Suggested Answers for Final Exam.

2003.1.17

1a False. The least squares estimators are still efficient.

1b False. Even with hetoroscedastic errors, the OLS estimators are unbiased. However, the standard errors are biased.

1c False. The estimators are unbiased so long as u_i is uncorrelated with X_i.

1d True.

1e False. One usually obtains high R^2 in models with highly correlated regressors.

2a Since $\text{Var}(\bar{u}_g) = \frac{\text{Var}(\sum_{i=1}^{n_g} u_{gi})}{n_g^2} = \frac{\sigma^2}{n_g}$, model (1) is heteroscedastic.

2b $\text{Var}(u^*) = k^2\text{Var}(\bar{u}_g) = \frac{k^2\sigma^2}{n_g}$, we can choose k to be $\sqrt{n_g}$, then $\text{Var}(u^*) = \sigma^2$ is homoscedastic.

2c $\sqrt{n_g}$ and $\sqrt{n_g}\bar{X}_g$.

2d From the transformed model in (b), we minimize

$$\sum_{g=1}^{G} \hat{u^*}_g^2 = \sum_{g=1}^{G} k^2 \hat{\bar{u}}_g^2 \equiv \sum_{g=1}^{G} w_g \hat{\bar{u}}_g^2$$

Therefore, $w_g = k^2 = n_g$.

3a

$$\min_{\hat{\beta}} \ \frac{1}{\sigma^2}(Y_1 - \hat{\beta})^2 + \frac{1}{2\sigma^2}(Y_2 + \hat{\beta})^2$$

The first order condition is

$$-2(Y_1 - \hat{\beta}) + (Y_2 + \hat{\beta}) = 0$$

Then $\hat{\beta}_{WLS} = \frac{2Y_1 - Y_2}{3}$.

$$\begin{aligned}
\hat{\beta}_{WLS} &= \frac{2(\beta + u_1) - (-\beta + u_2)}{3} = \beta + \frac{2u_1 - u_2}{3} \\
\text{E}(\hat{\beta}_{WLS}) &= \beta + \frac{2\text{E}(u_1) - \text{E}(u_2)}{3} = \beta \\
\text{Var}(\hat{\beta}_{WLS}) &= \text{E}[(\hat{\beta}_{WLS} - \beta)^2] = \frac{4\sigma^2 + 2\sigma^2}{9} = \frac{2}{3}\sigma^2
\end{aligned}$$

3b You can obtain $\hat{\beta}_{OLS}$ by the standard fromula, or derive it step by step as follows.

$$\min_{\hat{\beta}} (Y_1 - \hat{\beta})^2 + (Y_2 + \hat{\beta})^2$$

The first order condition is

$$-2(Y_1 - \hat{\beta}) + 2(Y_2 + \hat{\beta}) = 0$$

Then $\hat{\beta}_{OLS} = \frac{Y_1 - Y_2}{2}$.

$$\begin{aligned}
\hat{\beta}_{OLS} &= \frac{(\beta + u_1) - (-\beta + u_2)}{2} = \beta + \frac{u_1 - u_2}{2} \\
\mathrm{E}(\hat{\beta}_{OLS}) &= \beta + \frac{\mathrm{E}(u_1) - \mathrm{E}(u_2)}{2} = \beta \\
\mathrm{Var}(\hat{\beta}_{OLS}) &= \mathrm{E}[(\hat{\beta}_{OLS} - \beta)^2] = \frac{\sigma^2 + 2\sigma^2}{4} = \frac{3}{4}\sigma^2
\end{aligned}$$

3c From (a) and (b), $\mathrm{Var}(\hat{\beta}_{WLS}) = \frac{2}{3}\sigma^2 < \frac{3}{4}\sigma^2 = \hat{\beta}_{OLS}$, $\mathrm{Var}(\hat{\beta}_{WLS})$ is smaller. $\hat{\beta}_{WLS}$ is more efficient.

4a

$$vN = \frac{n}{n-1} d$$

4b $vN \in (0, 4)$.

4c Compute the mean and variance of vN for $n = 100$. $\mathrm{E}(vN) = \frac{200}{99} = 2.02$, $V(vN) = 4 \cdot 100^2 \frac{98}{101 \cdot 99^3} = 0.04$. Therefore, $vN \sim N(2.02, 0.04)$. 2.88 would be in the far right side of the normal distribution since $t = \frac{2.88 - 2.04}{0.2} = 4.2$. We can **reject** the null hypothese that there is no autocorrelation. Since 2.88 is on the right side, there is **negative** serial correlation.