Suggested Answers for Problem Set 2

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1a $\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}$,
$\hat{\alpha}_1 = \bar{Y} - \hat{\alpha}_2 \bar{x} = \bar{Y}$, where $x_i = X_i - \bar{X}$, $\bar{x} = \frac{1}{n}\sum X_i - \bar{X} = 0$.
$\text{Var}(\hat{\beta}_1) = \frac{\sum X_i^2}{n \sum x_i^2}\sigma^2$,
$\text{Var}(\hat{\alpha}_1) = \frac{\sum x_i^2}{n \sum (x_i - \bar{x})^2}\sigma^2 = \frac{\sigma^2}{n}$.
Therefore, neither the estimates nor the variances for $\hat{\beta}_1$ and $\hat{\alpha}_1$ are equal.

1b $\hat{\beta}_2 = \frac{\sum x_i y_i}{\sum x_i^2}$,
$\hat{\alpha}_2 = \frac{\sum (x_i - \bar{x}) y_i}{\sum (x_i - \bar{x})^2} = \frac{\sum x_i y_i}{\sum x_i^2}$,
$\text{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum x_i^2}$,
$\text{Var}(\hat{\alpha}_2) = \frac{\sigma^2}{\sum x_i^2}$.
Therefore, the estimates and variance for $\hat{\beta}_2$ and $\hat{\alpha}_2$ are identical.

2 If we multiply each X by 2, the residuals and fitted values for Y will not change. Initially, let $Y_i = \beta_1 + \beta_2 X_i = u_i$, then $\hat{\beta}_2 = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2}$, $\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}$, $\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$.

Now let $Z_i = 2X_i$ and let $Y_i = \alpha_1 + \alpha_2 Z_i + u_i$, then

$$\begin{aligned}
\hat{\alpha}_2 &= \frac{\sum (Z_i - \bar{Z})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} = \frac{\sum (2X_i - 2\bar{X})(Y_i - \bar{Y})}{\sum (2X_i - 2\bar{X})^2} = \frac{1}{2}\hat{\beta}_2 \\
\hat{\alpha}_1 &= \bar{Y} - \hat{\alpha}_2 \bar{Z} = \bar{Y} - \frac{1}{2}\hat{\beta}_2 2\bar{X} = \hat{\beta}_1 \\
\hat{Y} &= \hat{\alpha}_1 + \hat{\alpha}_2 Z_i = \hat{\beta}_1 + \frac{1}{2}\hat{\beta}_2 2X_i = \hat{\beta}_1 + \hat{\beta}_2 X_i
\end{aligned}$$

The analysis is analogous if a constant is added to X_i.

3a All of these estimators are unbiased.

3b $\text{Var}[\hat{\mu}_1] = \frac{\sigma^2}{n}$, $\text{Var}[\hat{\mu}_2] = \sigma^2$, $\text{Var}[\hat{\mu}_3] = \frac{n\sigma^2}{4(n-1)}$. Therefore, $\hat{\mu}_1$ is the most efficient.

3c Only $\text{Var}[\hat{\mu}_1]$ converges to 0, it converges in mean square which implies plim $\hat{\mu}_1 = \mu$. Therefore, $\hat{\mu}_1$ is consistent.

4 The log-likelihood function is

$$\ln L = -n \ln \theta - \frac{\sum X_i}{\theta}$$

The first order condition is

$$\frac{\partial \ln L}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum X_i}{\theta^2} = 0$$

Therefore, the ML estimator of $\hat{\theta} = \frac{\sum X_i}{n} = \bar{X}$.

5a True. Unbiasedness of the estimators does not require a normally distributed population error term. It does require that the error term has an expected value of zero.

5b True. Since $\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}$, if the slope coefficient is zero, then the intercept is estimated by the sample mean of Y, $\bar{Y}$.

6 If $r^2 = 1$, all the points lie on a straight line. So, whether we minimize sum of squares of vertical distances (regression of Y on X) or horizontal distances (regression of X on Y) does not matter. Both are zero in this case.

On the other hand, if the simple regression line of Y against X coincides with the simple regression line of X againist Y, then $\hat{\beta}_2^{YX} = \frac{1}{\hat{\beta}_2^{XY}}$ where $\hat{\beta}_2^{YX}$ is the slope coefficient of regressing Y on X and $\hat{\beta}_2^{XY}$ is the slope coefficient of regressing X on Y. Therefore,

$$\hat{\beta}_2^{YX} \hat{\beta}_2^{XY} = \frac{\sum x_i y_i}{\sum x_i^2} \frac{\sum x_i y_i}{\sum y_i^2} = r^2 = 1.$$

7a For (Y_i, X_i), we have $u_i \sim IN(0, 1)$. For $(\bar{Y}_i, \bar{X}_i)$, the erros are $IN(0, 2)$ since the errors are $u_1 + u_2$, $u_3 + u_4$, $\cdots$. For (Y_i^*, X_i^*), the errors are correlated but normally distributed with variance $\frac{1}{2}$. Note that the erros are $\frac{u_1+u_2}{2}$, $\frac{u_2+u_3}{2}$, $\frac{u_3+u_4}{2}$ etc. The consecutive erros are correlated. There covariance is $\frac{1}{4}$.

7b Yes, since the classical assumptions are statisfied with annual data.