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Are stationarity and cointegration restrictions really necessary for the intertemporal budget constraint?

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Abstract

Time series related to fiscal and external deficits are commonly subjected to stationarity and cointegration tests to assess if the deficits are sustainable. Such tests are incapable of rejecting sustainability. The intertemporal budget constraint proves to be satisfied if either the debt series or the revenue and with-interest spending series are integrated of *arbitrarily high* order, i.e., stationary after differencing arbitrarily often. Revenues and spending do not have to be cointegrated. Rejections of low-order difference-stationarity and of cointegration are thus consistent with the intertemporal budget constraint. Error-correction-type policy reaction functions are suggested as more promising for understanding deficit problems.

JEL classification: C12; C22; E60; F34; H60

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1. Introduction

Unit root and cointegration tests are commonly employed to examine if time series are consistent with an intertemporal budget constraint (IBC). In the fiscal policy literature, these methods are widely used to examine the government budget constraint—the sustainability of public debts and deficits. In the international literature, the same tools are

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used to examine the sustainability of external debts and current account deficits. Standard empirical strategies focus on testing if the debt series is difference-stationary or if revenues and spending are suitably cointegrated. Rejections are interpreted as evidence against sustainability, usually citing Trehan and Walsh (1988, 1991), Quintos (1995), or related papers for the alleged necessity of such conditions.[1]

This paper explains why standard unit root and cointegration tests are incapable of rejecting the consistency of data sets with the IBC. I prove that if the relevant debt variable is stationary after *any finite number* of differencing operations, then the IBC is satisfied. The IBC is also satisfied if revenues and with-interest spending are difference-stationary of arbitrary order, and this *without* cointegration requirement.

Compared to other sufficient conditions, my propositions are extensions of well-known first- and second-order integration and cointegration results, notably conditions in Hamilton and Flavin (1986), Trehan and Walsh (1988), Wickens and Uctum (1993), Quintos (1995), and Ahmed and Rogers (1995).

With regard to necessity, my findings disagree with—and may appear to contradict—much-cited results in the literature, notably the necessity of difference-stationary debt (Trehan and Walsh, 1988; Ahmed and Rogers, 1995) and the necessity of cointegration-type linkages between revenues and spending in Quintos (1995). The proofs in these papers are mathematically correct, but they restrict the class of admissible alternatives in a way that rules out higher-order integration. I show that there are broad classes of stochastic processes that violate much-cited stationarity and cointegration conditions for sustainability but nonetheless do satisfy the IBC.

An implication for applied work is that the common practice of judging a policy to be unsustainable on the basis of unit root and cointegration tests is invalid.

Trehan and Walsh's (1991) cointegration condition that links debt to primary deficits is examined separately and also generalized. Trehan and Walsh's condition implies an error-correction mechanism that can be interpreted as fiscal reaction function, thus providing a bridge to the literature on fiscal behavior (e.g., Bohn, 1998; Canzoneri et al., 2001). Error-correction conditions yield sustainability without finite-order integrated debt series. This proves by example that difference-stationarity of any order is not *necessary* for the IBC.

The paper is organized as follows. Section 2 sets up the budget constraint and proves that high-order difference-stationarity of debt suffices for the IBC. Section 3 first collects the implications for the integration and cointegration of various budget components, and then examines error-correction-type conditions. The concluding Section 4 discusses implications for empirical testing and suggests alternative strategies.

2. Budget equations and budget constraints

For clarity, I use labels motivated by fiscal policy. The budget identity

$$B_t = G_t^0 - T_t + (1 + r_t) \cdot B_{t-1} \tag{1}$$

[1]Afonso (2005) provides a recent review of empirical work that documents the widespread use of unit root and cointegration tests. The paper illustrates how such tests are typically used to assess sustainability.

describes how public debt B_t at the end of period t depends on the non-interest spending G_t^0, revenues T_t, the interest rate r_t, and the previous period's debt B_{t-1}.[2] The difference

$$\Delta B_t \equiv B_t - B_{t-1} = G_t^0 - T_t + r_t \cdot B_{t-1} \tag{2}$$

is known as the with-interest deficit. The component

$$DEF_t \equiv G_t^0 - T_t$$

is the primary (or non-interest) deficit. These variables may be defined in nominal terms, in real terms, or be deflated by suitable scale variables like GDP or population, provided the accumulation factor r_t is measured appropriately.[3]

Some assumptions on interest rates are needed to move from the budget identity to a budget constraint. Common in the literature are:

A1. The interest rate is positive and constant: $r_t = r > 0$.
A2. The interest rate is uncorrelated over time with a positive and constant conditional expectation $\mathrm{E}_t r_{t+1} = r > 0$.

If either assumption holds, the budget identity implies the expectational difference equation

$$B_t = \rho \cdot \mathrm{E}_t[T_{t+1} - G_{t+1}^0 + B_{t+1}], \tag{3}$$

where $\rho = 1/(1+r) < 1$. A third alternative is:

A3. The interest rate is any stationary stochastic process with mean $r > 0$, subject only to implicit restrictions that may be required for $G_t = G_t^0 + (r_t - r)B_{t-1}$ (adjusted spending) to have similar properties (to be specified) as ordinary non-interest spending.

Assumption A3 implies $B_t - B_{t-1} = G_t^0 - T_t + r_t B_{t-1} = G_t - T_t + rB_{t-1}$, so Eq. (3) applies for adjusted instead of actual spending. To obtain a uniform notation, define $G_t = G_t^0$ in cases A1 and A2. Then

$$B_t = \rho \cdot \mathrm{E}_t[T_{t+1} - G_{t+1} + B_{t+1}] \tag{4}$$

with $\rho < 1$ applies in all three cases.[4] For either specification, the IBC is the expected present value condition

$$(\mathrm{IBC}): \quad B_t = \sum_{i=1}^{\infty} \rho^i \mathrm{E}_t(T_{t+i} - G_{t+i}). \tag{5}$$

[2] Alternative interpretations are that B represents a country's net external liabilities, T exports, G imports, and r the interest rate on external liabilities. Or, B may represent an agent's net debt, T the agent's income, and G the agent's outlays.

[3] For example, if real variables are used, r is the real interest rate; if GDP-ratios are used, r is the (real or nominal) interest rate minus the (real or nominal) growth rate.

[4] Assumption A1 seems most common in the literature. Trehan–Walsh (1991) use A2. Quintos (1995) uses A3. The list is not meant to be exhaustive, but to illustrate two points: first, simplifying assumptions *are* needed to obtain a linear difference equation like (4). Second, because there are many ways to derive this equation, the specifics (e.g., if A1, A2, or A3 are adopted) are inessential. Some unit root restrictions can be derived with more general stochastic discount factors (e.g., see Ahmed and Rogers (1995)), but only with auxiliary assumptions that would distract from the paper's time series focus. I adopt (4) to conform to the empirical literature.

The IBC follows from (4) if and only if the transversality condition

$$\text{(TC)}: \quad \lim_{n\to\infty} \rho^n \mathrm{E}_t[B_{t+n}] = 0 \tag{6}$$

is satisfied. This defines the analytical framework.[5]

The key task for time series applications is to find classes of stochastic processes that either imply (IBC) or are inconsistent with (IBC).

Let a stochastic process X_t be called integrated of order m, denoted $X_t \sim I(m)$, if the mth-difference $\Delta^m X_t$ is covariance-stationary with finite mean, absolutely summable moving-average representation, and non-zero spectrum at frequency zero. The integrated process X_t may include a deterministic trend polynomial of up to mth-order. Throughout, convergence of random variables refers to convergence in mean square (m.s.); convergence in probability is implied.

The mathematical intuition for the first result is rather simple. The n-period-ahead conditional expectation of an mth-order integrated variable is at most an mth-order polynomial of the time horizon n. The discounting in the transversality condition is exponential in n. Exponential growth is known to dominate polynomial growth of any order. Hence the discount factor ρ^n in (TC) will asymptotically dominate $\mathrm{E}_t[B_{t+n}]$ whenever debt is difference-stationary with arbitrary order of integration. This motivates:

Proposition 1. *If a debt series is integrated of order m* ($B_t \sim I(m)$) *for any finite* $m \geqslant 0$, *then debt satisfies (TC) and debt, revenues, and spending satisfy (IBC).*

Proof. For $m = 0$, $\mathrm{E}_t[B_{t+n}] \xrightarrow{m.s.} \mathrm{E}[B_t]$ has a finite limit as $n \to \infty$ for given t, hence $\rho^n \to 0$ implies $\rho^n \mathrm{E}_t[B_{t+n}] \xrightarrow{m.s.} 0$. For $m \geqslant 1$, expand B_{t+n} as m-fold sum

$$B_{t+n} = B_t + \sum_{i=1}^{n} \Delta B_{t+i} = B_t + \sum_{i=1}^{n} (\Delta B_t + \sum_{j=1}^{i} (\Delta^2 B_{t+j})) = B_t + n\Delta B_t + \sum_{i=1}^{n} i\Delta^2 B_{t+(n+1-i)}$$

$$= \cdots = \sum_{k=0}^{m-1} p_k(n)\Delta^k B_t + \sum_{i=1}^{n} p_{m-1}(i)\Delta^m B_{t+(n+1-i)},$$

where the weights $p_k(n)$ are kth-order polynomial functions of n. They are obtained recursively for all integers n as $p_0(n) = 1$, $p_1(n) = n$, and $p_k(n) = \sum_{j=1}^{n} p_{k-1}(j)$ for $k \geqslant 2$. Note that $p_k(n) \geqslant 0 \forall (n,k)$. Because $\Delta^m B_t$ is stationary, $\mathrm{E}_t[\Delta^m B_{t+n}] \xrightarrow{m.s.} \mu_b \equiv \mathrm{E}[\Delta^m B_t]$ as $n \to \infty$. Define $Q(n) = \sum_{k=0}^{m-1} p_k(n)\Delta^k B_t$ and $Y_t(n) = (1/n^m)\sum_{i=1}^{n} p_{m-1}(i)[\Delta^m B_{t+n+1-i}]$ to write discounted debt for any horizon n as $\rho^n \mathrm{E}_t B_{t+n} = \rho^n Q(n) + \rho^n n^m \cdot \mathrm{E}_t Y_t(n)$. In $Q(n)$, the $\Delta^k B_t$-terms are constants (given t). For any polynomial $p_k(n)$, $\rho^n p_k(n) \to 0$ as $n \to \infty$. Hence $\rho^n Q(n) \to 0$. In $Y_t(n)$, the scale factor $1/n^m$ ensures that $q(n) \equiv (1/n^m)\sum_{i=1}^{n} p_{m-1}(i)$ has a finite limit $q(n) \to q$, which implies $\mathrm{E}_t Y_t(n) \xrightarrow{m.s.} q \cdot \mu_b$ as $n \to \infty$. Because $\rho^n n^m \to 0$, $\rho^n n^m \cdot \mathrm{E}_t Y_t(n) \xrightarrow{m.s.} 0$ and therefore $\rho^n \mathrm{E}_t[B_{t+n}] \xrightarrow{m.s.} 0$, proving (TC). (TC) and (4) imply (IBC). □

[5] The infinite-sum and limit notation in (5)–(6) is commonly used but imprecise because the limits involve random variables (conditional expectations). Below, I will interpret limit operations as convergence in mean square, which implies convergence in probability.

Notable special cases of this proposition are:

1. Hamilton–Flavin (1986): A stationary debt is sufficient for (TC), the case $B_t \sim I(0)$.
2. Trehan and Walsh (1988): A stationary with-interest deficit $\Delta B_t \sim I(0)$ is sufficient for (TC), the case $B_t \sim I(1)$.
3. Quintos (1995): A difference-stationary with-interest deficit $\Delta B_t \sim I(1)$ is sufficient for (TC), the case $B_t \sim I(2)$.

Quintos (1995)calls the $B_t \sim I(2)$ case 'weak' sustainability, as distinct from 'strong' sustainability in case $B_t \sim I(1)$. In this spirit, *absurdly weak* may be an appropriate label for the *m*th-order sustainability condition in Proposition 1.

Readers may wonder at this point about the well-known necessity proofs in this literature, statements that certain unit root conditions are necessary for the IBC. When examining such proofs, one invariably finds that necessity applies only within certain classes of stochastic processes. Notably, Hamilton–Flavin (1986), Trehan–Walsh (1988) and Ahmed– Rogers (1995) assume difference-stationarity (with deterministic trends as alternative) and they do not contemplate higher orders of integration. Quintos (1995) assumes difference-stationary revenues and with-interest spending, which means that debt is integrated of at most second-order.[6] Assumptions in such proofs that rule out higher-order integration may look like innocuous limitations of scope. Given Proposition 1, such auxiliary assumptions are evidently important and they appear to be unduly restrictive. Regrettably, these necessity proofs are widely misinterpreted as general results.

The economic implications of finding that debt satisfies Proposition 1 are similar for all orders of integration. Strong, weak, and absurdly weak (*m*th-order) sustainability all imply (TC) and (IBC). Most notions of sustainability—all except Hamilton–Flavin's case—also allow debt to be non-stationary in levels, which means that the debt series would violate any upper bound that might be imposed by (additional) economic considerations.

Regarding bounds, the paper's focus on the infinite-horizon IBC is not meant to dispute that more stringent bounds on the path of debt are sometimes of economic interest. Fiscal applications may, for example, involve a bounded tax rate, and international applications may feature a bounded capacity to export. Such bounds may in turn imply upper bounds on debt, either directly or after suitable scaling (e.g., for debt/GDP). In such cases, testing for $m = 0$ versus $m \geqslant 1$ can be economically insightful.

Testing the null hypothesis of difference-stationarity ($m = 1$), in contrast, seems economically uninteresting. With debt limits, $m = 1$ is not sufficient for sustainability. Without debt limits, a higher order of integration suffices, so rejecting $m = 1$ (or $m \leqslant 1$) provides no evidence against the IBC. Rejecting $m = 2$ or any other *m*-value would not

[6] In a short paper building on Quintos, Bergman (2001) asserts a high-order integration condition similar to Proposition 1, but without proof and erroneously presented as sufficient *and necessary*. (See Section 3.3 for a counterexample to necessity; the example also contradicts Bergman's necessity claim.) Bergman unfortunately adopts Quintos' assumptions of difference-stationary revenues and with-interest spending, which implicitly restrict debt to be $I(2)$ and thus preclude a valid analysis of higher-order integration. Quintos employs an insightful order-in-probability intuition but concludes (misleadingly) that $\Delta B_t = \mathrm{Op}(T^{1/2})$ is necessary for (TC).

provide negative evidence either, because it would not rule out higher-order integration. Proposition 1 thus invalidates empirical testing strategies that infer a violation of (IBC) from the absence of low-order integration.[7]

Finally, Proposition 1 suggests that the assumption $\rho < 1$ is quite powerful and that the level of interest rates—given a positive sign—is remarkably unimportant. Even a very small positive discount rate is enough to dominate the transversality condition for all finitely integrated stochastic processes for debt.

3. Cointegration and error-correction

Proposition 1 implies the sufficiency of a variety of unit root and cointegration conditions on related variables. Three sets of conditions deserve comment—two of them equivalent to restrictions on debt and covered under Proposition 1, the third requiring separate treatment.

3.1. Revenues and with-interest spending

First, consider restrictions on revenues T_t and on government spending *with interest*,

$$G_t^r \equiv G_t^0 + r_t \cdot B_{t-1}.$$

Best known is perhaps Quintos' (1995) condition for 'weak' sustainability, which assumes that (T_t, G_t^r) are each $I(1)$ and cointegrated with some vector $(1, -b)$. In this case, $T_t - bG_t^r = \varepsilon_t \sim I(0)$ implies

$$\Delta B_t = G_t^r - T_t = \varepsilon_t + (1-b)G_t^r \sim I(1)$$

so $B_t \sim I(2)$ (or $I(1)$ if $b = 1$), which implies (IBC).

Quintos also argues that $0 < b \leqslant 1$ and $B_t \sim \mathrm{Op}(T^{1/2})$ are necessary for weak sustainability, where $\mathrm{Op}(T^{1/2})$ denotes convergence of order $\frac{1}{2}$ in probability. (This means $m \leqslant 2$ when dealing with integrated series.) Restrictions on b and restrictions on the order of integration (or of convergence in probability) are actually unnecessary. The following conditions on (T_t, G_t^r) suffice:

Proposition 2. *Suppose $G_t^r \sim I(m_G)$ and $T_t \sim I(m_T)$, possibly with different orders of integration and not necessarily cointegrated. Then $B_t \sim I(m)$ with $m \leqslant \max(m_G, m_T) + 1$, so (TC) and (IBC) hold.*

Proof. $\Delta^{m_G} G_t^r = \varepsilon_t \sim I(0)$ and $\Delta^{m_T} T_t = \eta_t \sim I(0)$ are stationary by assumption. If $m_G < m_T$, then $\Delta^{m_T+1} B_t = \Delta^{m_T}(\Delta B_t) = (\Delta^{m_T - m_G}\varepsilon_t) - \eta_t \sim I(0)$. If $m_G > m_T$, then $\Delta^{m_T+1} B_t = \varepsilon_t - (\Delta^{m_G - m_T}\eta_t) \sim I(0)$. In both cases $\Delta^{\max(m_T, m_G)+1} B_t \sim I(0)$, so $B_t \sim I(m)$ with $m = \max(m_G, m_T) + 1$. If $m_G = m_T$, $\Delta^{m_T} G_t^r - \Delta^{m_T} T_t \sim I(0)$ is stationary, but a lower order of differencing may suffice if (T_t, G_t^r) are cointegrated; either way, $B_t \sim I(m)$ with $m \leqslant \max(m_G, m_T) + 1$. In all cases, $B_t \sim I(m)$ implies (TC) and (IBC) from Proposition 1. □

[7] A related, more basic point is that a *failure to reject* a null hypothesis does not prove the null. Own refereeing experiences suggest that some practitioners of unit root testing are remarkably sloppy about type-2 errors. The stronger statement here is that even if difference-stationarity is rejected, say, in a Kwiatkowski et al. (1992) test for stationarity, one cannot infer a violation of the IBC.

Quintos (1995) covers the case $m_G = m_T = 1$, which implies $m \leqslant 2$. In this case, debt displays at most quadratic growth even if even if $b \notin [0,1]$, so (TC) applies without restrictions on b.

3.2. Components of the budget identity

Second, consider restrictions on revenues, debt, and *non-interest* spending that ensure the stationarity of the linear combination $G_t^0 - T_t + r_t \cdot B_{t-1}$. Because this linear combination is the r.h.s. of budget identity (2), its stationarity is equivalent to $\Delta B_t \sim I(0)$ and implies (TC) by Proposition 1.

Stationarity of $G_t^0 - T_t + r_t \cdot B_{t-1}$ is implied, for instance, by cointegration of $(T_t, G_t^0, r_t B_{t-1})$ with cointegrating vector $(1,-1,-1)$, by cointegration of (T_t, G_t, rB_{t-1}) with vector $(1,-1,-1)$, or by cointegration of (T_t, G_t, B_{t-1}) with vector $(1,-1,-r)$. The same implication holds for suitable linear combinations, e.g., for the cointegration of (DEF_t, B_{t-1}) with vector $(1,r)$, for the cointegration of $(DEF_t, r_t B_{t-1})$ with vector $(1,1)$, or (with overlap to Quintos) for cointegration of (T_t, G_t^r) with vector $(1,-1)$.

All these cointegration conditions are sufficient for (TC) but far stronger than necessary. As in previous sections, it is straightforward to prove the sufficiency of higher-order versions. For example, suppose $(\Delta^m T_t, \Delta^m G_t, \Delta^m B_{t-1})$ are cointegrated with vector $(1,-1,-r)$ and arbitrary $m \geqslant 1$. Then $\Delta^{m+1} B_t \sim I(0)$ from Eq. (2), hence (TC) holds by Proposition 1. The example demonstrates that a lack of cointegration between (T_t, G_t, B_{t-1}) does not preclude the series' consistency with the IBC.

A secondary motivation for this section is to help distinguish adding-up restrictions that exploit the budget identity—discussed here—from error-correction-type conditions—to be examined next.

3.3. Primary deficits and debt

Third and finally, consider the case of cointegration between primary deficits and debt, now with an arbitrary weight on debt. Trehan–Walsh (1991) show that (TC) and (IBC) hold if (DEF_t, B_{t-1}) are cointegrated and if the quasi-difference

$$DEF_t - \lambda \cdot DEF_{t-1} \sim I(0) \tag{7}$$

is stationary with zero mean for some $\lambda \in [0, 1+r)$. This condition requires separate treatment—and indeed deserves attention—because for $\lambda > 1$ it implies a different convergence behavior for debt than the other unit root and cointegration conditions, and because it has an instructive economic interpretation.

Let $DEF_t + \alpha B_{t-1} = \varepsilon_t \sim I(0)$ with parameter $\alpha \neq 0$ denote the stationary linear combination of (DEF_t, B_t). Cointegration and assumption (7) together imply that the quasi-differenced debt

$$B_t - \lambda B_{t-1} = (DEF_{t+1} - \lambda DEF_t)/\alpha - (\varepsilon_{t+1} - \lambda \varepsilon_t)/\alpha \sim I(0).$$

is also stationary. For $\lambda \in [0,1)$, $B_t - \lambda B_{t-1} \sim I(0)$ implies $B_t \sim I(0)$. For $\lambda = 1$, $B_t - \lambda B_{t-1} \sim I(0)$ implies $B_t \sim I(1)$. Both cases are covered by Proposition 1.

The interesting case is $\lambda \in (1, 1+r)$. If $B_t - \lambda B_{t-1} \sim I(0)$ for some $\lambda \in (1, 1+r)$ but not $I(0)$ for any $\lambda \in [0,1]$, debt and primary deficit display *exponential* growth at the rate

$\lambda - 1 > 0$; and neither series is difference-stationary of any order.[8] For this parameter range, Trehan–Walsh's proof for (TC) relies critically on the inequality $\lambda/(1+r) < 1$ to ensure that $\rho^n \mathrm{E}_t[B_{t+n}] \approx B_t \cdot (\lambda/(1+r))^n \to 0$.

Importantly, this setting proves by counterexample that Propositions 1–2 are not necessary conditions for (TC). The lack of necessity for even these weak conditions reinforces the point that orders-of-integration conditions are unnecessary for sustainability.

To interpret Trehan–Walsh's conditions, note that the budget identity (1) and the cointegration assumption $DEF_t + \alpha B_{t-1} = \varepsilon_t \sim I(0)$ imply

$$B_t = DEF_t + (1 + r_t) \cdot B_{t-1} = (1 + r_t - \alpha) \cdot B_{t-1} + \varepsilon_t.$$

If $B_t - \lambda B_{t-1}$ is stationary for any λ, it must be stationary for $\lambda = 1 + r - \alpha$.[9] To satisfy $\lambda \in [0, 1+r)$, one needs $\alpha \in (0, 1+r]$. Thus, Trehan–Walsh are in effect examining an error-correction-type specification of the form

$$DEF_t = -\alpha B_{t-1} + \varepsilon_t, \tag{8}$$

with $\alpha > 0$.

Error-correction has a natural economic interpretation as a reaction function describing the behavior of the entity being studied. Specification (8) therefore fits well into the literature on fiscal behavior (e.g., Bohn, 1998; Canzoneri et al., 2001) and helps to clarify the econometrics of reaction functions with stationary ε_t-process:

- Reaction functions with $\alpha > r$ imply stationary debts and deficits (as $\lambda < 1$).
- Reaction functions with $0 < \alpha < r$ imply mildly explosive paths for debts and deficits (as $\lambda > 1$), but growing slowly enough to be consistent with (TC) and (IBC).
- Reaction functions with $\alpha = r$ imply a difference-stationary debt ($\lambda = 1$), stationary with-interest deficits, and hence satisfy the unit root and cointegration conditions discussed in Section 3.2.

The special case $\alpha = r$ is arguably the most studied scenario in the unit root literature. From the perspective of fiscal behavior—viewing α is as a continuous choice variable—this case is non-generic. The generic cases display either a stationary debt ($\alpha > r$) or an exponentially growing debt ($\alpha < r$).

Note that higher-order versions of (8) also imply sustainability. For example:

Proposition 3. *Suppose $DEF_t + \alpha B_{t-1} = z_t \sim I(m)$ for some $\alpha \in (0, 1+r]$. Suppose $r_t = r$ is constant. Then debt satisfies (TC).*

Proof. From (1), $B_{t+1} = (1 + r - \alpha) B_t + z_{t+1} = \lambda \cdot B_t + z_{t+1}$, where $\lambda = 1 + r - \alpha \in [0, 1+r)$. For $\lambda < 1$ and for $\lambda = 1$, this implies $B_t \sim I(m)$ and $B_t \sim I(m+1)$, respectively, so (TC) from Proposition 1. For $\lambda > 1$, consider $\rho^n \mathrm{E}_t[B_{t+n}] = (\rho\lambda)^n B_t + (\rho\lambda)^n \mathrm{E}_t[\sum_{i=1}^{n} \lambda^{-i} z_{t+i}]$. Because $z_t \sim I(m)$, the expression $\sum_{i=1}^{n} \lambda^{-i} z_{t+i}$ can be expanded into a linear combination of t-dated differences $\Delta^k z_t$ and stationary mth differences $\Delta^m z_{t+i}$,

[8] To see the latter, note that $B_t - \lambda B_{t-1} = \Delta B_t - (\lambda - 1) B_{t-1} \equiv \varepsilon_t^B \sim I(0)$ with $\lambda > 1$ implies $\Delta^m B_t - (\lambda - 1)\Delta^{m-1} B_{t-1} \equiv \Delta^{m-1} \varepsilon_t^B \sim I(0) \forall m \geqslant 1$. If $\Delta^m B_t$ were stationary for any $m \geqslant 1$, $\Delta^{m-1} B_t$ would also be stationary. By induction, stationary $\Delta^m B_t$ would imply a stationary B_t, contradicting $\lambda \notin [0, 1]$.

[9] For interest rates, Trehan–Walsh impose Assumption A2, so $r = E_{t-1} r_t$. If $\alpha > r$, stationarity applies for multiple λ-values, and cointegration holds in the trivial sense that both series are stationary.

analogous to the expansion in the proof of Proposition 1 (for $0 \leqslant k < m, 1 \leqslant i \leqslant n$). Because $\lambda^{-i} < 1$, the weights in the linear combination are bounded from above by polynomials. Because $\rho\lambda < 1$, discounting by $(\rho\lambda)^n$ implies $\rho^n \mathrm{E}_t[B_{t+n}] \xrightarrow{p} 0$. □

Proposition 3 demonstrates that the error-correction approach does not require stationary driving processes.[10] The economic intuition is that (TC) holds if a debtor is not oblivious to accumulating debt but responding positively ($\alpha > 0$). As in Trehan–Walsh (1991), the key technical condition is that all roots are strictly less than $1 + r$; and this condition is not affected if one allows for unit roots.

Practical challenges should be acknowledged at this point. Because the appropriate discount rate is typically a small number, it may be difficult to distinguish empirically between a unit root and a $(1 + r)$-root in the debt process, and between α-values that are slightly greater or slightly less than zero.

How serious are these challenges? Conceptually, I consider them a return to normalcy. Since the discovery of unit root testing, the *economic* analysis of debts and deficits has been overshadowed by the notion that sustainability questions can be answered conclusively by running data through a battery of time series tests. A finding that the econometric conditions are weak will hopefully encourage a return to economic thinking. This includes questions that go beyond the IBC and hence cannot be answered by testing IBC-conditions. For example, even if one finds that historical data seem to satisfy the IBC, one should not take for granted that lenders will necessarily extend credit. Lenders may well impose additional constraints to discourage opportunistic defaults—for example, upper bounds on debts and/or deficits. Conversely, if debt appears to be on an unsustainable path, one must wonder why lenders have not stopped lending already.

It is beyond the scope of this paper to offer complete answers to the economic questions surrounding sustainability. In other work, I have suggested estimating reaction functions for primary surpluses scaled by GDP (Bohn, 1998, 2005). Estimates of α in suitably specified regressions of primary surpluses/GDP on debt/GDP turned out to be sufficiently precise to yield useful insights.[11] Assessing sustainability is not a mechanical exercise, however, because applications typically encounter various specification issues, e.g., questions about the appropriate control variables and about discount rates (notably if debt management is endogenized). That is, one encounters the usual challenges of applied economic analysis. This paper's more modest objective is to ensure that such economic analysis is not preempted or stymied by (allegedly necessary) unit root and cointegration conditions.

[10] Constant r is assumed to simplify the proof. Extensions to variable interest rates (conditions A2, A3) are possible but would require more elaboration (dealing with a second source of noise) than warranted to make the simple point that (8) can be generalized. The proof suggests that (TC) also holds for debt processes with more than one root in the $(1, 1 + r)$ interval.

[11] The economic intuition for examining primary surpluses is that the primary surplus is the variable that keeps debt from growing at rate $(1 + r)$ and is therefore a sensitive indicator for how debt growth compares to $(1 + r)$. Finding $\alpha > r$ implies not only (IBC) but also a stationary debt-GDP ratio. This is promising for economic analysis, e.g., for deriving bounds. Finding $\alpha \leqslant 0$, in contrast, suggests that fiscal decision makers disregard debt when setting taxes and non-interest spending, raising questions why lenders are extending credit. Finding $0 < \alpha \leqslant r$ is consistent with (IBC) but unsettling from an economic perspective because it suggests an unbounded debt-GDP ratio. The references to my own work are not meant prescriptive but to illustrate that there are constructive alternatives to unit root testing.

To conclude, the statistical linkages between primary deficits and debt *are* of economic interest. The economic interest goes beyond accepting or rejecting cointegration, however, but centers around the parameters α and r. This suggests that research on fiscal and external deficits should focus more on questions of policy identification and stability (modeling $\{z_t\}$ and determining α) and on questions of discounting (determining r) than on testing for unit roots.

4. Discussion

The sufficiency of high-order integration for the IBC and the lack of necessity raise questions about widely used empirical strategies for testing sustainability. Tests for unit roots in debt and/or deficit series and tests for cointegration between revenues and spending are commonly used in applied work. Rejections of sustainability based such tests are invalid because the IBC may well be satisfied even if the components of the budget are not cointegrated and even if neither debts, nor deficits, revenues, or spending are difference-stationary.

What are the alternatives? An applied researcher might respond to Propositions 1–2 by pursuing a sequential strategy of repeated differencing and unit root testing, hoping for a rejection that would document sustainability. This is tempting because many economic time series turn out to be stationary in first or second differences. The correct design of a sequential strategy is challenging, however, not only because type-1 errors accumulate, but because Propositions 1–2 makes the sequence open-ended. Findings of low-order integration are statistically dubious in a sequential context where readers must suspect that testing would have continued in case of non-rejection.

Alternatively, repeated testing might end without rejection—either because a researcher gives up after a finite number of tests without reaching the true order of integration, or because the series is indeed not integrated, or because of type-2 errors. Because higher-order integration is not ruled out, a termination without rejection must be interpreted as inconclusive, not as evidence of non-sustainability. These statistical issues and the inability to ever reject sustainability limits the usefulness of unit root testing.

Two other strategies appear more promising, both from an economic and from a statistical perspective. One is to examine the behavior and functioning of the entity—government or nation—supposed to satisfy the IBC. One may ask, for example, if the entity's behavior is sufficiently "responsive" to debt that the primary balance has an error-correction representation. This is pursued in the literature on fiscal reaction functions.

A second strategy is to consider stronger conditions on policy, e.g., upper bounds on debt motivated by a limited capacity to service debt. Then stationarity in levels is the most relevant econometric condition, and additional restrictions may apply. Such additional considerations would lead away from the IBC, the subject of this paper. They are consistent, however, with this paper's main point: the IBC *per se* imposes very weak econometric restrictions.

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