Suggested Answers for Introduction to Quantatative Methods Final Exam.

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1a The eigenvalues of an idempotent matrix are either 1 or 0. Since A is of full rank, all the eigenvalues are 1. Therefore, $\text{trace}(A) = \sum_{i=1}^{K} \lambda_i = K$.

1b $|A| = \prod_{i=1}^{K} \lambda_i = 1$.

1c Since $\Lambda = \text{diag}(1, 1, \cdots, 1) = I$, then $A = C\Lambda C' = CC' = I$. $A - I = 0$.

2 This is a homogeneous system, to have a nonzero solution, we have

$$\begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$$

This gives $\lambda^2 + 8\lambda - 9 = 0$, or $\lambda = 1$ or $\lambda = -9$. For $\lambda = 1$, the solution is $(x_1, x_2, x_3) = \pm\frac{1}{\sqrt{6}}(1, 1, 2)$. For $\lambda = -9$, the solution is $(x_1, x_2, x_3) = \pm\frac{1}{\sqrt{94}}(3, 9, -2)$.

3a The first order condition is $2Ax + a = 0$, and $x = -\frac{1}{2}A^{-1}a$.

3b $x = -\frac{1}{2}\begin{bmatrix} 25 & 7 \\ 7 & 13 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = -\frac{1}{552}\begin{bmatrix} 13 & -7 \\ -7 & 25 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{5}{552} \\ -\frac{61}{552} \end{bmatrix}$.

3c The Hessian matrix is $2A$. We need to show that A is a positive definite matrix. One way of checking this is to find the eigenvlues of A.

$$\begin{vmatrix} 25-\lambda & 7 \\ 7 & 13-\lambda \end{vmatrix} = \lambda^2 - 38\lambda + 276 = 0$$

$$\lambda_1 + \lambda_2 = 38,\ \lambda_1\lambda_2 = 276$$

Therefore, $\lambda_1, \lambda_2 > 0$. A is positive definite.

4a All of these estimators are unbiased.

4b $\text{Var}[\hat{\mu}_1] = \frac{\sigma^2}{n}$, $\text{Var}[\hat{\mu}_2] = \sigma^2$, $\text{Var}[\hat{\mu}_3] = \frac{n\sigma^2}{4(n-1)}$. Therefore, $\hat{\mu}_1$ is the most efficient.

4c Only $\text{Var}[\hat{\mu}_1]$ converges to 0, it converges in mean square which implies $\text{plim}\ \hat{\mu}_1 = \mu$. Therefore, $\hat{\mu}_1$ is consistent.

5a $\hat{\beta} = (X'X)^{-1}X'(X\beta + \epsilon) = \beta + (X'X)^{-1}X'\epsilon$, $\text{E}[\hat{\beta}] = \beta + (X'X)^{-1}X'\text{E}[\epsilon] = \beta$. $\hat{\beta}$ is unbiased.

5b

$$\begin{aligned} M' &= I - \left(X(X'X)^{-1}X'\right)' = I - X(X'X)^{-1}X' = M, M \text{ is symmetric.} \\ MM &= \left(I - X(X'X)^{-1}X'\right)\left(I - X(X'X)^{-1}X'\right) \\ &= I - X(X'X)^{-1}X' - X(X'X)^{-1}X' + X(X'X)^{-1}X'X(X'X)^{-1}X' \\ &= I - X(X'X)^{-1}X' = M, M \text{ is idempotent.} \end{aligned}$$

5c

$$\begin{aligned} \hat{\epsilon}'X &= (My)'X = y'MX = y'(I - X(X'X)^{-1}X')X \\ &= y'(X - X(X'X)^{-1}X'X) = y'(X - X) = 0 \end{aligned}$$

5d We have $\mathrm{E}[\hat{\beta}] = \beta$, and

$$\begin{aligned} \mathrm{Var}[\hat{\beta}] &= \mathrm{E}[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] \\ &= \mathrm{E}[((X'X)^{-1}X'\epsilon)((X'X)^{-1}X'\epsilon)'] \\ &= (X'X)^{-1}X'\mathrm{E}[\epsilon\epsilon']X(X^{-1}X)^{-1} \\ &= (X'X)^{-1}X'\Sigma X(X^{-1}X)^{-1} \end{aligned}$$

Therefore, the distribution of $\hat{\beta}$ is

$$\hat{\beta} \sim N\left(\beta, (X'X)^{-1}X'\Sigma X(X^{-1}X)^{-1}\right)$$

6a 1,8,27 and c_1, c_2 and c_3.

6b Since A is symmetric, then $A' = A$. $(A^{-1})' = (A')^{-1} = A^{-1}$, so A^{-1} is also symmetric.

6c Let $A : n \times K$, $B : K \times n$, $C = AB$ and $D = BA$. Since $c_{ii} = a^i b_i = \sum_{k=1}^{K} a_{ik}b_{ki}$, and $d_{kk} = b^k a_k = \sum_{i=1}^{n} b_{ki}a_{ik}$,

$$\mathrm{tr}(AB) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{K} a_{ik}b_{ki} = \sum_{k=1}^{K}\sum_{i=1}^{n} b_{ki}a_{ik} = \sum_{k=1}^{K} d_{kk} = \mathrm{tr}(BA)$$