Part IV: Production and Supply

9. Production Functions
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Chapter 10
Cost Functions

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Definitions of Cost

- It is important to differentiate between accounting cost and economic cost.
- The accountant's view of cost stresses out-of-pocket expenses, historical costs, depreciation, and other bookkeeping entries.
- The economist's definition of cost is that the cost of any input is given by the size of the payment necessary to keep the resource in its present employment.
- Alternatively, the economic cost of using an input is what that input would be paid in its next bet use (opportunity cost).

Labor costs

- To accountants, expenditures on labor are current expenses and hence costs of production.
- For economists, labor is *explicit* costs. Labor services are contracted at some hourly wage rate, and it is usually assumed that this is also what the labor could earn in alternative employment.

Capital costs

- Accountants use the historical price of the capital and apply some depreciation rule to determine current costs.
- Economists regard the historical price of machine as a “sunk cost." Implicit cost of the machine is what someone else would be willing to pay for its use.
- The cost of one machine-hour is the rental rate for that machine in its best alternative use.

Costs of entrepreneurial services

- The owner of a firm is a residual claimant who is entitled to whatever extra revenues or losses are left after paying other input costs.
- To an accountant, these would be called *profits*.
- Economists ask whether owners also encounter opportunity costs by working at a particular firm or devoting some of their funds to its operation.
- Hence some part of the accounting profits generated by the firm would be categorized as entrepreneurial costs by economists.
- Economic profits would be smaller than accounting profits and might be negative.

Economic costs

- **Economic cost.** The *economic cost* of any input is the payment required to keep that input in its present employment. It is the remuneration the input would receive in its best alternative employment.
- Accounting data are often readily available, whereas the corresponding economic concepts may be difficult to measure.
- We use the decision-relevant concepts— economic costs— throughout the analysis.

Simplifying assumptions

- There are only two inputs: homogeneous labor (l), measured in labor-hours, and homogeneous capital (k), measured in machine-hours.
- Entrepreneurial costs are included in capital costs.
- Inputs are hired in perfectly competitive markets. Firms can buy all the labor and capital services they want at the prevailing rental rates (w and v).
- Total cost C for the firm is given by

$$\text{total cost} = C = wl + vk$$

Relationship Between Profit Maximization and Cost Minimization

- Suppose the firm takes the market price (p) for its output (q) as given, and its production function is $q = f(k, l)$. Then its profit is

$$\pi = R - C = pq - wl - vk = pf(k, l) - wl - vk,$$

economic profits are a function of k and l.

- We could examine how a firm would choose k and l to maximize profit. This would lead to a theory of supply and "derived demand" for capital and labor inputs.
- Assume that for some reason the firm has already chosen its output level (q_0) and wants to minimize its costs.

Cost-Minimizing Input Choices

- **Cost minimization.** To minimize the cost of producing a given level of output (q_0), a firm should choose that point on the q_0 isoquant at which the rate of technical substitution (RTS) of l and k is equal to the ratio w/v.

$$RTS = \frac{w}{v}$$

- It should equate the rate at which k can be traded for l in the production process to the rate at which they can be traded in the marketplace.

Mathematical analysis

- We seek to minimize total cost given $q = f(k, l) = q_0$. Setting up the lagrangian,

$$\mathscr{L} = wl + vk + \lambda[q_0 - f(k, l)],$$

the first-order conditions are

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial l} &= w - \lambda \frac{\partial f}{\partial l} = 0, \\ \frac{\partial \mathscr{L}}{\partial k} &= v - \lambda \frac{\partial f}{\partial k} = 0, \\ \frac{\partial \mathscr{L}}{\partial \lambda} &= q_0 - f(k, l) = 0, \end{aligned}$$

or, dividing the first two equations

$$\frac{w}{v} = \frac{f_l}{f_k} = RTS(\text{of } l \text{ for } k).$$

- Cross-multiplying the previous equation gives

$$\frac{f_k}{v} = \frac{f_l}{w}$$

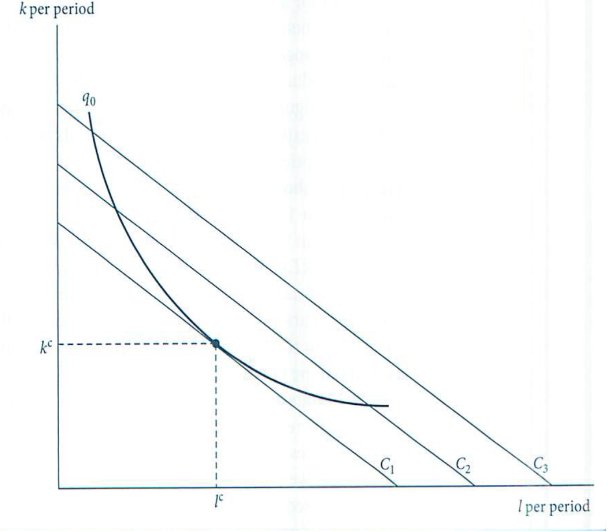
For costs to be minimized, the marginal productivity per dollar spent should be the same for all inputs. Or,

$$\frac{w}{f_l} = \frac{v}{f_k} = \lambda$$

The extra cost of obtaining an extra unit of output by hiring either added labor or added capital input should be the same.

- The common marginal cost is measured by the Lagrangian multiplier from the cost-minimization problem.

Figure 10.1 Minimization of Costs of Producing q_0



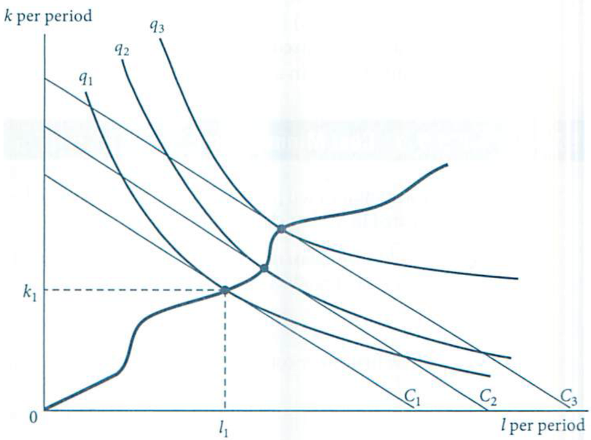
Contingent demand for inputs

- Cost minimization leads to a demand for capital and labor that is contingent on the level of output being produced.
- The demand for an input is a derived demand. It is based on the level of the firm's output.

Firm's expansion path

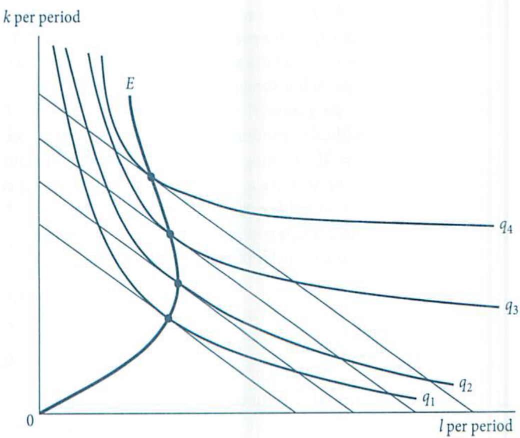
- The firm can determine the cost-minimizing combinations of k and l for every level of output.
- If input costs of v and w remain constant for all amounts of k and l, we can trace the locus of cost-minimizing choices, which is called the firm's expansion path.

Figure 10.2 Firm's expansion path



- The expansion path does not have to be a straight line. The use of some inputs may increase faster than others as output expands. This depends on the shape of the isoquants.

Figure 10.3 Input Inferiority



- The expansion path does not have to be upward sloping. If the use of an input falls as output expands, that input is an inferior input.

Example 10.1 Cost Minimization

- **Cobb-Douglas:** $q = k^{\alpha} l^{\beta}$. The Lagrangian is

$$\mathscr{L} = vk + wl + \lambda(q_0 - k^{\alpha} l^{\beta})$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial k} &= v - \lambda \alpha k^{\alpha-1} l^{\beta} = 0 \\
\frac{\partial \mathscr{L}}{\partial l} &= w - \lambda \beta k^{\alpha} l^{\beta-1} = 0 \\
\frac{\partial \mathscr{L}}{\partial \lambda} &= q_0 - k^{\alpha} l^{\beta} = 0
\end{aligned}$$

- Dividing the second of these by the first yields

$$\frac{w}{v} = \frac{\beta}{\alpha} \cdot \frac{k}{l} = RTS$$

  which again shows that costs are minimized when the ratio of the inputs' prices is equal to the RTS.
- Because the Cobb-Douglas function is homothetic, the RTS depends only on the ratio of the two inputs.
- The expansion path will be a straight line through the origin.

- **CES.** $q = f(k,l) = (k^\rho + l^\rho)^{\gamma/\rho}$. The Lagrangian is

$$\mathscr{L} = vk + wl + \lambda[q_0 - (k^\rho + l^\rho)^{\gamma/\rho}]$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial k} &= v - \lambda(\gamma/\rho)(k^\rho + l^\rho)^{(\gamma-\rho)/\rho}\rho k^{\rho-1} = 0 \\
\frac{\partial \mathscr{L}}{\partial l} &= w - \lambda(\gamma/\rho)(k^\rho + l^\rho)^{(\gamma-\rho)/\rho}\rho l^{\rho-1} = 0 \\
\frac{\partial \mathscr{L}}{\partial \lambda} &= q_0 - (k^\rho + l^\rho)^{\gamma/\rho} = 0
\end{aligned}$$

- Dividing the first two of these equations gives

$$\frac{w}{v} = \left(\frac{l}{k}\right)^{\rho-1} = \left(\frac{k}{l}\right)^{1-\rho} = \left(\frac{k}{l}\right)^{1/\sigma}$$

$$\text{or } \frac{k}{l} = \left(\frac{w}{v}\right)^{\sigma}$$

where $\sigma = 1/(1-\rho)$ is the elasticity of substitution.

- With the Cobb-Douglas ($\sigma = 1$), the cost-minimizing capital-labor ratio changes directly in proportion to changes in the ratio of wages to capital rental rates.
- In cases with greater substitutability ($\sigma > 1$), changes in the ratio of wages to rental rates causes a greater than proportional increase in the cost-minimizing capital-labor ratio.

Cost Functions

- **Total cost function.** The *total cost function* shows that for any set of input costs and for any output level, the minimum cost incurred by the firm is

$$C = C(v, w, q)$$

Average and marginal cost functions

$$\begin{aligned} \text{average cost} &= AC(v, w, q) = \frac{C(v, w, q)}{q} \\ \text{marginal cost} &= MC(v, w, q) = \frac{\partial C(v, w, q)}{\partial q} \end{aligned}$$

Graphical analysis of costs

- Suppose k_1 units of capital input and l_1 units of labor input are required to produce one unit of output. Then

$$C(v,w,1) = vk_1 + wl_1$$

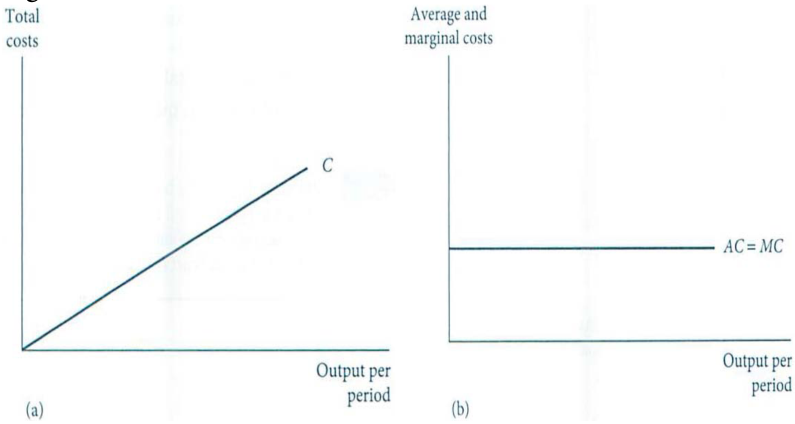
- To produce m units of output under constant returns to scale, mk_1 units of capital and ml_1 units of labor are required. Hence

$$C(v,w,m) = vmk_1 + wml_1 = m(vk_1 + wl_1) = mC(v,k,1)$$

and

$$\begin{aligned} AC(v,w,q) &= \frac{C(v,w,q)}{q} = \frac{qC(v,w,1)}{q} = C(v,w,1) \\ MC(v,w,q) &= \frac{\partial qC(v,w,1))}{\partial q} = C(v,w,1) \end{aligned}$$

Figure 10.4 Cost Curves in the Constant Returns-to-Scale Case



Cubic total cost

- For the cost curve in Figure 10.5a, it is assumed that initially the total cost curve is concave.
- Although initially costs increase rapidly for increases in output, that rate of increase slows as output expands into the midrange of output.
- Beyond this middle range, the total cost curve becomes convex, and costs begin to increase progressively more rapidly.

Figure 10.5 Total, Average, and Marginal Cost Curves for the Cubic Total Cost Curve Case

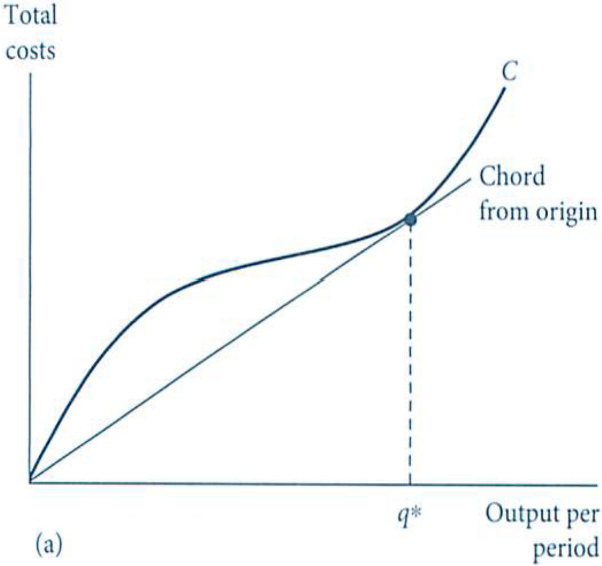
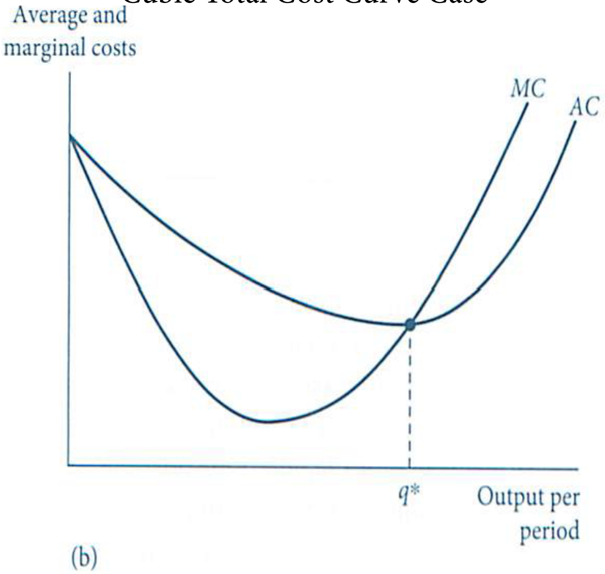


Figure 10.5 Total, Average, and Marginal Cost Curves for the Cubic Total Cost Curve Case



- As long as $AC > MC$, average must be decreasing; When $MC > AC$, average cost will increase. $AC = MC$ at q^*,
- AC curve has a U-shape and it reaches a low point at q^*, where AC and MC intersect.
- The point of minimum average cost reflects the *minimum efficient scale* for the particular production process being examined.
- To find the minimum AC, we have

$$\frac{\partial AC}{\partial q} = \frac{\partial (C/q)}{\partial q} = \frac{q \cdot (\partial C/\partial q) - C \cdot 1}{q^2} = \frac{q \cdot MC - C}{q^2} = 0,$$

implying $MC = C/q = AC$. Thus $AC = MC$ when AC is minimized.

Shifts in Cost Curves

Example 10.2 Some Illustrative Cost Functions

1. **Fixed Proportions:** $q = f(k,l) = \min(\alpha k, \beta l)$. Production will occur at the vertex of the L-shape isoquants where $q = \alpha k = \beta l$. Hence total costs are

$$C(v,w,q) = vk + wl = v\left(\frac{q}{\alpha}\right) + w\left(\frac{q}{\beta}\right) = q\left(\frac{v}{\alpha} + \frac{w}{\beta}\right),$$

Increasing in input prices increase total costs, and technical improvements that take the form of increasing α and β reduce costs.

- Because of the constant returns-to-scale nature of this production function, it takes the special form

$$C(v,w,q) = qC(v,w,1)$$

2. **Cobb-Douglas:** $q = f(k, l) = k^{\alpha} l^{\beta}$. Cost minimization requires that

$$
\begin{aligned}
\frac{w}{v} &= \frac{\beta}{\alpha} \cdot \frac{k}{l} \\
\text{and so } k &= \frac{\alpha}{\beta} \cdot \frac{w}{v} \cdot l
\end{aligned}
$$

Substituting into the production yields

$$
\begin{aligned}
q &= k^{\alpha} l^{\beta} = \left(\frac{\alpha}{\beta} \cdot \frac{w}{v} \right)^{\alpha} l^{\alpha+\beta} \\
\text{or } l^{c}(v, w, q) &= q^{1/(\alpha+\beta)} \left(\frac{\beta}{\alpha} \right)^{\alpha/(\alpha+\beta)} w^{-\alpha/(\alpha+\beta)} v^{\alpha/(\alpha+\beta)}
\end{aligned}
$$

Similarly,

$$
k^{c}(v, w, q) = q^{1/(\alpha+\beta)} \left(\frac{\alpha}{\beta} \right)^{\beta/(\alpha+\beta)} w^{\beta/(\alpha+\beta)} v^{-\beta/(\alpha+\beta)}
$$

- The total costs are

$$C(v,w,q) = vk^c + wl^c = q^{1/(\alpha+\beta)} B v^{\alpha/(\alpha+\beta)} w^{\beta/(\alpha+\beta)}$$

  where $B = (\alpha+\beta)\alpha^{-\alpha/(\alpha+\beta)}\beta^{-\beta/(\alpha+\beta)}$.
- Whether the cost function is a convex, linear, or concave function of output depends on whether $\alpha+\beta<1$, $\alpha+\beta=1$, or $\alpha+\beta>1$
- An increase in any input price increases costs.
- The cost function is homogeneous of degree 1 in the input prices.

3. **CES.** $q = f(k,l) = (k^\rho + l^\rho)^{\gamma/\rho}$. To derive the total cost function, we use the cost-minimization condition, solve for each input individually, and eventually get

$$\begin{aligned} C(v,w,q) &= vk + wl = q^{1/\gamma}\left(v^{\rho/(\rho-1)} + w^{\rho/(\rho-1)}\right)^{(\rho-1)/\rho} \\ &= q^{1/\gamma}\left(v^{1-\sigma} + w^{1-\sigma}\right)^{1/(1-\sigma)} \end{aligned}$$

where the elasticity of substitution is given by $\sigma = 1/(1-\rho)$.

- Total cost is determined by scale parameter γ, and the cost function increases in both of the input prices.
- The function is homogeneous of degree 1 in those prices.
- The inputs are given equal weights— hence their prices are equally important in the cost function.

Properties of cost functions

1. *Homogeneity.* The total cost functions are homogeneous of degree 1 in the input prices.

- A doubling of all input prices will double the cost of producing any given output level.
- A pure, uniform inflation in all input costs will not change a firm's input decisions. Inflation will shift the cost curves up.

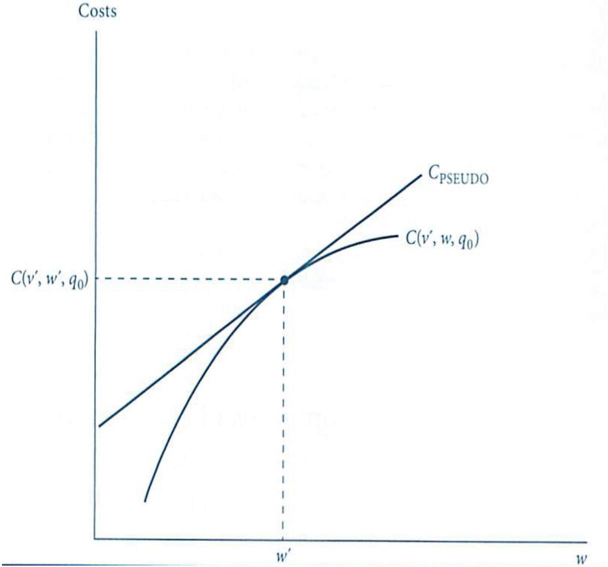
2. *Total cost functions are nondecreasing in q, v and w.*

- Cost functions are derived from a cost-minimization process.
- Any decline in costs from an increase in one of the function's arguments would lead to a contradiction.

3. *Total cost are concave in input prices.*

- A cost-minimizing firm would change the input mix it uses when the input prices change, hence the total cost function must have the concave shape shown in Figure 10.6.
- One implication is that costs will be lower when a firm faces input prices that fluctuate around a given level than when they remain constant at that level.
- With fluctuating input prices, the firm can adapt its input mix to take advantage of such fluctuations by using a lot of, say, labor when its price is low and economizing on that input when its price is high.

Figure 10.6 Cost Functions Are Concave in Ino=put Prices



4. *Properties carrying over to average and marginal costs.* Some of these properties carry over to related average and marginal cost functions.

- Homogeneity is one property that carries over directly. Because $C(tv, tw, q) = tC(v, w, q)$, we have

$$\begin{aligned} AC(tv, tw, q) &= \frac{C(tv, tw, q)}{q} = \frac{tC(v, w, q)}{q} = tAC(v, w, q) \\ MC(tv, tw, q) &= \frac{\partial C(tv, tw, q)}{\partial q} = \frac{t\partial C(v, w, q)}{\partial q} = tMC(v, w, q). \end{aligned}$$

- However, the effects of changes in v, w, and q on average and marginal costs are sometimes ambiguous.

- Because total costs must not decrease when an input price increases, it is clear that average cost is increasing in w and v.
- But the case for marginal cost is more complex. An increase in an inferior input's price will actually cause marginal cost to decrease.
- Because MC function can be derived by differentiation from the Lagrangian for cost minimization, we can use Young's theorem to show

$$\frac{\partial MC}{\partial v} = \frac{\partial(\partial \mathscr{L}/\partial q)}{\partial v} = \frac{\partial^2 \mathscr{L}}{\partial v \partial q} = \frac{\partial^2 \mathscr{L}}{\partial q \partial v} = \frac{\partial k}{\partial q}$$

Hence, if capital is inferior ($\partial k/\partial q < 0$), an increase in v will actually reduce MC.

Input substitution

- A change in the price of an input will cause the firm to alter its input mix. Recall the formula for elasticity of substitution

$$\sigma = \frac{d(k/l)}{d(RTS)} \cdot \frac{RTS}{k/l} = \frac{d\ln(k/l)}{d\ln RTS}$$

- But the cost-minimization principle says that RTS (of l for k) $= w/v$ at an optimum. This gives a new version of the elasticity of substitution:

$$s = \frac{d(k/l)}{d(w/v)} \cdot \frac{w/v}{k/l} = \frac{d\ln(k/l)}{d\ln(w/v)}$$

- In the two-input case, s must be non-negative; an increase in w/v will be met by an increase in k/l.
- Large s indicate that firms change their input proportions significantly in response to changes in relative input prices.

Substitution with many inputs

- Suppose there are many inputs to the production process $(x_1, x_2, \cdots, x_n)$ that can be hired at competitive rental rates $(w_1, w_2, \cdots, w_n)$. Then the elasticity of substitution between any two inouts (s_{ij}) is defined as follows.

$$s_{ij} = \frac{\partial(x_i/x_j)}{\partial(w_j/w_i)} \cdot \frac{w_j/w_i}{x_i/x_j} = \frac{\ln(x_i/x_j)}{\ln(w_j/w_i)},$$

  where output and all other input prices are held constant.
- s_{ij} is a more flexible concept than s in the two-input case. It allows the firm to alter the usage of inputs other than x_i and x_j when input prices w_i and w_j change.
- e.g., it allows a researcher to study how the ratio of energy to capital input changes when relative energy prices increase while permitting the firm to make any adjustment to labor input (whose price hasn't changed) for cost minimization.

Quantitative size of shifts in cost curves

- The increase in costs when input prices increase will be largely influenced by the relative significance of the input in the production process.
- The extent of cost increases also depends on the ability of firms to substitute another input for the one that has risen in price.
- For example, increases in copper prices in the late 1960s had little impact on electric utilities' costs of distributing electricity because they could easily substitute aluminum for copper cables.

Technical change

- Technical improvements allow the firm to produce a given output with fewer inputs. This will shift total costs downward.
- Suppose the production function exhibits constant returns to scale and is given by

$$q = A(t)f(k,l)$$

where $A(0) = 1$. Total costs in the initial period are

$$C_0(v,w,q) = qC_0(v,w,1)$$

- The same inputs that produced one unit of output in period zero will produce $A(t)$ units in period t, we know that

$$C_0(v,w,1) = C_t(v,w,A(t)) = A(t)C_t(v,w,1).$$

Then the total cost function in period t is

$$C_t(v,w,q) = qC_t(v,w,1) = \frac{qC_0(c,w,1)}{A(t)} = \frac{C_0(v,w,q)}{A(t)}$$

- Hence total costs decrease over time as the rate of technical change.

Example 10.3 Shifting the Cobb-Douglas Cost Function

- The Cobb-Douglas cost function is

$$C(v,w,q) = vk^c + wl^c = q^{1/(\alpha+\beta)}Bv^{\alpha/(\alpha+\beta)}w^{\beta/(\alpha+\beta)}$$

where $B = (\alpha+\beta)\alpha^{-\alpha/(\alpha+\beta)}\beta^{-\beta/(\alpha+\beta)}$.

- Assume that $\alpha = \beta = 0.5$, the total cost function is greatly simplified:

$$C(v,w,q) = 2qv^{0.5}w^{0.5}$$

- If we assume $v = 3, w = 12$ then the relationship is

$$\begin{aligned} C(3,12,q) &= 2q\sqrt{36} = 12q, \\ AC &= \frac{C}{q} = 12 \\ MC &= \frac{\partial C}{\partial q} = 12. \end{aligned}$$

- **Changes in input prices.** If $v = 3, w = 27$, wages increase to 27, then the relationship is

$$
\begin{aligned}
C(3, 27, q) &= 2q\sqrt{81} = 18q, \\
AC &= \frac{C}{q} = 18 \\
MC &= \frac{\partial C}{\partial q} = 18.
\end{aligned}
$$

- Note that an increase in wages of 125%(from 12 to 27) increased costs by only 50% ($12q$ to $18q$), both because labor represents only 50% all costs and because the change in input prices encouraged the firm to substitute capital for labor.

- **Technical progress.** Assume that the Cobb-Douglas production function is

$$q = A(t)k^{0.5}l^{0.5} = e^{.3t}k^{0.5}l^{0.5}.$$

That is, we assume that the rate of technical change is 3% per year. Using the results of the previous section

$$C_t(v, w, q) = \frac{C_0(v, w, q)}{A(t)} = 2qv^{0.5}w^{0.5}e^{-0.3t}.$$

- If input prices remain the same, costs fall at the rate of technical improvement, that is, 3% per year.
- After 20 years, costs will be (with $v = 3, w = 12$)

$$\begin{aligned} C_{20}(3, 12, q) &= 2q\sqrt{36}e^{-0.60} = 12q \cdot 0.55 = 6.6q \\ AC_{20} &= 6.6, \\ MC_{20} &= 6.6. \end{aligned}$$

Contingent demand for inputs and Shephard's lemma

- Because the process of cost minimization holds quantity produced constant, this demand for inputs will be "contingent" on the quantity being produced.
- The Shephard's lemma states that the contingent demand function for any input is given by the partial derivative of the total-cost function with respect to that input's price.
- Shepard's lemma is one result of the envelope theorem, which says that the change in the optimal value in a constrained optimization problem with respect to one of the parameters of the problem can be found by differentiating the Lagrangian for that optimization problem with respect to this changing parameter.

- In the cost-minimization case, the Lagrangian is

$$\mathscr{L} = vk + wl + \lambda[q - f(k, l)]$$

and the envelope theorem applied to either input is

$$\frac{\partial C(v, w, q)}{\partial v} = \frac{\partial \mathscr{L}(v, w, q, \lambda)}{\partial v} = k^c(v, w, q)$$
$$\frac{\partial C(v, w, q)}{\partial w} = \frac{\partial \mathscr{L}(v, w, q, \lambda)}{\partial w} = l^c(v, w, q)$$

where superscript “c" denotes the feature that input demand is contingent on quantity produced q.

Example 10.4 Contingent Input Demand Functions

1. **Fixed Proportions:** $C(v, w, q) = q(v/\alpha + w/\beta)$. Contingent demand functions are:

$$\begin{aligned} k^c(v, w, q) &= \frac{\partial C(v, w, q)}{\partial v} = \frac{q}{\alpha} \\ l^c(v, w, q) &= \frac{\partial C(v, w, q)}{\partial w} = \frac{q}{\beta} \end{aligned}$$

2. **Cobb-Douglas:** $C(v,w,q) = q^{1/(\alpha+\beta)} B v^{\alpha/(\alpha+\beta)} w^{\beta/(\alpha+\beta)}$

$$k^c(v,w,q) = \frac{\partial C}{\partial v} = \frac{\alpha}{\alpha+\beta} \cdot q^{1/(\alpha+\beta)} B v^{-\beta/(\alpha+\beta)} w^{\beta/(\alpha+\beta)}$$

$$= \frac{\alpha}{\alpha+\beta} \cdot q^{1/(\alpha+\beta)} B \left(\frac{w}{v}\right)^{\beta/(\alpha+\beta)}$$

$$l^c(v,w,q) = \frac{\partial C}{\partial w} = \frac{\beta}{\alpha+\beta} \cdot q^{1/(\alpha+\beta)} B v^{\alpha/(\alpha+\beta)} w^{-\alpha/(\alpha+\beta)}$$

$$= \frac{\beta}{\alpha+\beta} \cdot q^{1/(\alpha+\beta)} B \left(\frac{w}{v}\right)^{-\alpha/(\alpha+\beta)}$$

If we assume $\alpha = \beta = 0.5$ (so $B = 2$), these reduce to

$$k^c(v,w,q) = 0.5 \cdot q \cdot 2 \cdot \left(\frac{w}{v}\right)^{0.5} = q\left(\frac{w}{v}\right)^{0.5}$$

$$l^c(v,w,q) = 0.5 \cdot q \cdot 2 \cdot \left(\frac{w}{v}\right)^{-0.5} = q\left(\frac{w}{v}\right)^{-0.5}$$

3. **CES:** $C(v,w,q) = q^{1/\gamma}\left(v^{1-\sigma} + w^{1-\sigma}\right)^{1/(1-\sigma)}$. The contingent demand functions are

$$
\begin{aligned}
k^c(v,w,q) = \frac{\partial C}{\partial v} &= \frac{1}{1-\sigma} \cdot q^{1/\gamma}\left(v^{1-\sigma} + w^{1-\sigma}\right)^{\sigma/(1-\sigma)} (1-\sigma) v^{-\sigma} \\
&= q^{1/\gamma}\left(v^{1-\sigma} + w^{1-\sigma}\right)^{\sigma/(1-\sigma)} v^{-\sigma}, \\
l^c(v,w,q) = \frac{\partial C}{\partial w} &= \frac{1}{1-\sigma} \cdot q^{1/\gamma}\left(v^{1-\sigma} + w^{1-\sigma}\right)^{\sigma/(1-\sigma)} (1-\sigma) w^{-\sigma} \\
&= q^{1/\gamma}\left(v^{1-\sigma} + w^{1-\sigma}\right)^{\sigma/(1-\sigma)} w^{-\sigma}.
\end{aligned}
$$

These function collapse when $\sigma = 1$ (the Cobb-Douglas case), but we can study examples with wither more ($\sigma = 2$) or less ($\sigma = 0.5$) substitutability.

- If we assume constant returns to scale ($\gamma = 1$) and $v = 3, w = 12$, and $q = 40$, then contingent demands for the inputs when σ=2 are

$$
\begin{aligned}
k^c(3,12,40) &= 40(3^{-1} + 12^{-1})^{-2}3^{-2} = 25.6 \\
l^c(3,12,40) &= 40(3^{-1} + 12^{-1})^{-2}12^{-2} = 1.6
\end{aligned}
$$

The level of capital input is 16 times the amount of labor input.

- With less substitutability ($\sigma = 0.5$), contingent demands are

$$
\begin{aligned}
k^c(3,12,40) &= 40(3^{0.5} + 12^{0.5})^{1}3^{-0.5} = 120 \\
l^c(3,12,40) &= 40(3^{0.5} + 12^{0.5})^{1}12^{-0.5} = 60
\end{aligned}
$$

Capital input is only twice as large as labor input.

- For the case of $\sigma = 0.5$, if the price of labor increases to $w = 27$. the contingent demands are

$$
\begin{aligned}
k^c(3, 27, 40) &= 40(3^{0.5} + 27^{0.5})^1 3^{-0.5} = 160 \\
l^c(3, 27, 40) &= 40(3^{0.5} + 27^{0.5})^1 27^{-0.5} = 160/3 = 53.3
\end{aligned}
$$

- The cost savings from substitution can be calculated by comparing total costs when using the initial input combination$(= 3 \cdot 120 + 27 \cdot 60 = 1980)$ to total costs with the optimal combination $(=3 \cdot 160 + 27 \cdot 160/3 = 1920)$.
- Hence moving to optimal input combination reduces total costs by only about 3%, compared to over 20% in the Cobb-Douglas case where $\sigma = 1$. while tt

Shephard's lemma and the elasticity of substitution

- Shepherd's lemma can be used to derive information about input substitution directly from the total cost function.

$$s_{ij} = \frac{\partial \ln(x_i/x_j)}{\partial \ln(w_j/w_i)} = \frac{\partial \ln(C_i/C_j)}{\partial \ln(w_j/w_i)},$$

  where C_i and C_j are the partial derivatives of the total cost function with respect to the input prices.

- Once the total cost function is known (perhaps through econometric estimation), information about substitutability among inputs can thus be readily obtained from it.

Short-Run, Long-Run Distinction

- In the short run, economic actors have only limited flexibility in their actions.
- Assume the capital input is held constant at k_1, the firm is free to vary only its labor input. The production function becomes $q = f(k_1, l)$.

Short-run total costs

- Short-run total cost for the firm is

$$SC = vk_1 + wl,$$

where the S indicates that we are analyzing short-run costs with the level of capital input fixed.

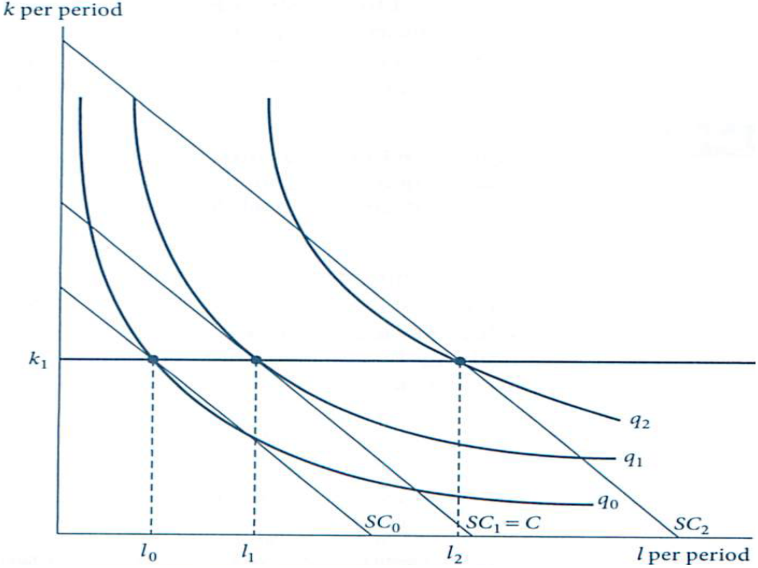
Fixed and variable costs

- **Short-run fixed and variable costs.** *Short-run fixed costs* are costs associated with inputs that can not be varied in the short run. *Short-run variable costs* are costs of those inputs that can be varied to change the firm's output level.

Nonoptimality of short-run costs

- Short-run costs are not the minimal costs for producing the various output levels.
- The firm does not have the flexibility of input choice.
- To vary its output in the short run, the firm must use nonoptimal input combinations.
- The RTS will not be equal to the ratio of input prices.

Figure 10.7 “Nonoptimal" Input Choice Must Be Made in the Short Run



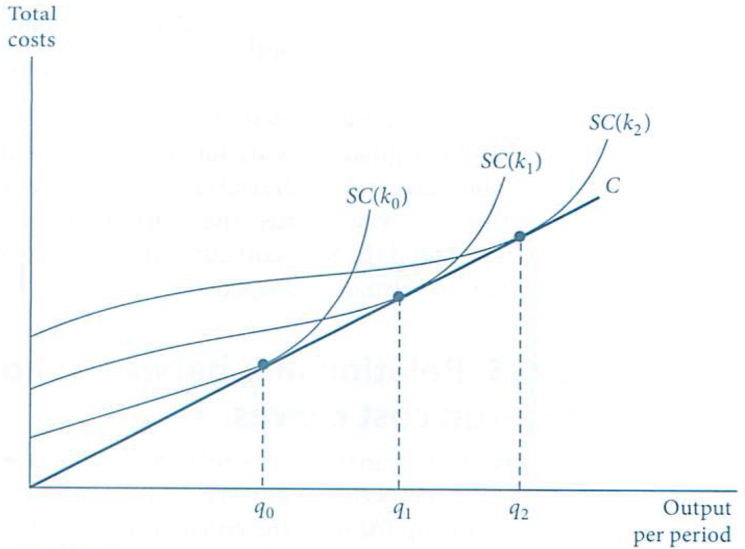
Short-Run Marginal and Average Costs

- The *short-run average total cost* (SAC) function and the *short-run marginal cost* (SMC) are defined as

$$SAC = \frac{\text{total costs}}{\text{total output}} = \frac{SC}{q}$$

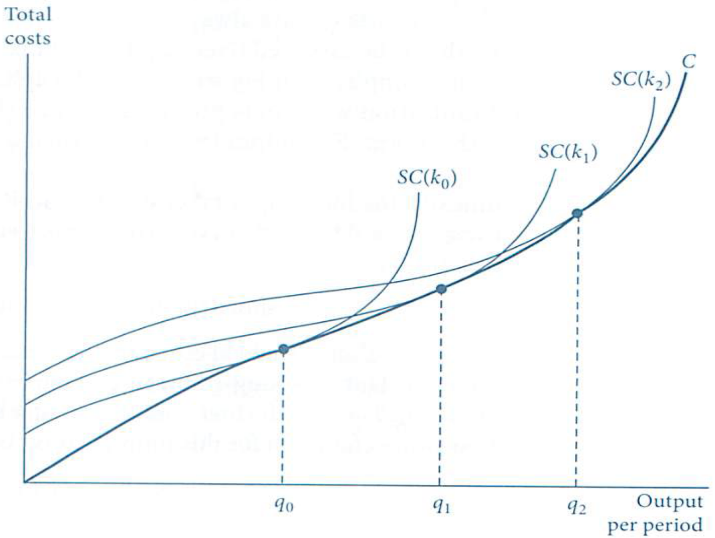
$$SMC = \frac{\text{change in total costs}}{\text{change in output}} = \frac{\partial SC}{\partial q}$$

Figure 10.8 Two Possible Shapes for Long-Run Total Cost Curves



(a) Constant returns to scale

Figure 10.8 Two Possible Shapes for Long-Run Total Cost Curves



(b) Cubic total cost curve case

- Long-run total costs (C) are always less than short-run total costs (SC), except at that output level (q_1) for which the assumed fixed capital input (k_1) is appropriate to long-run cost-minimization.
- For $q_1 : SC = C$. For all other quantities: $SC > C$.
- The long-run total cost curves in Figure 10.8 are said to be an "envelope" of their respective shirt-run curves.
- These short-run total curves can be represented by

$$\text{short} - \text{run total cost} = SC(v, w, q, k),$$

  and the family of short-run total cost curves is generated by allowing k to vary while holding v and w constant.
- The long-run total curve C must obey the first-order condition that $\dfrac{\partial SC(v, w, q, k)}{\partial k} = 0.$

Example 10.5 Envelope Relations and Cobb-Douglas Cost Functions

- Start with the Cobb-Douglas production function $q = k^\alpha l^\beta$, but we hold capital input constant at k_1. Thus in the short run,

$$q = k_1^\alpha l^\beta, \text{ or } l = q^{1/\beta} k_1^{-\alpha/\beta},$$

  and total costs are given by

$$SC(v, w, q, k_1) = vk_1 + wl = vk_1 + wq^{1/\beta} k_1^{-\alpha/\beta}.$$

- To derive long-run costs, we require that k be chosen to minimize total costs:

$$\frac{\partial SC(v, w, q, k)}{\partial k} = v + \frac{-\alpha}{\beta} \cdot wq^{1/\beta} k^{-(\alpha+\beta)/\beta} = 0$$

- This equation can be solved for k and substituted into the equation of SC to return to the Cobb-Douglas cost function:

$$C(v, w, q) = Bq^{1/(\alpha+\beta)} v^{1\alpha/(\alpha+\beta)} w^{\beta/(\alpha+\beta)}.$$

- **Numerical example.** Let $\alpha = \beta = 0.5$, $v = 3$ and $w = 12$, then the short-run cost function is

$$SC(3, 12, q, k_1) = 3k_1 + 12q^2 k_1^{-1}.$$

- In example 10.1 the cost-minimizing level of capital input for $q = 40$ was $k = 80$. The short-run cost function for $q = 40$ and $k = 80$ is

$$\begin{aligned} SC(3, 12, q, 80) &= 3 \cdot 80 + 12q^2 \cdot \frac{1}{80} = 240 + \frac{3q^2}{20} \\ &= 240 + 240 = 480. \end{aligned}$$

- The long-run cost function is

$$C(3, 12, q) = 2 \cdot q \cdot 3^{1/2} 12^{1/2} = 12q.$$

Table 10.1 Difference Between Short-Run and Long-Run Total Cost, k = 80

TABLE 10.1 DIFFERENCE BETWEEN SHORT-RUN AND LONG-RUN TOTAL COST, $k = 80$

q	$C = 12q$	$SC = 240 + 3q^2/20$
10	120	255
20	240	300
30	360	375
40	480	480
50	600	615
60	720	780
70	840	975
80	960	1,200

- Table 10.1 shows that, for output levels other than q = 40, short-run costs are larger than long-run costs and that this difference is proportionally larger the farther one gets from the output level for which k = 80 is optimal.

Table 10.2 Unit Costs in the Long Run and the Short Run, $k = 80$

TABLE 10.2 UNIT COSTS IN THE LONG RUN AND THE SHORT RUN, $k = 80$

q	AC	MC	SAC	SMC
10	12	12	25.5	3
20	12	12	15.0	6
30	12	12	12.5	9
40	12	12	12.0	12
50	12	12	12.3	15
60	12	12	13.0	18
70	12	12	13.9	21
80	12	12	15.0	24

- Here $AC = MC = 12$, the short-run equivalents (where $k = 80$) are

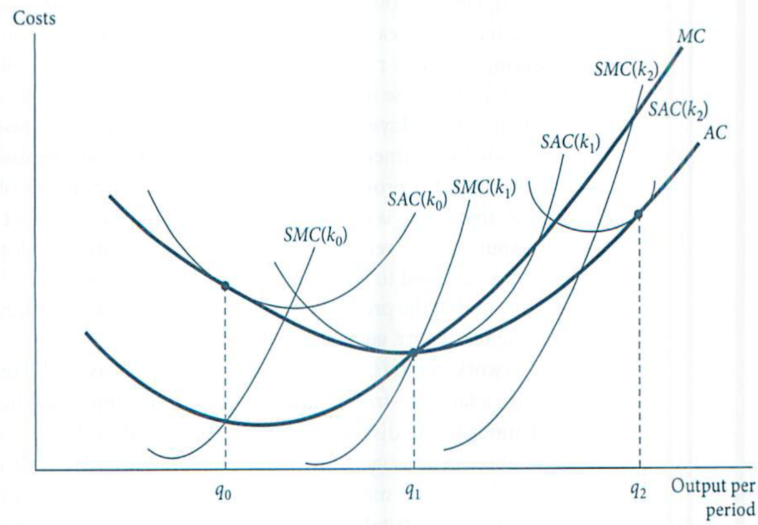
$$SAC = \frac{SC}{q} = \frac{240}{q} + \frac{3q}{20},$$
$$SMC = \frac{\partial SC}{\partial q} = \frac{6q}{20}.$$

Graphs of per-unit cost curves

- At the minimum point of the AC curve (q_1), the MC curve crosses the AC curve, $MC = AC$ at this point.
- The SAC curve is tangent to the AC curve at q_0, q_1, q_2.
- SMC intersects SAC also at q_1 where

$$AC = MC = SAC = SMC$$

Figure 10.9 Average and Marginal Cost Curves for the Cubic Cost Curve Case



Extensions: The Translog Cost Function

- The Cobb-Douglas implicitly assumes that $\sigma = 1$ between any two inputs.
- The CES permits σ to take any value, but it requires that the elasticity of substitution be the same between any two inputs.
- Empirical economists would prefer to let the data show what the actual substitution possibilities among inputs are, they have tried to find more flexible functional forms.
- One especially popular such form is the translog cost function, first made popular by Fuss and McFadden (1978).

E10.1 The translog with two inputs

- From Example 10.2, the Cobb-Douglas cost function in the two-input case is

$$C(v,w,q) = Bq^{1/(\alpha+\beta)} v^{\alpha/(\alpha+\beta)} w^{\beta/(\alpha+\beta)}.$$

  Take the natural logarithm we have

$$\ln C(v,w,q) = \ln B + \frac{1}{\alpha+\beta}\ln q + \frac{\alpha}{\alpha+\beta}\ln v + \frac{\beta}{\alpha+\beta}\ln w.$$

- The translog function generalizes this by permitting second-order terms in input prices:

$$\begin{aligned}\ln C(v,w,q) &= \ln q + a_0 + a_1 \ln v + a_2 \ln w \\ &\quad + a_3(\ln v)^2 + a_4(\ln w)^2 + a_5 \ln v \ln w,\end{aligned}$$

  where this function implicitly assumes constant returns to scale (because the coefficient of $\ln q$ is 1).

Some properties of the translog cost functions are:

- For the function to be homogeneous of degree 1 in input prices, it must be the case that $a_1 + a_2 = 1$ and $a_3 + a_4 + a_5 = 0$.
- This function includes the Cobb-Douglas as the special case $a_3 = a_4 = a_5 = 0$, hence can be used to test statistically whether the Cobb-Douglas is appropriate.
- Input shares can be computed using the result that $s_i = \frac{\partial \ln C}{\partial \ln w_i}$. In the two-input case, this yields

$$
\begin{aligned}
s_k &= \frac{\partial \ln C}{\partial \ln v} = a_1 + 2a_3 \ln v + a_5 \ln w, \\
s_l &= \frac{\partial \ln C}{\partial \ln w} = a_2 + 2a_4 \ln w + a_5 \ln v.
\end{aligned}
$$

- Calculating the elasticity of substitution in the translog case by using the result in Problem 10.11 that

$$s_{kl} = e_{k^c,w} - e_{l^c,w}$$

where

$$\begin{aligned} e_{k^c,w} &= \frac{\partial \ln k^c}{\partial \ln w} = \frac{\partial \ln C_v}{\partial \ln w} = \frac{\partial \ln\left(\frac{C}{v}\cdot\frac{v}{C}\frac{\partial C}{\partial v}\right)}{\partial \ln w} = \frac{\partial \ln\left(\frac{C}{v}\cdot\frac{\partial \ln C}{\partial \ln v}\right)}{\partial \ln w} \\ &= \frac{\partial\left(\ln C - \ln v + \ln\left(\frac{\partial \ln C}{\partial \ln v}\right)\right)}{\partial \ln w} \\ &= \frac{w}{C}\cdot\frac{\partial C}{\partial w} - 0 + \frac{\partial \ln s_k}{\partial s_k}\cdot\frac{\partial^2 \ln C}{\partial \ln v \partial \ln w} \\ &= s_l + \frac{a_5}{s_k} \end{aligned}$$

Note that $\frac{\partial \ln C}{\partial \ln w} = \frac{w}{C}\cdot l = s_l$, $\frac{\partial \ln C}{\partial \ln v} = \frac{v}{C}\cdot k = s_k$. Observe that in the Cobb-Douglas case ($a_5 = 0$), $e_{k^c,w} = s_l$.

- Similar manipulations yields

$$
\begin{aligned}
e_{l^c,w} &= \frac{\partial \ln l^c}{\partial \ln w} = \frac{\partial \ln C_w}{\partial \ln w} = \frac{\partial \ln\left(\frac{C}{w}\cdot\frac{w}{C}\frac{\partial C}{\partial w}\right)}{\partial \ln w} = \frac{\partial \ln\left(\frac{C}{w}\cdot\frac{\partial \ln C}{\partial \ln w}\right)}{\partial \ln w} \\
&= \frac{\partial\left(\ln C - \ln w + \ln\left(\frac{\partial \ln C}{\partial \ln w}\right)\right)}{\partial \ln w} \\
&= \frac{w}{C}\cdot\frac{\partial C}{\partial w} - 1 + \frac{\partial \ln s_l}{\partial s_l}\cdot\frac{\partial^2 \ln C}{\partial \ln w^2} \\
&= s_l - 1 + \frac{2a_4}{s_l} = -s_k + \frac{2a_4}{s_l}
\end{aligned}
$$

Note that $s_k + s_l = 1$ and observe that in the Cobb-Douglas case $(a_4 = 0)$ $e_{l^c,w} = -s_k$.

- Bringing these two elasticities together yields

$$\begin{aligned} s_{kl} &= e_{k^c,w} - e_{l^c,w} \\ &= s_l + s_k + \frac{a_5}{s_k} - \frac{2a_4}{s_l} = 1 + \frac{s_l a_5 - 2 s_k a_4}{s_k s_l} \end{aligned}$$

  Again, in the Cobb-Douglas case we have $s_{kl} = 1$, as should have been expected.

- The Allen elasticity of substitution (see Problem 10.12) for the translog function is

$$A_{ij} = 1 + \frac{a_5}{s_k s_l}$$

  This function can also be used to calculate that the (contingent) cross-price elasticity of demand is

$$e_{k^c,w} = s_l A_{ij} = s_l + \frac{a_5}{s_k},$$

  as was shown previously.

The many-input translog cost function

- Most empirical studies include more than two inputs. The translog cost function is easy to generalize to these situations.
- Assume that there are n inputs, each with a price of $w_i (i = 1, \cdots, n)$ then the function is

$$
\begin{aligned}
C(w_1, \cdots, w_n, q) &= \ln q + a_0 + \sum_{i=1}^{n} a_i \ln w_i \\
&+ 0.5 \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \ln w_i \ln w_j,
\end{aligned}
$$

where we have assumed constant returns to scale. The function requires $a_{ij} = a_{ji}$ so each term for which $i \neq j$ appears twice in the final double sum.

- For this function to be homogeneous of degree one in the input prices, it must be the case that

$$\sum_{i=1}^{n} a_i = 1, \sum_{i=1}^{n} a_{ij} = 0$$

Two useful properties of this function are:

- Input shares take the linear form

$$s_i = a_i + \sum_{j=1}^{n} a_{ij} \ln w_j.$$

- The elasticity of substitution between any two inouts in the translog function is given by

$$s_{ij} = 1 + \frac{s_j a_{ij} - s_i a_{jj}}{s_i s_j}$$

E10.3 Some applications

- The translog cost function has become the main choice for empirical studies of production. Two factors account for this popularity.
- First, the function allows a fairly complete characterization of substitution patterns among inputs.
- Second, the function's format incorporates input prices in a flexible way so that one can be reasonably sure that he or she has controlled fir such prices in regression analysis.

Examples:

- Westbrook and Buckley (1990) studied the responses that shippers made to changing relative prices of moving goods that resulted from deregulation of the railroad and trucking industries in the United States.
- The authors look specifically at the shipping of fruits and vegetables from the western states to Chicago and New York.
- They find relatively high substitution elasticities among shipping options and so conclude that deregulation had significant welfare benefits.

- Doucouliagos and Hone (2000) provide a similar analysis of deregulation of dairy prices in Australia.
- They show that changes in the prices of raw milk caused dairy processing firms to undertake significant changes in input usage.
- They also show that the industry adopted significant new technologies in response to the price change.
- Latzko's (1999) uses the translog function to judge returns to scale of the U.S. mutual funds industry.
- He finds that the elasticity of total costs with respect to the total assets managed by the fund is less than 1 for all but the largest funds (those with more than $4 billion in assets).
- The author concludes that money management exhibits substantial returns to scale.

- Garcia and Thomas (2001) look at water supply systems in local French community to estimate economies of scale.
- They conclude thtat there are significant operating economies of scale in such systems and that some merging of systems would make sense.
- Yatchew (2000) reaches a similar conclusion about electricity distribution in small communities in Ontario, Canada.
- He finds that there are economies of scale for electrical distribution systems serving up to about 20,000 customers.
- Some efficiencies might be obtained from merging systems that are much smaller than this size.