Suggested Answers for Midterm Exam. I

Nov. 2, 2001

1a False. Statistical significance and practical significance are different. They don't imply each other.

1b False. It states that in the class of linear unbiased estimators, the least-squares estimators have the minimum variance.

1c False. Since $\text{Var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum(X_i - \bar{X})^2}$. If X-values are closer to their sample mean, $\sum x_i^2$ is smaller. Then $\text{Var}(\hat{\beta}_2)$ is greater. $\hat{\beta}_2$ is not precisely estimated.

1d True. This is the first equation in normal equations. However, this is not true if an intercept is not included in the model.

2

$$
\begin{aligned}
\text{E}(A) &= \sum x_i^2 \text{Var}(\hat{\beta}_2) = \sigma^2 \\
\text{E}(B) &= \text{E}(\sum u_i^2) - 2\text{E}(\bar{u}\sum u_i) + \text{E}(\bar{u}^2) \\
&= n\sigma^2 - 2n\frac{1}{n}\sigma^2 + \frac{1}{n^2}n\sigma^2 = (n-1)\sigma^2 \\
\text{E}(C) &= -2\text{E}\left(\sum k_i u_i \sum x_i(u_i - \bar{u})\right) \\
&= -2\text{E}\left(\sum k_i u_i \sum x_i u_i - \sum k_i u_i (\sum x_i)\bar{u}\right) \\
&= -2\left(\sum k_i x_i \sigma^2 - \sum x_i \frac{\sum k_i}{n}\sigma^2\right) = -2\sigma^2 \\
\text{E}\left(\sum \hat{u}_i^2\right) &= (n-2)\sigma^2
\end{aligned}
$$

3 The log-likelihood function is

$$\ln L = -n\ln\theta - \frac{\sum X_i}{\theta}$$

The first order condition is

$$\frac{\partial \ln L}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum X_i}{\theta^2} = 0$$

Therefore, the ML estimator of $\hat{\theta} = \frac{\sum X_i}{n} = \bar{X}$.

4a Let $\hat{\hat{\beta}}_1$ and $\hat{\hat{\beta}}_2$ be the new estimators and $X_i^* = \frac{X_i}{10^3}$ then

$$\begin{aligned}
\hat{\hat{\beta}}_2 &= \frac{\sum x_i^* y_i}{\sum x_i^{*2}} = \frac{\frac{\sum x_i y_i}{10^3}}{\frac{\sum x_i^2}{10^6}} = 10^3 \hat{\beta}_2 \\
\hat{\hat{\beta}}_1 &= \bar{Y} - \hat{\hat{\beta}}_2 \bar{X}_i^* = \bar{Y} - 10^3 \hat{\beta}_2 \frac{\bar{X}}{10^3} = \hat{\beta}_1 \\
\hat{u}_i^* &= Y_i - \hat{\hat{\beta}}_1 - \hat{\hat{\beta}}_2 X_i^* = Y_i - \hat{\beta}_1 - 10^3 \hat{\beta}_2 \frac{X_i}{10^3} = \hat{u}_i \\
\mathrm{Var}(\hat{\hat{\beta}}_2) &= \frac{\sum \hat{u}_i^{*2}/(n-2)}{\sum x^{*2}} = \frac{\sum \hat{u}_i^2/(n-2)}{\frac{\sum x^2}{10^6}} = 10^6 \mathrm{Var}(\hat{\beta}_2)
\end{aligned}$$

4b Similar to (a), and we further have $Y^* = \frac{Y}{10^3}$, therefore

$$\begin{aligned}
\hat{\hat{\beta}}_2 &= \frac{\sum x^* y^*}{\sum x^{*2}} = \frac{\frac{\sum xy}{10^6}}{\frac{\sum x^2}{10^6}} = \hat{\beta}_2 \\
\hat{\hat{\beta}}_1 &= \bar{Y}^* - \hat{\hat{\beta}}_2 \bar{X}^* = \frac{\bar{Y}}{10^3} - \hat{\beta}_2 \frac{\bar{X}}{10^3} = \frac{\hat{\beta}_1}{10^3} \\
\hat{u}_i^* &= Y_i^* - \hat{\hat{\beta}}_1 - \hat{\hat{\beta}}_2 X_i^* = \frac{Y_i}{10^3} - \frac{\hat{\beta}_1}{10^3} - \hat{\beta}_2 \frac{X_i}{10^3} = \frac{\hat{u}_i}{10^3} \\
\mathrm{Var}(\hat{\hat{\beta}}_2) &= \frac{\sum \hat{u}_i^{*2}/(n-2)}{\sum x^{*2}} = \frac{\frac{\sum \hat{u}_i^2/(n-2)}{10^6}}{\frac{\sum x^2}{10^6}} = \mathrm{Var}(\hat{\beta}_2)
\end{aligned}$$

5a $R^2 = \frac{\sum \hat{y}_i^2}{\sum y_i^2} = \frac{1.5297859}{28.1255874} = 0.0544.$

5b The t statistic for $\hat{\beta}_2$ is 4.45. Therefore, it is significant at a 99% level of confidence. You can also see this from the small p-value of $\hat{\beta}_2$.

5c $F = \frac{1.5297859}{.077313376} = 19.79 = t^2.$

5d The son's wage will increase by $10 * 0.156 = 1.56\%$.