Introduction to Quantatative Methods
Final Exam.
September 10, 2001

1. (15%) Suppose A is a symmetric and idempotent matrix with full rank K, then
   (a) What is the trace of A?
   (b) What is the determinant of A?
   (c) What is the difference between A and the identity matrix I?

2. (15%) Determine those valuse of λ for which the following set of equations may posses a nonzero solution:

$$\begin{aligned} 3x_1 + x_2 - \lambda x_3 &= 0 \\ 4x_1 - 2x_2 - 3x_3 &= 0 \\ 2\lambda x_1 + 4x_2 + \lambda x_3 &= 0 \end{aligned}$$

   For each permissible value of λ, determine the solution such that $x_1^2 + x_2^2 + x_3^2 = 1$.

3. (15%) Find the vector x that minimize

$$y = x'Ax + a'x - 10,$$

   where A is a $K \times K$ symmetric matrix, a and x are both $K \times 1$ vectors.
   (a) Write the first order condtion for minimization and derive the solution of x.
   (b) Suppose matrix A is $\begin{bmatrix} 25 & 7 \\ 7 & 13 \end{bmatrix}$ and $a = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, calculate the optimal x.
   (c) Check the second order condition for the minimization of y.

4. (15%) Let $x_1, x_2, \cdots, x_n$ be a sample of size n from a normal distribution $N(\mu, \sigma^2)$. Consider the following point estimator of μ:

$$\begin{aligned} \hat{\mu}_1 &= \bar{x}, \text{ the sample mean} \\ \hat{\mu}_2 &= x_1 \\ \hat{\mu}_3 &= \frac{x_1}{2} + \frac{1}{2(n-1)}(x_2 + x_3 + \cdots + x_n) \end{aligned}$$

   (a) Which of these estimators are unbiased?
   (b) Which of these is the most efficient?

(c) Which of these are consistent.

5. (20%) We can write a regression model in the matrix form

$$\underset{n\times 1}{y} = \underset{n\times K}{X} \underset{K\times 1}{\beta} + \underset{n\times 1}{\epsilon}$$

Suppose ϵ is normally distribution with zero mean and covariance matrix Σ. It can be shown that the least square estimator of β is $\hat{\beta} = (X'X)^{-1}X'y$.

(a) Show that $\hat{\beta}$ is unbiased.

(b) Let $\hat{\epsilon} \equiv y - X\hat{\beta} = My$, where $M = I - X(X'X)^{-1}X'$. Show that M is both symmetric and idempotent.

(c) Show that $\hat{\epsilon}'X = 0$. In other words, $\hat{\epsilon}$ and X are orthogonal.

(d) Derive the distribution of $\hat{\beta}$?

6. (20%) Briefly answer or prove the following questions.

(a) (5%) Suppose A is a 3×3 symmetric matrix, A's eigenvalues are $1, 2, 3$ and A's eigenvectors are c_1, c_2 and c_3. What are the eigenvalues and eigenvectors of A^3.

(b) (5%) Using the fact that $(A^{-1})' = (A')^{-1}$ to show that, if A is symmetric, then A^{-1} is also symmtric.

(c) (10%) Let A be a $n \times K$ matrix and B be a $K \times n$ matrix, prove that $\text{trace}(AB) = \text{trace}(BA)$.