Introduction to Quantatative Methods
Final Exam.
September 16, 2005

1. (10%) For the matrices

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 \\ 1 & 5 \\ 6 & 2 \end{bmatrix}$$

compute AB, $A'B'$ and BA.

2. (15%) Suppose A is a symmetric and idempotent matrix with full rank K, then
   (a) What is the trace of A?
   (b) What is the determinant of A?
   (c) What is the difference between A and the identity matrix I?

3. (15%) Briefly answer or prove the following questions.
   (a) (5%) Suppose A is a 2×2 symmetric matrix, A's eigenvalues are $1, 2$ and A's eigenvectors are c_1 and c_2. What are the eigenvalues and eigenvectors of A^2.
   (b) (5%) Using the fact that $(AB)' = B'A'$ to show $(A^{-1})' = (A')^{-1}$.
   (c) (5%) (True or False ?) If A is positive definite, then A^{-1} is also positive definite.

4. (15%) Find the vector x that minimize

$$y = x'Ax + 2a'x - 10,$$

where A is a $K \times K$ symmetric matrix, a and x are both $K \times 1$ vectors.
   (a) What is the first order condtion for optimization? Derive the solution for x.
   (b) Suppose matrix A is $\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$ and $a = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, calculate the optimal x.
   (c) Is the optimum in (b) a maximum or a minimum?

5. (15%) Let $\{z_i\}$ be a sequence of i.i.d. (independently and identically distributed) random variables with $\mathrm{E}(z_i) = \mu \neq 0$ and $\mathrm{Var}(z_i) = \sigma^2 > 0$, and let $\bar{z}_n$ be the sample mean.
   (a) (5%) What is the limiting distribution of $\sqrt{n}(\bar{z}_n - \mu)$?

(b) (10%) What is the limiting distribution of $\sqrt{n}(\sqrt{\bar{z}_n} - \sqrt{\mu})$?

6. (30%) We can write a regression model in the matrix form

$$\underset{n \times 1}{y} = \underset{n \times K}{X} \underset{K \times 1}{\beta} + \underset{n \times 1}{u}$$

Suppose u is distributed with zero mean and covariance matrix Σ, i.e. $\mathrm{E}(uu') = \Sigma$, and $\mathrm{E}(X'u) = 0$. It can be shown that the least square estimator of β is $\hat{\beta} = (X'X)^{-1}X'y$.

(a) (5%) Show that Σ is symmetric and positive definite.

(b) (5%) What is the covariance matrix of Pu, where P is a $n \times n$ matrix?

(c) (5%) Find a $n \times n$ matrix P such that the covariance matrix of Pu is an identity matrix I_n. (Hint: think about the diagonalization and spectral decomposition of Σ.)

(d) (5%) Show that $\hat{\beta}$ is consistent.

(e) (5%) Let $\hat{u} \equiv y - X\hat{\beta} = My$, where $M = I - X(X'X)^{-1}X'$. Show that M is both symmetric and idempotent.

(f) (5%) Show that $X'\hat{u} = 0$. In other words, $\hat{u}$ and X are orthogonal.

Suggested Answers for Introduction to Quantatative Methods Final Exam.

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1

$$
\begin{aligned}
AB &= \begin{bmatrix} 23 & 25 \\ 14 & 30 \end{bmatrix} \\
A'B' &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 \\ 4 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 10 \\ 22 & 23 & 26 \\ 10 & 8 & 20 \end{bmatrix} \\
BA &= (A'B')' = \begin{bmatrix} 10 & 22 & 10 \\ 11 & 23 & 8 \\ 10 & 26 & 20 \end{bmatrix}
\end{aligned}
$$

2a The eigenvalues of an idempotent matrix are either 1 or 0. Since A is of full rank, all the eigenvalues are 1. Therefore, $\text{trace}(A) = \sum_{i=1}^{K} \lambda_i = K$.

2b $|A| = \prod_{i=1}^{K} \lambda_i = 1$.

2c Since $\Lambda = \text{diag}(1, 1, \cdots, 1) = I$, then $A = C\Lambda C' = CC' = I$. $A - I = 0$.

3a 1,4 and c_1, c_2.

3b Since $AA^{-1} = I$, then $(AA^{-1})' = (A^{-1})'A' = I' = I$. Therefore, $(A')^{-1} = (A^{-1})'$.

3c The eigenvalues of A^{-1} are the reciprocals of eigenvalues of A which are all positive because A is positive definite. Therefore, A^{-1} is also positive definite.

4a The first order condition is $2Ax + 2a = 0$, and $x = -A^{-1}a$.

4b $x = -\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = -\frac{1}{8} \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

4c The Hessian matrix is $2A$. We need to check whether A is positive or negative definite. One way to check this is to find the eigenvlues of A.

$$
\begin{aligned}
\begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} &= \lambda^2 + 6\lambda + 8 = 0 \\
\lambda_1 + \lambda_2 &= -6, \quad \lambda_1\lambda_2 = 8
\end{aligned}
$$

Therefore, $\lambda_1, \lambda_2 < 0$. A is negative definite. Optimum in (b) is a maximum.

5a From the Central Limit Theorem,

$$\sqrt{n}(\bar{z}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$$

5b Let $a(x) = \sqrt{x}$, from Delta method we have

$$\sqrt{n}(a(\bar{z}_n) - a(\mu)) \xrightarrow{d} N\left(0, \left(\frac{da(\mu)}{d\mu}\right)^2 \sigma^2\right)$$

and

$$\sqrt{n}(\sqrt{\bar{z}_n} - \sqrt{\mu}) \xrightarrow{d} N\left(0, \frac{\sigma^2}{4\mu}\right).$$

6a $\Sigma = \mathrm{E}(uu')$, then $\Sigma' = (\mathrm{E}(uu'))' = \mathrm{E}(uu') = \Sigma$, Σ is symmetric.
Let a be a $n \times 1$ column vector, then

$$\begin{aligned} a'\Sigma a &= \mathrm{E}(a'uu'a) = \mathrm{E}(\epsilon'\epsilon), \quad \epsilon \equiv u'a \\ &= \mathrm{E}(\epsilon^2) > 0 \end{aligned}$$

Σ is positive definite.

6b

$$\mathrm{Var}(Pu) = \mathrm{E}(Puu'P') = P\Sigma P'$$

6c Since $\Sigma = C\Lambda C'$ where columns of C are eigenvectors and the diagonal elements of Λ are eignevalues of Σ. Because Σ is positive definite, its eigenvalues are positive and thus $\Lambda^{1/2}$ exists.

We want to find a P such that

$$PC\Lambda^{1/2}\Lambda^{1/2}C'P' = I_n$$

Note that $C'C = I$, if $P = \underline{\Lambda^{-1/2}C'}$, then

$$\Lambda^{-1/2}C'C\Lambda^{1/2}\Lambda^{1/2}C'C\Lambda^{-1/2} = I_n$$

6d

$$\begin{aligned} \hat{\beta} &= (X'X)^{-1}X'(X\beta + u) \\ &= \beta + \left(\frac{X'X}{n}\right)^{-1} \frac{X'u}{n} \\ &\xrightarrow{p} \beta + \big(\mathrm{E}(X'X)\big)^{-1} \cdot \mathrm{E}(X'u) = \beta. \end{aligned}$$

$\hat{\beta}$ is consistent.

6e

$$\begin{aligned} M' &= I - \big(X(X'X)^{-1}X'\big)' = I - X(X'X)^{-1}X' = M, M \text{ is symmetric.} \\ MM &= \big(I - X(X'X)^{-1}X'\big)\big(I - X(X'X)^{-1}X'\big) \\ &= I - X(X'X)^{-1}X' - X(X'X)^{-1}X' + X(X'X)^{-1}X'X(X'X)^{-1}X' \\ &= I - X(X'X)^{-1}X' = M, M \text{ is idempotent.} \end{aligned}$$

6f

$$\begin{aligned} \hat{X}'u &= X'My = X'(I - X(X'X)^{-1}X')y \\ &= X'y - X'y = 0 \end{aligned}$$