Suggested Answers for Problem Set 1

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1a

$$
\begin{aligned}
f(x) &= \int_0^1 f(x,y)dy = \int_0^1 \frac{1}{3}(4-x-y)\,dy = \frac{1}{3}\left[(4-x)y - \frac{1}{2}y^2\right]_0^1 \\
&= \frac{1}{3}(4-x-\frac{1}{2}-0) = \frac{1}{3}(3.5-x) \\
f(y) &= \frac{1}{3}(3.5-y).
\end{aligned}
$$

1b

$$
\begin{aligned}
f(x|y) &= \frac{f(x,y)}{f(y)} = \frac{4-x-y}{3.5-y}, 0 \le x, y \le 1 \\
f(y|x) &= \frac{f(x,y)}{f(x)} = \frac{4-x-y}{3.5-x}, 0 \le x, y \le 1
\end{aligned}
$$

1c

$$
\begin{aligned}
\mathrm{E}(X) &= \int_0^1 xf(x)dx = \int_0^1 \frac{1}{3}(3.5x - x^2)dx = \frac{1}{3}\left[\frac{3.5}{2}x^2 - \frac{x^3}{3}\right]_0^1 = 0.4722 \\
\mathrm{E}(Y) &= 0.4722
\end{aligned}
$$

1d

$$
\begin{aligned}
\mathrm{E}(X|Y=0.4) &= \int_0^1 xf(x|y=0.4)dx = \int x\frac{4-x-0.4}{3.5-0.4}dx = \frac{1}{3.1}\int_0^1 3.6x - x^2\,dx \\
&= \frac{1}{3.1}\left[1.8x - \frac{x^3}{3}\right]_0^1 = 0.4731
\end{aligned}
$$

2a No. P(X=1, Y=2)=0.2. But, $P(X=1)P(Y=2) = 0.4 \times 0.4 = 0.16$. Thus, X and Y are not independent.

2b $P(X=1) = 0.4$, $P(X=2) = 0.2$, $P(X=3) = 0.4$, $P(Y=2) = 0.4$, $P(Y=4) = 0.6$, $P(Y=6) = 0.4$.

2c $P(Y=2|X=1) = \frac{0.2}{0.4} = 0.5$, $P(Y=4|X=1) = 0$, and $P(Y=6|X=1) = 0.5$. Therefore, $\mathrm{E}(Y|X=1) = 4$ and $\mathrm{Var}(Y|X=1) = 0.5(2-4)^2 + 0.5(6-4)^2 = 4$.

3 Only (c) is a linear regression model. However, after some transformations, (a) and (b) can become linear regression model with the transformed dependent variable. For model (a), $\ln Y_i = \beta_1 + \beta_2 X_i = u_i$, and for model (c),

$$\begin{aligned}\frac{1 - Y_i}{Y_i} &= e^{\beta_1 + \beta_2 X_i - u_i} \\ \ln\left(\frac{1 - Y_i}{Y_i}\right) &= \beta_1 + \beta_2 X_i - u_i\end{aligned}$$

can both be written as linear regression model.