0.1p) - 5(0.1p)^2 = 0.05p^2$$

- The envelope theorem states that

$$\frac{d\pi^*(p)}{dp} = 0.1p = \frac{\partial \pi(p,q)}{\partial p}\bigg|_{q=q^*} = q\bigg|_{q=q^*} = q^*$$

Constrained Maximization

What if not all values for the x's are feasible?

- The values of x may all have to be positive.
- A consumer's choices are limited by the amount of purchasing power. available

Lagrange multiplier method

- One method for solving constrained maximization problems is the *Lagrange multiplier method*.
- Suppose that we wish to find the values of $x_1, x_2, \cdots, x_n$ that maximize: $y = f(x_1, x_2, \cdots, x_n)$
- Subject to a constraint: $g(x_1, x_2, \cdots, x_n) = 0$

The Lagrangian expression is

$$\mathcal{L} = f(x_1, x_2, \cdots, x_n) + \lambda g(x_1, x_2, \cdots, x_n)$$

- λ is an additional variable called the Lagrange multiplier.
- $\mathcal{L} = f$, because $g(x_1, x_2, \cdots, x_n) = 0$.

First-order conditions

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= f_1 + \lambda g_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= f_2 + \lambda g_2 = 0 \\
\cdots &= \cdots\cdots\cdots \\
\frac{\partial \mathcal{L}}{\partial x_n} &= f_1 + \lambda g_n = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= g(x_1, x_2, \cdots, x_n) = 0
\end{aligned}
$$

- The equations can generally be solved for $x_1, x_2, \cdots, x_n$ and λ.
- The solution will have two properties:
  - The x's will obey the constraint.
  - The x's will make the value of $\mathcal{L}$ (and therefore f) as large as possible.

Interpretation of the Lagrange multiplier

- The Lagrange multiplier λ has an important economic interpretation.
- The first-order conditions imply that

$$\frac{f_1}{-g_1} = \frac{f_2}{-g_2} = \cdots = \frac{f_n}{-g_n} = \lambda$$

  - The numerators measure the marginal benefit of one more unit of x_i.
  - The denominators reflect the added burden on the constraint of using more x_i

Lagrange multiplier as a benefit-cost ratio

- At the optimal x_i's, the ratio of the marginal benefit to the marginal cost of x_i should be the same for every x_i.
- λ is the common cost-benefit ratio for all x_i

$$\lambda = \frac{\text{marginal benefit of } \mathrm{x_i}}{\text{marginal cost of } \mathrm{x_i}}$$

- A high value of λ indicates that each x_i has a high cost-benefit ratio.
- A low value of λ indicates that each x_i has a low cost-benefit ratio.
- $\lambda = 0$ implies that the constraint is not binding.

Duality

- Any constrained maximization problem has an associated dual problem in constrained *minimization* that focuses attention on the constraints in the original (primal) problem.
- Individuals maximize utility subject to a budget constraint.

  Dual problem: individuals minimize the expenditure needed to achieve a given level of utility.
- Firms minimize the cost of inputs to produce a given level of output.

  Dual problem: firms maximize output for a given cost of inputs purchased.

Example 2.8 Optimal Fences and Constrained Maximization

- A farmer had a certain length of fence, P, and wishes to enclose the largest possible rectangular area, with x and y the lengths of the sides.
- This is a problem in constrained maximization.
- The problem is to choose x and y to maximize the area ($A = x \cdot y$), subject to the constraint that the perimeter is fixed at $P = 2x + 2y$

The Lagrangian expression is

$$\mathcal{L} = x \cdot y + \lambda(P - 2x - 2y)$$

The first-order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= y - 2\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= x - 2\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= P - 2x - 2y = 0 \end{aligned}$$

Therefore,

- $y/2 = x/2 = \lambda$, then $x = y$, the field should be square.
- $x = y$ and $y = 2\lambda$, then $x = y = P/4$ and $\lambda = P/8$.

Interpretation of the Lagrange multiplier

- Lagrange multiplier, λ, suggests that an extra yard of fencing would add $P/8$ to the area. It provides information about the implicit value of the constraint.

- For example, when $P = 400$, $x = y = 100$, $\lambda = 50$, and $A = 10000$. This implies that A will increase to 10050 when P increases to $400 + 1 = 401$.

- check: when $P = 401$, $x = y = \frac{401}{4} = 100.25$, then $A = 100.25^2 = 10050.0625$

Duality

- Choose x and y to minimize the amount of fence required to surround the field. The problem is to minimize

$$p = 2X + 2Y$$

subject to

$$A = x \cdot y$$

- Setting up the Lagrangian:

$$\mathcal{L}^D = 2x + 2y + \lambda^D (A - xy)$$

- The first-order conditions are

$$\frac{\partial \mathcal{L}^D}{\partial x} = 2 - \lambda^D \cdot y = 0$$
$$\frac{\partial \mathcal{L}^D}{\partial y} = 2 - \lambda^D \cdot x = 0$$
$$\frac{\partial \mathcal{L}^D}{\partial \lambda^D} = A - x \cdot y = 0$$

- Solving these equations yields

$$x = y = \sqrt{A}$$

- The Lagrangian multiplier is

$$\lambda^D = \frac{2}{x} = \frac{2}{y} = \frac{2}{\sqrt{A}}$$

Envelope Theorem in Constrained Maximization Problems

- Suppose that we want to maximize

$$y = f(x_1, \cdots, x_n, a)$$

Subject to the constraint:

$$g(x_1, \cdots, x_n, a) = 0$$

- One way to solve this problem is to et up the Lagrangian expression

$$\mathcal{L} = f(x_1, \cdots, x_n, a) + \lambda g(x_1, \cdots, x_n, a)$$

and solve the first-order conditions for the optimal, constrained values $x_1^*, \cdots, x_n^*$.

- These optimal values can then be substituted back into the original function f to yield a value function for the problem.
- For this value function, the envelope theorem states that

$$\frac{dy^*}{da} = \frac{\partial \mathcal{L}}{\partial a}(x_1^*, \cdots, x_n^*, a)$$

- The change in the maximal value of y that results when a changes can be found by partially differentiate the Lagrange expression and evaluating the resultant partial derivative at the optimal values of the x's.

Example 2.9 Optimal Fences and the Envelope Theorem

- In the fencing problem in Example 2.8, the value function is

$$A^* = x^* \cdot y^* = \frac{P}{4} \cdot \frac{P}{4} = \frac{P^2}{16}$$

- Since the Lagrangian expression is $\mathcal{L} = xy + \lambda(P - 2x - 2y)$, applying the envelope theorem yields

$$\frac{dA^*}{dP} = \frac{P}{8} = \frac{\partial \mathcal{L}}{\partial P} = \lambda$$

- The Lagrange multiplier in a constrained maximization problem shows the marginal gain in the objective function that can be obtained from a slight relaxation of the constraint.

Inequality Constraints

- In some economic problems the constraints need not hold exactly.
- For example, an individual's budget constraint requires that that he or she spend no more than a certain amount per period, but it is at least possible to spend less than his amount.
- Inequality constraints also arise in the values permitted for some variables in economic problems. For example, economic variables usually must be non-negative.

A two-variable example

$$\begin{array}{ll} \text{Maximize} & y = f(x_1, x_2) \\ \text{subject to} & g(x_1, x_2) \geq 0, x_1 \geq 0, x_2 \geq 0 \end{array}$$

Slack variables:

- One way to solve this optimization problem.Introduce three new variables (a, b, and c) that convert the inequalities into equalities.
- To ensure that the inequalities continue to hold, we square these new variables.

  $g(x_1, x_2) - a^2 = 0$; $x_1 - b^2 = 0$, and $x_2 - c^2 = 0$
- Any solution that obeys these three equality constraints will also obey the inequality constraints.

Solution using Lagrange multipliers

$$\mathcal{L} = f(x_1, x_2) + \lambda_1[g(x_1, x_2) - a^2] + \lambda_2[x_1 - b^2] + \lambda_3[x_2 - c^2]$$

F.O.C.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= f_1 + \lambda_1 g_1 + \lambda_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= f_1 + \lambda_1 g_2 + \lambda_3 = 0 \\
\frac{\partial \mathcal{L}}{\partial a} &= -2a\lambda_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial b} &= -2b\lambda_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial c} &= -2c\lambda_3 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_1} &= g(x_1, x_2) - a^2 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_2} &= x_1 - b^2 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_3} &= x_2 - c^2 = 0
\end{aligned}$$

Complementary slackness

- According to the third condition, either a or $\lambda_1 = 0$

  If $a = 0$, the constraint $g(x_1, x_2)$ holds exactly.

  If $\lambda_1 = 0$, the availability of some slackness of the constraint implies that its marginal value to the objective function is 0.

- Similar complementary slackness relationships also hold for x_1 and x_2.

- These results are sometimes called Kuhn-Tucker conditions, which show that solutions to problems involving inequality constraints will differ from those involving equality constraints in rather simple ways.

- This allows us to work primarily with constraints involving equalities.