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Chapter 5
Income and Substitution Effects, part II

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2020.10.30.

Demand Elasticities

- Thus far in this chapter we have been examining how individuals respond to changes in prices and income by looking at the derivatives of the demand function.
- However, focusing on derivative has one major disadvantage for empirical work: The sizes of derivatives depend directly on how variables are measured.
- This makes comparisons among goods or across countries and time periods difficult.
- For this reason, most empirical work in microeconomics uses some form of elasticity measure.

Marshallian demand elasticities

From the Marshallian demand function, $x(p_x, p_y, I)$,

- *Price elasticity of demand*, e_{x,p_x}.

$$e_{x,p_x} = \frac{\Delta x/x}{\Delta p_x/p_x} = \frac{\Delta x}{\Delta p_x} \cdot \frac{p_x}{x} = \frac{\partial x(p_x, p_y, I)}{\partial p_x} \cdot \frac{p_x}{x}$$

- *Income elasticity of demand*, $e_{x,I}$.

$$e_{x,I} = \frac{\Delta x/x}{\Delta I/I} = \frac{\Delta x}{\Delta I} \cdot \frac{I}{x} = \frac{\partial x(p_x, p_y, I)}{\partial I} \cdot \frac{I}{x}$$

- *Cross-price elasticity of demand*, e_{x,p_y} (Chapter 6).

$$e_{x,p_y} = \frac{\Delta x/x}{\Delta p_y/p_y} = \frac{\Delta x}{\Delta p_y} \cdot \frac{p_y}{x} = \frac{\partial x(p_x, p_y, I)}{\partial p_y} \cdot \frac{p_y}{x}$$

Price elasticity of demand

- The own price elasticity of demand is always negative, except for the unlikely case of Giffen's paradox.
- The size of the elasticity is important.
  - If $e_{x,p_x} < -1$, demand is elastic.
  - If $e_{x,p_x} > -1$, demand is inelastic.
  - If $e_{x,p_x} = -1$, demand is unit elastic.

Price elasticity and total spending

- Total spending on x is $p_x \cdot x$. The effects of a change in p_x on total spending is

$$\frac{\partial p_x \cdot x}{p_x} = p_x \cdot \frac{\partial x}{\partial p_x} + x = x(e_{x,p_x} + 1)$$

- If $0 > e_{x,p_x} > -1$, demand is inelastic. Price and total spending move in the same direction.
- If $e_{x,p_x} < -1$, demand is elastic. Price and total spending move in opposite directions.
- For the unit-elastic case ($e_{x,p_x} = -1$), total spending is constant no matter how price changes.

Compensated Price Elasticities

From compensated demand function, $x^c(p_x, p_y, U)$,

- *Compensated own-price elasticity of demand,* e_{x^c,p_x}

$$e_{x^c,p_x} = \frac{\Delta x^c / x^c}{\Delta p_x / p_x} = \frac{\Delta x^c}{\Delta p_x} \cdot \frac{p_x}{x} = \frac{\partial x(p_x, p_y, U)}{\partial p_x} \cdot \frac{p_x}{x^c}$$

- *Compensated cross-price elasticity of demand,* e_{x^c,p_y}

$$e_{x^c,p_y} = \frac{\Delta x^c / x^c}{\Delta p_y / p_y} = \frac{\Delta x^c}{\Delta p_y} \cdot \frac{p_y}{x} = \frac{\partial x(p_y, p_y, U)}{\partial p_y} \cdot \frac{p_y}{x^c}$$

- Multiplying p_x/x to the Slutsky equation yields

$$\frac{p_x}{x} \cdot \frac{\partial x}{\partial p_x} = e_{x,p_x} = \frac{p_x}{x} \cdot \frac{\partial x^c}{\partial p_x} - \frac{p_x}{x} \cdot x \cdot \frac{\partial x}{\partial I} = e_{x^c,p_x} - s_x e_{x,I}$$

  where $s_x = \frac{p_x x}{I}$ is the share of total income devoted to the purchase of x.

- The Slutsky equation shows that the compensated and uncompensated price elasticities will be similar if
  - The share of income devoted to x (s_x) is small.
  - The income elasticity of x ($e_{x,I}$) is small.
- Hence, there are many economic circumstances in which substitution effects constitutes the most important component of price responses.

Relationships among Demand Elasticities

Homogeneity

- Demand functions $x(p_x, p_y, I)$ are homogeneous of degree zero in all prices and income.
- Euler's theorem for homogenous functions shows that

$$0 = p_x \cdot \frac{\partial x}{\partial p_x} + p_y \cdot \frac{\partial x}{\partial p_y} + I \frac{\partial x}{\partial I},$$

  divide by x yields

$$0 = e_{x,p_x} + e_{x,p_y} + e_{x,I}.$$

- The net sum of all price elasticities together with the income elasticity for a particular good must sum to zero.

Engel aggregation

- Engel's law states that income elasticity of demand for food items is less than 1, while income elasticity of demand for all non-food items must be greater than 1.
- Differentiating the budget constraint ($I = p_x x + p_y y$) with respect to income

$$
\begin{aligned}
1 &= p_x \cdot \frac{\partial x}{\partial I} + p_y \cdot \frac{\partial y}{\partial I} \\
1 &= p_x \cdot \frac{\partial x}{\partial I} \cdot \frac{xI}{xI} + p_y \cdot \frac{\partial y}{\partial I} \cdot \frac{yI}{yI} = s_x e_{x,I} + s_y e_{y,I}
\end{aligned}
$$

- The weighted average on income elasticities for all goods that a person buys must be 1.

Cournot aggregation

- Differentiating the budget constraint with respect to p_x

$$\frac{\partial I}{\partial p_x} = 0 = p_x \cdot \frac{\partial x}{\partial p_x} + x + p_y \cdot \frac{\partial y}{\partial p_x}$$

  Multiply this equation by p_x/I yields

$$\begin{aligned} 0 &= p_x \cdot \frac{\partial x}{\partial p_x} \cdot \frac{p_x}{I} \cdot \frac{x}{x} + x \cdot \frac{p_x}{I} + p_y \cdot \frac{\partial y}{\partial p_x} \cdot \frac{p_x}{I} \cdot \frac{y}{y} \\ 0 &= s_x e_{x,p_x} + s_x + s_y e_{y,p_x}, \end{aligned}$$

  so the Cournot result is

$$s_x e_{x,p_x} + s_y e_{y,p_x} = -s_x$$

- The budget constraint imposes some limit on the degree to which the the cross-price elasticity (e_{y,p_x}) can be positive.

Example 5.5

Demand Elasticities:The Importance of Substitution Effects

Case 1. Cobb-Douglas ($\sigma = 1$). $U(x, y) = x^{\alpha} y^{\beta}, \alpha + \beta = 1.$

- Demand functions are

$$x(p_x, p_y, I) = \frac{\alpha I}{p_x}, y(p_x, p_y, I) = \frac{\beta I}{p_y} = \frac{(1-\alpha)I}{p_y}$$

- Elasticities are

$$\begin{aligned} e_{x,p_x} &= \frac{\partial x}{\partial p_x} \cdot \frac{p_x}{x} = \frac{-\alpha I}{p_x^2} \frac{p_x}{\alpha I / p_x} = -1 \\ e_{x,p_y} &= \frac{\partial x}{\partial p_y} \cdot \frac{p_y}{x} = 0 \cdot \frac{p_y}{x} = 0 \\ e_{x,I} &= \frac{\partial x}{\partial I} \cdot \frac{I}{x} = \frac{\alpha}{p_x} \cdot \frac{I}{\alpha I / p_x} = 1 \end{aligned}$$

For the Cobb-Douglas function, $s_x = \alpha, s_y = \beta$.

- **homogeneity**

$$e_{x,p_x} + e_{x,p_y} + e_{x,I} = 1 + 0 + 1 = 0$$

- **Engel aggregation**

$$s_x e_{x,I} + s_y e_{y,I} = \alpha \cdot 1 + \beta \cdot 1 = \alpha + \beta = 1$$

- **Cournot aggregation**

$$s_x e_{x,p_x} + s_y e_{y,p_x} = \alpha \cdot (-1) + \beta \cdot 0 = -\alpha = -s_x$$

- Using the Slutsky equation in elasticity form to derive the compensated price elasticity:

$$e_{x^c,p_x} = e_{x,p_x} + s_x e_{x,I} = -1 + \alpha \cdot 1 = \alpha - 1 = -\beta$$

Case 2. CES utility function ($\sigma = 2, \delta = 0.5$), $U(x, y) = x^{0.5} + y^{0.5}$

- The demand functions are

$$x(p_x, p_y, I) = \frac{I}{p_x(1 + p_x p_y^{-1})},$$
$$y(p_x, p_y, I) = \frac{I}{p_y(1 + p_x^{-1} p_y)}.$$

- The "share elasticity" of any good is

$$e_{s_x, p_x} = \frac{\partial s_x}{\partial p_x} \cdot \frac{p_x}{s_x}, \qquad s_x = \frac{p_x x}{I}$$
$$= \left(\frac{x}{I} + \frac{p_x}{I} \cdot \frac{\partial x}{\partial p_x} \right) \frac{p_x}{\frac{p_x x}{I}}$$
$$= 1 + e_{x, p_x}$$

- In this case,

$$s_x = \frac{p_x x}{I} = \frac{1}{1 + p_x p_y^{-1}}$$

The share elasticity is

$$e_{s_x, p_x} = \frac{\partial s_x}{\partial p_x} \frac{p_x}{s_x} = \frac{-p_y^{-1}}{(1 + p_x p_y^{-1})^2} \cdot \frac{p_x}{(1 + p_x p_y^{-1})^{-1}}$$

$$= \frac{-p_x p_y^{-1}}{1 + p_x p_y^{-1}}$$

We can arbitrary define the units of goods so that initially $p_x = p_y$, then

$$e_{x, p_x} = e_{s_x, p_x} - 1 = \frac{-1}{1+1} - 1 = -1.5$$

Hence demand is more elastic in this case than in the Cobb-Douglas example.

Case 3.

CES utility function ($\sigma = 0.5, \delta = -1$), $U(x, y) = -x^{-1} - y^{-1}$

- The share of good x is

$$s_x = \frac{1}{1 + p_y^{0.5} p_x^{-0.5}}$$

so the share elasticity is

$$e_{s_x,p_x} = \frac{\partial s_x}{\partial p_x} \cdot \frac{p_x}{s_x} = \frac{0.5 p_y^{0.5} p_x^{-1.5}}{(1 + p_y^{0.5} p_x^{-0.5})^2} \cdot \frac{p_x}{(1 + p_y^{0.5} p_x^{-0.5})^{-1}}$$

$$= \frac{0.5 p_y^{0.5} p_x^{-0.5}}{1 + p_y^{0.5} p_x^{-0.5}}$$

When $p_x = p_y$, the own-price elasticity is

$$e_{x,p_x} = e_{s_x,p_x} - 1 = \frac{0.5}{2} - 1 = -0.75$$

- and the compensated price elasticity is

$$e_{x^c,p_x} = e_{x,p_x} + s_x e_{x,I} = -0.75 + 0.5(1) = -0.25$$

- The own-price elasticity in this case is smaller than in Case 1 and Case 2 because the substitution effect is smaller. The main variations in these cases are caused by differences in the size of the substitution effect.
- In general (Problem 5.9),

$$e_{x^c,p_x} = -(1 - s_x)\sigma$$

Consumer Surplus

- An important problem in applied economics is to devise a monetary measure of the utility gains and losses that individuals experience when prices change. For example,
  - welfare loss when a market is monopolized
  - welfare gains when technical progress reduces prices
  - welfare costs of incorrectly priced resources
  - the costs of excess protections taken in fear of lawsuits
  - excess burden of a tax
- To make such calculations, economists use empirical data from studies of market demand in combination with the theory that underlies that demand.

Consumer welfare and the expenditure function

- We want to measure the change in welfare an individuals experiences if the price of good x increases from p_x^0 to p_x^1.
- To reach a utility of U_0, it requires expenditures of

$$E(p_x^0, p_y, U_0)$$

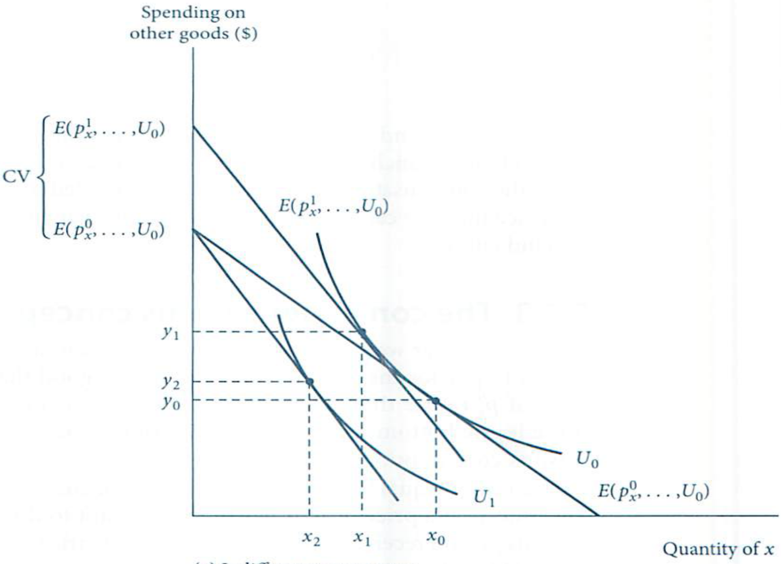
- To reach the same utility once the price of x increases, the required expenditure becomes

$$E(p_x^1, p_y, U_0)$$

- To compensate for the price increase, this person requires a compensation (*compensating variation, CV*) of

$$CV = E(p_x^1, p_y, U_0) - E(p_x^0, p_y, U_0)$$

Figure 5.8 Showing Compensating Variation



(a) Indifference curve map

Using the compensated demand curve to show CV

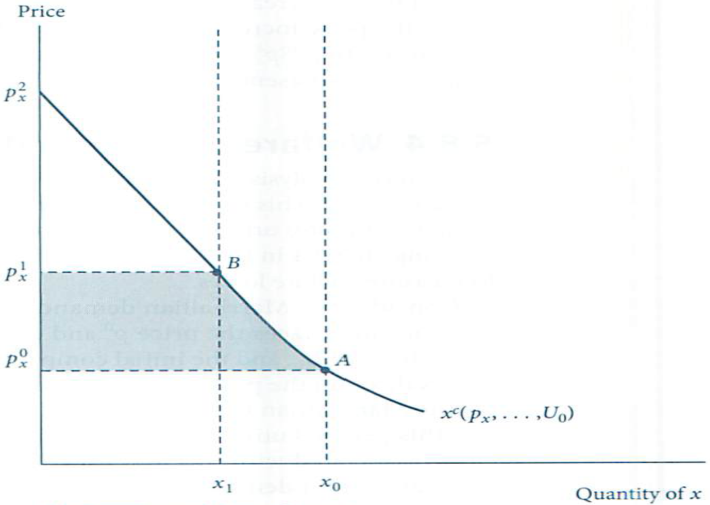
- The derivative of the expenditure function with respect to p_x is the compensated demand function.

$$x^c(p_x, p_y, U) = \frac{\partial E(p_x, p_y, U)}{\partial p_x}$$

- The compensation required can be found by integrating across a sequence of small increments to price from p_x^0 to p_x^1:

$$CV = \int_{p_x^0}^{p_x^1} \frac{\partial E(p_x, p_y, U_0)}{\partial p_x} dp_x = \int_{p_x^0}^{p_x^1} x^c(p_x, p_y, U_0) dp_x$$

Figure 5.8 Showing Compensating Variation



(b) Compensated demand curve

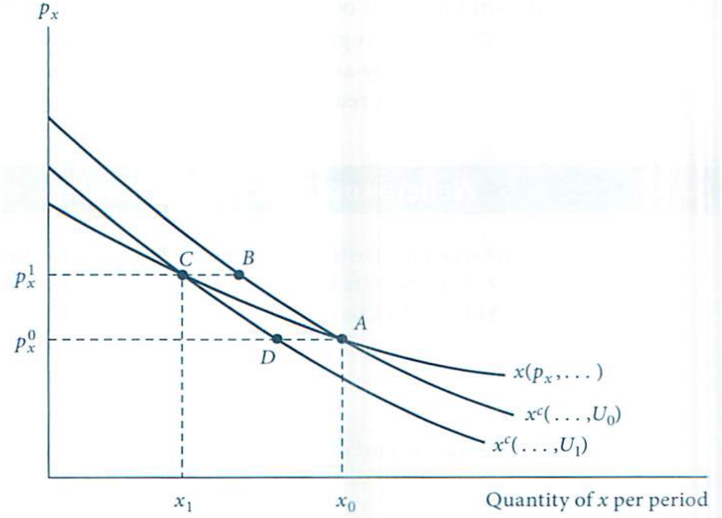
The consumer surplus concept

- Another way to look at this issue is to measure *consumer surplus*, which is the area below the compensated demand curve and above the market price.
- It is the extra benefit the person receives by being able to make market transactions at the prevailing market price.
- When the price increases from p_x^0 to p_x^1, the consumer surplus “triangle" decreases in size from $p_x^2 A p_x^0$ to $p_x^2 B p_x^1$.

Welfare changes and the Marshallian demand curve

- Instead of compensated demand curves, most empirical work on demand actually estimates ordinary (Marshallian) demand curves.
- The changes in the area below a Marchallian demand curve may be a good way to measure welfare losses.
- Should we measures the welfare loss from the price increase as area $p_x^1 BA p_x^0$ (with utility level U_0, compensated variation, $CV = E(p_x^1, p_y, U_0) - E(p_x^0, p_y, U_0)$)
- Or, should we measures the welfare loss from the price increase as area $p_x^1 CD p_x^0$? (with utility level U_1, equivalent variation, $EV = E(p_x^1, p_y, U_1) - E(p_x^0, p_y, U_1)$)
- Marshallian demand curve provides a convenient compromise between CV and EV, which is area $p_x^1 CA p_x^0$.

Figure 5.9 Welfare Effects of Price Changes and the Marshallian Demand Curve



Example 5.6 Welfare Loss from a Price Increase

- From Example 5.3, the compensated demand function for good x was given by

$$x^c(p_x, p_y, V) = \frac{V p_y^{0.5}}{p_x^{0.5}}$$

Hence the welfare cost of price increase from \$1 to \$4 is

$$CV = \int_1^4 V p_y^{0.5} p_x^{-0.5} dp_x = 2V p_y^{0.5} p_x^{0.5} \Big|_{p_x=1}^{p_x=4}$$

  - For $V = 2, p_y = 4$, $CV = 8$.
  - For utility after the price increase $V = 1, p_y = 4$, then $CV = 4$.

- If instead we use the Marshallian demand function

$$x(p_x, p_y, I) = 0.5Ip_x^{-1}$$

the loss would be

$$loss = \int_1^4 0.5Ip_x^{-1}dp_x = 0.5I \ln p_x \Big|_1^4$$

With $I = 8$,

$$loss = 4\ln 4 - 4\ln 1 = 4 \cdot 1.39 = 5.55$$

which is between the two alternative measures based on the compensated demand functions.

Revealed Preference and the Substitution Effect

- The principal unambiguous prediction that can be derived from the utility-maximization model is that the slope (price elasticity) of the compensated demand curve is negative.
- The reliance on a hypothesis about an unobservable utility function represented a weak foundation on which to base a theory of demand.
- An alternative approach, the theory of revealed preference, was proposed by Paul Samuelson in the late 1940s.
- It defines a principle of rationality based on observed reactions to differing budget constraints and uses it to approximate an individual's utility function.

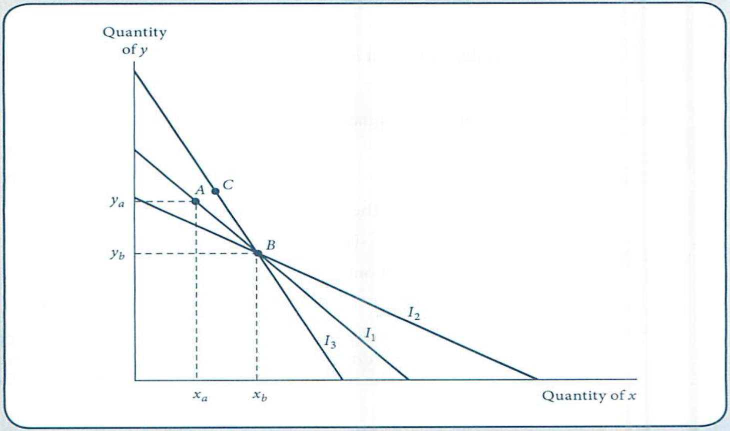
Graphical approach

Consider two bundles of goods: A and B

- If the individual can afford to purchase either bundle but chooses A, we say that A had been revealed preferred to B.
- The principle of rationality states that under any other price-income arrangement, B can never be revealed preferred to A.
- If B in fact chosen at another price-income configuration, it must be because the individual could not afford A.

Figure 5.10 Demonstration of the Principle of Rationality in the Theory of Revealed Preference

With income I_1 the individual can afford both points A and B. If A is selected, then A is revealed preferred to B. It would be irrational for B to be revealed preferred to A in some other price–income configuration.



Revealed Preference and the negativity of the Substitution Effect

- Suppose that an individual is *indifferent* between two bundles: C (x_C, and y_C) and D (x_D and y_D).
- Let p_x^C and p_y^C be the prices at which bundle C is chosen and p_x^D and p_y^D be the prices at which bundle D is chosen.
- Because the individual is indifferent between C and D, it must be the case that

$$\begin{aligned} p_x^C x_C + p_y^C y_c &\leq p_x^C x_D + p_y^C y_D \\ p_x^D x_D + p_y^D y_D &\leq p_x^D x_C + p_y^D y_C \end{aligned}$$

Rewriting these together yields

$$p_x^C(x_C - x_D) + p_y^C(y_C - y_D) \leq 0$$
$$p_x^D(x_D - x_C) + p_y^D(y_D - y_C) \leq 0$$

- Adding these together yields

$$(p_x^C - p_x^D)(x_C - x_D) + (p_y^C - p_y^D)(y_C - y_D) \leq 0$$

Suppose that only the price of x changes, assuming that $p_y^C = p_y^D$, then

$$(p_x^C - p_x^D)(x_C - x_D) \leq 0$$

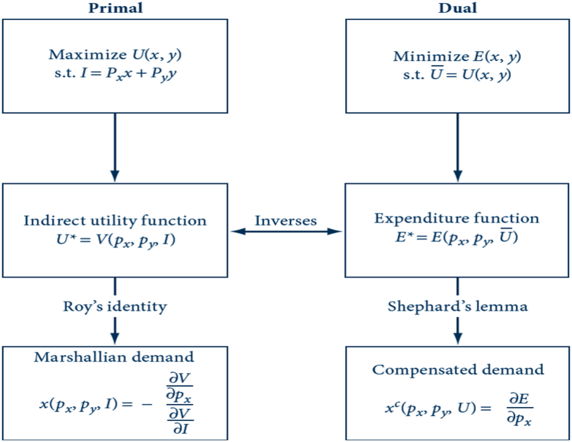
- The price and quantity move in the opposite direction when utility is held constant. This is precisely a statement about the nonpositive nature of the substitution effect

$$\frac{\partial x^c(p_x, p_y, V)}{\partial p_x} = \left.\frac{\partial x}{\partial p_x}\right|_{U=constant} \leq 0$$

- This result does not require the existence of a quasi-concave utility function.

Extensions: Demand Concept and the Evaluation of Price Indices

Figure E5.1 Relationships among Demand Concepts



- In this extension we will look at how the concepts of demand can shed light on the accuracy of the consumer price index (CPI), the primary measure of inflation.
- The CPI is a "Market basket" index of the cost of living.
- The cost of the market basket initially would be

$$I_0 = p_x^0 x_0 + p_y^0 y_0$$

and the cost in period 1 would be

$$I_1 = p_x^1 x_0 + p_y^1 y_0.$$

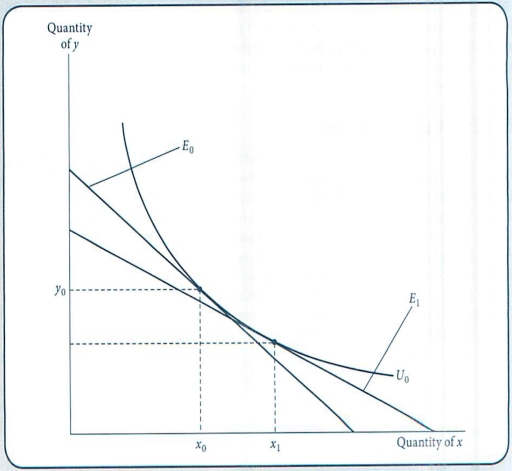
- The change in the cost of living would then be measured by I_1/I_0.

E5.1 Expenditure functions and substitution bias

- Market basket price indices suffer from "substitution bias."
- Because the indices do not permit individuals to make substitutions in the market basket in response to changes in relative prices, they will tend to overstate the welfare losses that people incur from increasing prices.
- This exaggeration is illustrated in Figure E5.2.
- Aizocorbe and Jackman (1993) find that this difficulty with a market basket index may exaggerate the level of inflation shown by the CPI by approximately 0.2 percent per year.

Figure E5.2 Substitution Bias in the CPI

Initially expenditures are given by E_0, and this individual buys x_0, y_0. If p_x/p_y decreases, utility level U_0 can be reached most cheaply by consuming x_1, y_1 and spending E_1. Purchasing x_0, y_0 at the new prices would cost more than E_1. Hence holding the consumption bundle constant imparts an upward bias to CPI-type computations.



E5.2 Roy's identity and new goods bias

- When new goods are introduced, it takes some time for them to be integrated into the CPI. For example, it takes 15 years for cell-phones to enter the basket (Hausman 1999, 2003).
- Market basket indices will fail to reflect the welfare gains that people experience from using new goods.
- To measure these costs, Hausman sought to measure a "virtual" price (p^*) at which the demand for cell phones would be zero and then argue that the introduction of the good at its market price represented a change in the consumer surplus that could be measured.
- Hence the author was faced with the problem of how to get from Marshallian demand function to the expenditure function.

- The indirect utility function is $V(p_x, p_y, I)$. Applying the envelope theorem to this value function yields

$$\begin{aligned}\frac{dV(p_x,p_y,I)}{dp_x} &= \frac{\partial \mathcal{L}}{\partial p_x} = -\lambda x(p_x,p_y,I),\\ \frac{dV(p_x,p_y,I)}{dI} &= \frac{\partial \mathcal{L}}{\partial I} = \lambda\end{aligned}$$

- These equations allow us to extract the Marshall demand function as

$$x(p_x,p_y,I) = \frac{-\partial V/\partial p_x}{\partial V/\partial I}$$

This is called "Roy's identity."

- Using his estimate of the Marshallian demand function, Hausman integrated the equation above to obtain the implied indirect utility function and then calculated its inverse, the expenditure function. (Check Figure E5.1)
- It yielded large estimate for the gain in consumer welfare from cell phones— a present value in 1999 of more than $100 billion.
- Delay's in the inclusion of such goods into the CPI can result in a misleading measure of consumer surplus.

E5.3 Other complaints about the CPI

- Other complaints focus on the consequences of using incorrect prices to compute the index.
- When the quality of a good improves, people are made better off, although this may not show up in the good's price.
- The opening of "big box" retailers such as Costco and Home Depot reduced the prices that consumers paid for various goods.
- Including these new retail outlets into the sample scheme for the CPI took several years.

E5.4 Exact price indices

- Suppose there are only two goods and we wish to know how purchasing power has changed between period 1 and period 2.
- If the expenditure function is given by $E(p_x, p_y, U)$, then

$$I_{1,2} = \frac{E(p_x^2, p_y^2, \bar{U})}{E(p_x^1, p_y^1, \bar{U})}$$

shows how the cost of attaining the target utility level $\bar{U}$ has changed between the two periods.

- Without knowing the representative person's utility function, we would not know the specific form of the expenditure function.

- For the Cobb-Douglas utility function $U(x, y) = x^{\alpha} y^{1-\alpha}$, the expenditure function is

$$E(p_x, p_y, U) = p_x^{\alpha} p_y^{1-\alpha} U / \alpha^{\alpha}(1-\alpha)^{1-\alpha} = k p_x^{\alpha} p_y^{1-\alpha} U$$

Thus,

$$I_{1,2} = \frac{(p_x^2)^{\alpha}(p_y^2)^{1-\alpha}}{(p_x^1)^{\alpha}(p_y^1)^{1-\alpha}}$$

- The utility target cancels out in the construction of the cost-of-living index.

E5.5 Development of exact price indices

- Feenstra and Reinsdorf (2000) show that the almost ideal demand system in the Extensions to Chapter 4 implies an exact price index (I) that takes a “Divisia" form:

$$\ln(I) = \sum_{i=1}^{n} w_i \Delta \ln p_i$$

where w_i are weights attached to each good's price.

- This is the price index implies by the Cobb-Douglas utility function because

$$\begin{aligned} \ln(I_{1,2}) &= \alpha \ln p_x^2 + (1-\alpha)\ln p_y^2 - \alpha \ln p_x^1 - (1-\alpha)\ln p_y^1 \\ &= \alpha \Delta \ln p_x + (1-\alpha)\Delta \ln p_y \end{aligned}$$