Tight bounds on American option prices

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\textbf{A B S T R A C T}

In contrast to the constant exercise boundary assumed by Broadie and Detemple (1996) [Broadie, M., Detemple, J., 1996. American option valuation: New bounds, approximations, and comparison of existing methods. Review of Financial Studies 9, 1211–1250], we use an exponential function to approximate the early exercise boundary. Then, we obtain lower bounds for American option prices and the optimal exercise boundary which improve the bounds of Broadie and Detemple (1996). With the tight lower bound for the optimal exercise boundary, we further derive a tight upper bound for the American option price using the early exercise premium integral of Kim (1990) [Kim, I.J., 1990. The analytic valuation of American options. Review of Financial Studies 3, 547–572]. The numerical results show that our lower and upper bounds are very tight and can improve the pricing errors of the lower bound and upper bound of Broadie and Detemple (1996) by 83.0% and 87.5%, respectively. The tightness of our upper bounds is comparable to some best accurate/efficient methods in the literature for pricing American options. Moreover, the results also indicate that the hedge ratios (deltas and gammas) of our bounds are close to the accurate values of American options.

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\end{center}

1. Introduction

One stream of the American option pricing literature is to derive lower and/or upper bounds for American option values,\textsuperscript{1} see e.g. Perrakis (1986), Chen and Yeh (2002), and Chung and Chang (2007).\textsuperscript{2} Notably Broadie and Detemple (1996) provide tight lower and upper bounds for American call prices based on the assumption that the early exercise boundary is a constant. In this paper, we provide even tighter bounds for American option prices by making a more realistic and flexible assumption that the early exercise boundary follows an exponential function.\textsuperscript{3} Since the optimal exercise boundary is a monotone function in time-to-maturity, the assumed exponential boundary can improve the constant boundary in Broadie and Detemple (1996).

We first derive a tight lower bound for the American call option as the price of a capped (barrier) call option with an exponential exercise policy. Since all admissible exponential exercise policies generate lower price bounds, a tight lower bound for the American call option can be obtained based on maximizing the values of the capped options over the parameters of the exponential exercise barrier. Given the derivative information of the capped call price with respect to parameters of the exponential function, the optimization problem can be easily solved by an iterative procedure.

Next, we obtain a tight lower bound for the optimal exercise boundary based on our tight lower bound for the American call option price. The idea is intuitive and can be described as follows. Note that the optimal exercise boundary $B_u$ is the intersection point (i.e. the value-matching point) of the early exercise value

\begin{itemize}
  \item $B_u$: Intersection point of the early exercise value
  \item $C_u$: Capped call value
  \item $C_u$: Capped put value
\end{itemize}

\textsuperscript{1} Although Omberg (1987) and Ingersoll (1998) suggest using an exponential function to approximate the early exercise boundary, they do not provide detailed formulae to solve the lower bound and the optimal exercise boundary. In contrast, we follow the approach of Broadie and Detemple (1996) and derive the essential formulae for solving the lower bound and the optimal exercise boundary.

\textsuperscript{2} For European option pricing bound literature, please see Chung and Wang (2008) and the references therein.
and the holding value for American call option as the underlying asset price \((S_t)\) increases from below. Since our lower price bound is always smaller than American option price in the holding region (i.e. the region where \(S_t \leq B_t\)), the intersection point of the early exercise value and the holding value for the capped call option under the exponential exercise policy must be less than the optimal exercise boundary \(B_t\). Please see Fig. 1 for the ease of understanding our explanation.

Equipped with our tight lower bound for optimal exercise boundary and the integral representation formula for the early exercise premium in Kim (1990), Jacka (1991), and Carr et al. (1992), a tight upper bound for the American call price is then derived in this paper.

Finally, following the method in Broadie and Detemple (1996), we combine both lower and upper bounds, with an optimization and regression exercise, to derive two accurate approximations for American option prices. This optimization regression is based on a set of simulated option contracts, and once equipped with the estimated coefficients, it is possible to approximate American option prices accurately.

Based on the numerical experiments in Tables 1 and 2, we find that our method can improve the errors of the lower bound and upper bound of Broadie and Detemple (1996) by 83.0% and 87.5%, respectively. Moreover, our option pricing bounds are so tight that their accuracy is comparable to the best accurate/efficient methods for pricing American options in the literature. For example, the integral representation formula of Kim (1990), Jacka (1991), and Carr et al. (1992) has been successfully implemented in a series of papers, such as Huang et al. (1996), Ju (1998), and Ibáñez (2003). These papers offer the best speed-accuracy trade-offs in the literature. Our numerical results indicate that the errors of our upper bounds for pricing long term American put options are only 0.049% which is close to the errors of Ju (1998), i.e. 0.032% (see Table 3 of this study).

The rest of this article proceeds as follows. Section 2 provides lower bounds for the American call option price and the optimal exercise boundary based on the assumption that the optimal exercise boundary follows an exponential function. Section 3 applies the lower bound of the optimal exercise boundary to the integral representation formula of Kim (1990), Jacka (1991), and Carr et al. (1992) to obtain an upper bound for the American option price. Section 4 shows the numerical results to analyze the tightness of our lower and upper bounds. Section 5 concludes the paper.

Moreover, it is straightforward to show that a tighter lower bound for the American call price can result in a tighter lower bound for the optimal exercise boundary. Since our lower bound is closer than the lower bound of Broadie and Detemple (1996), to the American call price, our lower bound for the optimal exercise boundary is also tighter than that of Broadie and Detemple (1996).

6 Ju (1998) approximates the early exercise boundary as a multipiece exponential function and substitute it to the early exercise premium integral, derived by Kim (1990), to price American options. Closed-form formulae can be derived and the bases and exponents of the multipiece exponential function can be obtained backward by using value-matching and smooth-pasting conditions. Thus a two-dimensional Newton-Raphson method must be used to solve the bases and exponents at different times (e.g. see Eqs. (13) and (14) of Ju (1998)).

7 Ibáñez (2003) introduces a new algorithm to implement the decomposition formula of Kim (1990). He proposes an adjustment of Kim’s (1990) discrete-time early exercise premium so that these premiums monotonically converge and therefore it is appropriate to apply them in Richardson extrapolation. Moreover, Ibáñez (2003) also derives the correct order for the error term when applying extrapolation, which is then used to control the error of the extrapolated prices.

8 Since the pricing error of the proposed method is small, it may be also suitable for pricing long term American-style employee stock options (e.g. see Leon and Vællo-Sebastia (2009)).
2. Lower bounds for the American call option value and the optimal exercise boundary

Consider the pricing of an American call option with a strike price \( K \) and a fixed maturity date \( T \). Following Black and Scholes (1973) model, we assume that the underlying asset price \( S \) under the risk-neutral world satisfies

\[
dS_t = (r - q)S_t dt + \sigma S_t dW_t,
\]

where \( W_t \) is a standard Brownian motion process. The volatility \( \sigma \), the risk-free rate \( r \), and the dividend yield rate \( q \) are assumed constant. Let \( C(S) \) denote the American call option price, where the parameters \( K, T, \sigma, r, \) and \( q \) are omitted for simplicity.

It is well-known that the valuation of American options is a free boundary problem (see McKean (1965)) and the optimal exercise boundary \( B^* \) must be solved simultaneously with the valuation problem. Although it is difficult and time consuming to solve \( B^* \), the asymptotic behavior of \( B^* \) has been derived in the literature. For example, Merton (1973) proves that the optimal exercise boundary for the perpetual (i.e., \( T \rightarrow \infty \)) American put option is a constant. Using Merton's technique, one can easily show that the optimal exercise boundary for the perpetual American call option

Table 3
Price of American put options (K = 100, T = 3 years, σ = 0.2, r = 0.08).

<table>
<thead>
<tr>
<th>(S, q)</th>
<th>True value</th>
<th>EXP3</th>
<th>LB2</th>
<th>UB2</th>
<th>LBA2</th>
<th>LUBA2</th>
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<tr>
<td>(80, 0.12)</td>
<td>25.6578</td>
<td>25.6570</td>
<td>25.6572</td>
<td>25.6578</td>
<td>25.6572</td>
<td>25.6577</td>
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<tr>
<td>(90, 0.12)</td>
<td>20.0832</td>
<td>20.0817</td>
<td>20.0829</td>
<td>20.0833</td>
<td>20.0832</td>
<td>20.0832</td>
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<tr>
<td>(100, 0.12)</td>
<td>15.4984</td>
<td>15.4970</td>
<td>15.4982</td>
<td>15.4984</td>
<td>15.5005</td>
<td>15.4984</td>
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<td>(110, 0.12)</td>
<td>11.8032</td>
<td>11.8022</td>
<td>11.8031</td>
<td>11.8032</td>
<td>11.8061</td>
<td>11.8032</td>
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<tr>
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<tr>
<td>(80, 0.08)</td>
<td>22.2050</td>
<td>22.2064</td>
<td>22.1963</td>
<td>22.2091</td>
<td>22.2022</td>
<td>22.2032</td>
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<tr>
<td>(100, 0.08)</td>
<td>11.7039</td>
<td>11.7066</td>
<td>11.6953</td>
<td>11.7054</td>
<td>11.7057</td>
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<tr>
<td>(110, 0.08)</td>
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<td>(120, 0.08)</td>
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<td>5.9323</td>
<td>5.9247</td>
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<td>(80, 0.04)</td>
<td>20.3501</td>
<td>20.3511</td>
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<td>13.4968</td>
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<td>5.9147</td>
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<td>6.9235</td>
<td>6.9379</td>
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<tr>
<td>(110, 0)</td>
<td>4.1550</td>
<td>4.1571</td>
<td>4.1473</td>
<td>4.1583</td>
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<td>4.1549</td>
</tr>
<tr>
<td>(120, 0)</td>
<td>2.5103</td>
<td>2.5119</td>
<td>2.5044</td>
<td>2.5122</td>
<td>2.5095</td>
<td>2.5103</td>
</tr>
</tbody>
</table>
| RMS | 0.0316% | 0.1138% | 0.0136% | 0.0487% | 0.0247% | 0.0040%

Table 4
Prices of (short-term) standard American put options.

<table>
<thead>
<tr>
<th>(S, q)</th>
<th>True value</th>
<th>PEXT</th>
<th>LB2</th>
<th>UB2</th>
<th>LBA2</th>
<th>LUBA2</th>
</tr>
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<td>35</td>
<td>0.0833</td>
<td>0.0062</td>
<td>0.0062</td>
<td>0.0062</td>
<td>0.0062</td>
<td>0.0062</td>
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<tr>
<td>35</td>
<td>0.3333</td>
<td>0.2004</td>
<td>0.2003</td>
<td>0.2002</td>
<td>0.2004</td>
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<tr>
<td>35</td>
<td>0.5833</td>
<td>0.4328</td>
<td>0.4327</td>
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<td>0.4327</td>
</tr>
<tr>
<td>35</td>
<td>0.0833</td>
<td>0.8523</td>
<td>0.8523</td>
<td>0.8519</td>
<td>0.8524</td>
<td>0.8516</td>
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<tr>
<td>40</td>
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<td>1.5799</td>
<td>1.5799</td>
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<td>1.5799</td>
</tr>
<tr>
<td>40</td>
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<td>1.9910</td>
<td>1.9905</td>
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<tr>
<td>45</td>
<td>0.0833</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0002</td>
<td>5.0000</td>
</tr>
<tr>
<td>45</td>
<td>0.3333</td>
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<td>5.0883</td>
<td>5.0871</td>
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<tr>
<td>45</td>
<td>0.5833</td>
<td>5.2670</td>
<td>5.2670</td>
<td>5.2647</td>
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<td>0.0833</td>
<td>0.0775</td>
<td>0.0774</td>
<td>0.0774</td>
<td>0.0775</td>
<td>0.0775</td>
</tr>
<tr>
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<td>0.6976</td>
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<td>1.2196</td>
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<td>1.2199</td>
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<tr>
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<td>0.0833</td>
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<td>1.3101</td>
<td>1.3098</td>
<td>1.3103</td>
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<tr>
<td>45</td>
<td>0.3333</td>
<td>2.4827</td>
<td>2.4825</td>
<td>2.4811</td>
<td>2.4830</td>
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<tr>
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<td>3.1702</td>
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<td>5.0598</td>
<td>5.0588</td>
<td>5.0601</td>
<td>5.0595</td>
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<td>6.2437</td>
<td>6.2423</td>
<td>6.2402</td>
<td>6.2446</td>
<td>6.2440</td>
</tr>
</tbody>
</table>
| RMS | 0.0236% | 0.0618% | 0.0136% | 0.0336% | 0.0045%

The parameters are adopted from Table 2 of Ju (1998). The "true value" is calculated from the Binomial Black and Scholes method with Richardson extrapolation (BBSR) and the length of each time step is 0.0001 years. EXP3 represents the America put price estimate using the three-point multipiece exponential boundary method of Ju (1998). LB2 and UB2 are lower and upper bounds proposed in this paper. LBA2 and LUBA2 are approximations based on our bounds.

The parameters are adopted from Table 1 of Ibáñez (2003). The "true value" is calculated from the Binomial Black and Scholes method with Richardson extrapolation (BBSR) and the length of each time step is 0.0001 years. PEXT represents the extrapolated America put price from three Bermudan prices with 4, 5, and 6 exercise dates with the modified early exercise premium derived by Ibáñez (2003). LB2 and UB2 are lower and upper bounds proposed in this paper. LBA2 and LUBA2 are approximations based on our bounds.

is $\mu_K(\beta - 1)$, where $\beta = \frac{1}{2} - \frac{r - q}{\sigma^2}$ and $\frac{(r - q)^2}{2\sigma^4}$. On the other hand, when $r > q$ and the time to maturity approaches zero (i.e. $t \rightarrow 0$), Evans et al. (2002, Eq. (1.11)) show that the early exercise boundary near expiration satisfies

$$B_t^r \sim \frac{r}{q} K \left( 1 + \sigma_0 \sqrt{v / (T - t)} \right),$$

where $\sigma_0$ is a constant determined by a transcendental equation.
Inspiring by the above results, we approximate the early exercise boundary of an American call using an exponential function, i.e., $B_t = \text{Le}^{\text{e}^{-t}}$. The payoff of the approximated American call is either $\max(\text{Le}^{\text{e}^{-t}} - K, 0)$, if the underlying asset price first hits the exponential exercise boundary from below at time $\tau$ where $t \leq \tau < T$, or $\max(S_t - K, 0)$, otherwise. The American call under the exponential exercise boundary is essentially a capped (barrier) option and its value ($C^e_t(S_t, L, a)$) is given by

$$C^e_t(S_t, L, a) = \text{Le}^{\text{e}^{-t}} \left[ \frac{1}{\sqrt{T-t}} N(d_0) + \frac{1}{\sqrt{T-t}} N(d_0) + 2\beta \sigma \sqrt{T-t} \right] - K \left[ \frac{1}{\sqrt{T-t}} N(d_0) + \frac{1}{\sqrt{T-t}} N(d_0) + 2\beta \sigma \sqrt{T-t} \right] + S_t \text{e}^{-\text{e}^{-t}} \left[ N(d_1; (\text{Le}^{\text{e}^{-t}}) - \sigma \sqrt{T-t}) + N(d_1; (\text{Le}^{\text{e}^{-t}}) - \sigma \sqrt{T-t}) \right] - N(d^-(K) - \sigma \sqrt{t-T}) - \lambda^{d_{-1}} \left[ N(d^-(K) - \sigma \sqrt{T-t}) + N(d^-(K) - \sigma \sqrt{T-t}) \right] - \beta e^{-\text{e}^{-t}} \left[ N(d^+(K) - \sigma \sqrt{T-t}) \right] - N(d^-(K) - \sigma \sqrt{T-t}) - \lambda^{d_{+1}} \left[ N(d^+(K) - \sigma \sqrt{T-t}) + N(d^+(K) - \sigma \sqrt{T-t}) \right],$$

(2)

where $N(z)$ is the cumulative distribution function of a standard normal variable, 

$$\lambda_i = S_i / (\text{Le}^{\text{e}^{-t}}), \quad \beta = - \left( r - q + a - \frac{1}{2} \sigma^2 \right) / \sigma^2,$$

$$d_0 = \frac{1}{\sqrt{T-t}} \left[ \ln(\lambda) - \beta \sigma^2(T-t) \right]$$

$$d_1 = \frac{1}{\sqrt{T-t}} \left[ \ln(\lambda) - \beta \sigma^2(T-t) \right],$$

$$d^+ (x) = \frac{1}{\sqrt{T-t}} \left[ \ln(\lambda) - \ln(L) + \ln(x) - (r - q + a) / \sigma^2 \right],$$

$$d^- (x) = \frac{1}{\sqrt{T-t}} \left[ \ln(\lambda) - \ln(L) + \ln(x) - (r - q - a - \frac{1}{2} \sigma^2) \right].$$

Please note the Eq. (2) holds only when both the current stock price and the strike price are below the exponential exercise boundary, i.e., $\text{Le}^{\text{e}^{-t}} > \max(S_t, K)$. For completeness, we define $C^e_t(S_t, L, a) = \max(\min(S_t, \text{Le}^{\text{e}^{-t}}) - K, 0)$ when $\text{Le}^{\text{e}^{-t}} < \max(S_t, K)$.

Since the policy of exercising when the underlying asset price reaches the exponential exercise boundary is admissible (but not the optimal) policy for the American option, the above formula for $C^e_t(S_t, L, a)$ gives an immediate lower bound of the American option price $C_t(S_t)$. i.e. $C^e_t(S_t, L, a) \leq C_t(S_t)$ for any $(L, a)$. Moreover a lower bound is still obtained after maximizing over $(L, a)$ subject to the constraint that $\text{Le}^{\text{e}^{-t}} > \max(S_t, K)$ i.e., $\max_{\text{Le}^{\text{e}^{-t}} > \max(S_t, K)} C^e_t(S_t, L, a) \leq C_t(S_t)$, Denote the optimal solution $(\tilde{L}_t(S_t), a(S_t))$ as 

$$(\tilde{L}_t(S_t), a(S_t)) = \arg \max_{\text{Le}^{\text{e}^{-t}} > \max(S_t, K)} C^e_t(S_t, L, a).$$

Hence

$$C^e_t(S_t) = \max_{\text{Le}^{\text{e}^{-t}} > \max(S_t, K)} C^e_t(S_t, L, a) \leq C_t(S_t).$$

(4)

Note that our lower bound in Eq. (4) improves over the naive lower bound of the European call value, denoted as $C_t(S_t)$ because $C_t(S_t) = \lim_{\text{Le}^{\text{e}^{-t}} \to \infty} C^e_t(S_t, L, a)$. Moreover, since the constant exercise boundary of Broadie and Detemple (1996) is a special case of our exponential exercise boundary with $\mu = 0$, our lower bound is also tighter than their lower bound.

Solving $(\tilde{L}_t(S_t), a(S_t))$ is a bi-variate differentiable optimization problem for any given $S_t$. We apply an iterative procedure to solve the optimization problem. With the formulæ of derivatives $\partial C^e_t(S_t, L, a) / \partial L$ and $\partial C^e_t(S_t, L, a) / \partial a$, the optimal solution should satisfy

$$|\partial C^e_t(S_t, L, a) / \partial L|^2 + |\partial C^e_t(S_t, L, a) / \partial a|^2 = 0.$$

(5)

We first give an initial guess of the level $L_0$ and then find the optimal solution of the growth rate $a_0$ by minimizing the value of the left hand side (LHS) of Eq. (5). Using $a_0$ we find the optimal solution of $L_0$ by minimizing the value of the left hand side of Eq. (5). We repeat the above procedure to generate a series of $L_t$ and $a_t$ until Eq. (5) is satisfied. The derivatives $\partial C^e_t(S_t, L, a) / \partial L$ and $\partial C^e_t(S_t, L, a) / \partial a$ are also given in Proposition 1 of Appendix A.

Following the same idea of Broadie and Detemple (1996), the lower price bound based on the exponential exercise policy can give a lower bound for the optimal exercise boundary. As the asset price $S_t$ approaches $\text{Le}^{\text{e}^{-t}}$ from below, we can evaluate the derivatives of the exponential barrier option price with respect to $L$ and $a$, respectively:

$$D_t(L, a, t) = \lim_{S_t \to \text{Le}^{\text{e}^{-t}}} \frac{\partial C^e_t(S_t, L, a)}{\partial L},$$

$$D_t(L, a, t) = \lim_{S_t \to \text{Le}^{\text{e}^{-t}}} \frac{\partial C^e_t(S_t, L, a)}{\partial a}.$$

(6)

The formulae of $D_t(L, a, t)$ and $D_t(L, a, t)$ are also given in Proposition 1 of Appendix A. Let $H(L, a, t) = [D_t(L, a, t)]^2 + [D_t(L, a, t)]^2$ and denote by $L^*_t$ and $a^*_t$ the solutions to the equation

$$H(L, a, t) = 0.$$

(7)

It is worth emphasizing that Eq. (7) does not need to be solved recursively, i.e., Eq. (7) can be solved for $L^*_t$ and $a^*_t$ without having first solved for $L^*_t$ and $a^*_t$ for $s \in (t, T)$. We solve Eq. (7) using the same iterative procedure as the one for solving Eq. (5).

It should be noted that $\text{Le}^{\text{e}^{-t}}, t \in [0, T]$, provides a lower boundary for the optimal exercise boundary. Although a rigorous proof of this statement is difficult, the intuition behind it is straightforward. The solution of Eq. (7), $(L^*_t, a^*_t)$, can generate the exponential exercise barrier and thus the lower bound for American call prices through Eq. (2) when the asset price $S_t$ approaches $\text{Le}^{\text{e}^{-t}}$ from below. Please note that Eq. (2) is the formula to calculate the value option of a capped call and its value equals $\text{Le}^{\text{e}^{-t}} - K$ when $S_t = \text{Le}^{\text{e}^{-t}}$. Therefore, the solution of Eq. (7) satisfies

$$\text{Le}^{\text{e}^{-t}} - K = C_t(\text{Le}^{\text{e}^{-t}}, L^*_t, a^*_t).$$

Since $C^e_t(S_t, L, a)$ generates the lower bounds of the American call prices, $\text{Le}^{\text{e}^{-t}}$ should be the lower-biased early exercise boundary for the American call. Fig. 1 illustrates the above inference and

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9 The results indicate that the first derivative of the early exercise boundary with respect to time is in between 0 (perpetual) and $-\infty$ (short maturity).

10 If $\text{Le}^{\text{e}^{-t}} < S_t$, by definition the capped option with the exponential boundary has been exercised and its value equals $\text{Le}^{\text{e}^{-t}} - K, 0$. Thus the optimization is solved under the constraint that $\text{Le}^{\text{e}^{-t}} > S_t$.

11 The initial guess is chosen as $L_0 = \max(K, \mu)$.

12 The number of iterations is typically within 10 and thus the computation is quick.

13 The authors thank an anonymous referee for providing the reference of Ibáñez and Paraskevopoulos (forthcoming), which motivates us to derive the following explanation to show that $\text{Le}^{\text{e}^{-t}}$ is indeed a low-biased exercise boundary for the optimal exercise boundary $E_t$. 

shows that $L_c^e\chi^{(T-t)}$ is smaller than the optimal exercise boundary $B'_c$. In addition, from the figure, we find that if a tighter lower bound for the American call price is derived, a tighter lower bound for the optimal exercise boundary can be obtained.

3. An upper bound for the American call option value

Under Black and Scholes (1973) model, Kim (1990), Jacka (1991), and Carr et al. (1992) derive the following formula for the price of an American call option:

$$C_t^c(S_t) = c_t^c(S_t) + \int_t^T \left[ q_s \frac{e^{-r(T-t)}}{\sigma^t} N(d_1(S_t, B'_c, s - t)) - r Ke^{-r(T-t)} N(d_2(S_t, B'_c, s - t)) \right] ds,$$

(8)

where

$$d_1(x, y, t) = \frac{\ln(x/y) + (r - q + \sigma^2/2)t}{\sigma^t},$$

$$d_2(x, y, t) = d_1(x, y, t) - \sigma^t.$$

The second term in the right hand side of Eq. (8) is called the early exercise premium in the literature. The critical exercise boundary solves the following integral equation for $B'_c$ for all $s \in [t, T]$:

$$B'_c - K = c_t^c(B'_c) + \int_s^T \left[ q'_B e^{-q'_B(T-s)} N(d_1(B'_c, B'_c, \tau - s)) - r Ke^{-r(T-t)} N(d_2(B'_c, B'_c, \tau - s)) \right] d\tau.$$

(9)

Once $B'_c$ is obtained, the price of the American option can be calculated easily using Eq. (8). However, solving for $B'_c$ is usually a time-consuming process because it needs to be solved recursively, i.e. before solving for $B'_c$, one needs to first solve for $B'_c$ for $s \in [t, T]$.

Because the early exercise premium in the above formula is decreasing in the boundary, Carr et al. (1992) show that it is possible to bound the American call value analytically. For example, substituting $B'_c = K$ into Eq. (8) yields an upper bound of the American call. Actually substituting any lower estimates of the critical exercise boundary into Eq. (8) will result in an upper bound of the early exercise premium and thus an upper of the American call price. As a result we can substitute our tight lower bounds for optimal exercise boundary into the premium integral of Kim (1990), Jacka (1991), and Carr et al. (1992) to obtain a tight upper bound of the American call price. In other words, the American call option price is bounded above by the following formula:

$$C_t^c(S_t) = c_t^c(S_t) + \int_t^T \left[ q_s \frac{e^{-r(T-t)}}{\sigma^t} N(d_1(S_t, L_s^e\chi^{(T-t)}, s - t)) - r Ke^{-r(T-t)} N(d_2(S_t, L_s^e\chi^{(T-t)}, s - t)) \right] ds,$$

(10)

where $L_s^e\chi^{(T-t)}$ is the lower bound on the optimal exercise boundary given by the solution to Eq. (7).

4. Numerical results and discussions

In this section we compare our lower and upper bounds with those of Broadie and Detemple (1996). The comparison is based on the speed of computation and the accuracy of the results. Besides the option pricing bounds, Broadie and Detemple (1996) also propose two approximations for the American option prices. The first approximation is based on the lower bound (LBA) and the second approximation is based on the lower and upper bounds (LUBA). Following Broadie and Detemple (1996), we also develop two approximations based on our lower and upper bounds. Details of our LBA and LUBA are given in Appendix B.

4.1. Comparing the tightness of bounds

We first compare the tightness of our lower and upper bounds with those of Broadie and Detemple (1996). The results reported include (1) the lower bound of Broadie and Detemple (1996) (LB1), (2) our lower bound (LB2), (3) the upper bound of Broadie and Detemple (1996) (UB1), (4) our upper bound (UB2), (5) the approximation based on Broadie and Detemple's lower bound (LB1A), (6) the approximation based on our lower bound (LBA2), (7) the approximation based on Broadie and Detemple's lower and upper bounds (LUBA1), and (8) the approximation based on our lower and upper bounds (LUBA2). The parameters used are adopted from Tables 1 and 2 of Broadie and Detemple (1996). There are 40 options considered in Tables 1 and 2. In this paper, we calculate “true” option values by the Binomial Black and Sholes model with Richardson extrapolation (BBSR) where the length of each time step is 0.0001 years.

As expected, the results in Tables 1 and 2 suggest that our lower and upper bounds are tighter than Broadie and Detemple’s (1996) bounds since their bounds are special cases of our bounds with $\alpha = 0$. For short term options, Table 1 shows that the difference between our upper bound and lower bound is generally small. For example, the maximum difference between our upper bound and lower bound is only $0.0134$. The average difference is only $0.0049$ and the difference is smaller than 1 cent for 16 out 20 cases. In contrast, the average difference of Broadie and Detemple’s (1996) bounds is about nine times ($0.0425$) of our average difference. Similarly, Table 2 also indicates that our bounds are far tighter than Broadie and Detemple’s (1996) bounds for long term options. Even for long term options, the average difference of our bounds is only $0.0174$ which is smaller than the bid-ask spreads observed in the option market.

Based on the numerical results from 40 American call option contracts reported in Tables 1 and 2, we find that the average pricing errors of our lower bounds and upper bounds are 0.0957% and 0.0318%, respectively. In contrast, the average pricing errors of Broadie and Detemple’s (1996) lower bound and upper bound are 0.5641% and 0.2549%, respectively. Thus our method can improve the errors of the lower bound and upper bound of Broadie and Detemple (1996) by 83.0% and 87.5%, respectively.

Tables 1 and 2 also show that the pricing error of our LBA2 (an approximation based on our lower bound) is smaller than that of LBA1 which is based on the lower bound of Broadie and Detemple (1996). The average pricing errors of our LBA2 based on 40 options in Tables 1 and 2 are 0.0170%, which improve a lot over LBA1 (0.1319%). However, while our bounds are far more accurate than the bounds of Broadie and Detemple (1996), the approximation based on our bounds (LUBA2) is not necessarily more accurate than their LUBA1. Nevertheless, note that the results in Tables 1 and 2 are illustrative because they are only based on 40 options. This

14 In fact, according to Ibáñez (2008), American option prices in an incomplete market setting can be decomposed into three components. The first part is priced by arbitrage, the second part depends on a risk orthogonal to the first part, and third part is the early exercise premium.

15 More specifically, we first employ the BBS method with the length of each time step equaling 0.0001 (0.0002) years to calculate the option values $C_1$ ($C_2$). Then apply the Richardson extrapolation $C = 2C_1 - C_2$ to deriving the approximate option value, which is the termed the BBSR estimate with the length of each time step equaling 0.0001 years in this paper.
Deltas of American call options (maturity $T = 0.5$ years).

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<th>Option parameter</th>
<th>Asset price</th>
<th>LB1</th>
<th>LB2</th>
<th>UB1</th>
<th>UB2</th>
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<th>LBA2</th>
<th>LUBA1</th>
<th>LUBA2</th>
<th>True value</th>
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All options have $K = 100$, LB1 and UB1 (LB2 and UB2) are deltas calculated from Broadie and Detemple’s (our) lower and upper bounds. LBA1 and LUBA1 (LBA2 and LUBA2) are deltas based on the approximations of Broadie and Detemple’s (our) bounds. The “true” value is computed from the extended tree method described in Pelsser and Vorst (1994) using the Binomial Black and Scholes method where the length of each time step is 0.0001 years.

4.2. Comparing the accuracy of hedge ratios based on bounds

One possible application of our pricing bounds in Eqs. (4) and (10) is to use them to calculate the hedge ratios for American options. For example, deltas of our lower bound and upper bound can be respectively defined as:

\[ \Delta_{LB1} = \frac{v_L - v_{LB2}}{v_L - v_{UB2}} \]
\[ \Delta_{UB1} = \frac{v_{UB1} - v_{UB2}}{v_L - v_{UB2}} \]

18 Note that the “true” American option values in Ju (1998) are based on the binomial model with 10,000 time steps. Since the considered American puts are long term ($T = 3$ years) options, a binomial model with 10,000 time steps is not accurate enough as the benchmark values. Thus, we recalculate the “true” American option values using the BBSR method with 10,000 time steps per year.

19 Note that the computational time of Ibáñez (2003) is close to that of a six-point recursive scheme of Huang et al. (1996) because both methods involve the calculations of Bermudan option prices with 4, 5, and 6 exercise dates. Moreover, the computational time of our LUBA2 is also close to that of Broadie and Detemple’s (1996) LUBA1 (see Fig. 2 of this study). According to Table 3 of Ju (1998), the computational time is of the same magnitude for the methods of Huang et al. (1996), Broadie and Detemple (1996), and Ju (1998) (see columns 5, 6, and 12 of Ju’s Table 3). Thus we would expect that the computational time of the proposed method is similar to that of Ju (1998) and Ibáñez (2003).

20 Since $\beta$ and $\gamma$ are functions of $S_t$, it is impossible to derive analytical solutions of hedge ratios for the lower bound. However, the numerical derivatives of our lower bound are accurate and easy to compute.
All options have $K = 100$. LB1 and UB1 (LB2 and UB2) are deltas calculated from Broadie and Detemple's (our) lower and upper bounds. LBA1 and LUBA1 (LBA2 and LUBA2) are deltas based on the approximations of Broadie and Detemple's (our) bounds. The "true value" column is computed from the extended tree method described in Pelsser and Vorst (1994) using the Binomial Black and Scholes method where the length of each step is 0.0001 years.

<table>
<thead>
<tr>
<th>Option parameter</th>
<th>Asset price</th>
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<th>LB2</th>
<th>UB1</th>
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$\text{RMS}$ exclude the 5th option 1.2248% 0.1343% 0.3397% 0.0339% 0.8714% 0.0843% 0.4094% 0.0893% 0.0322%

All options have $K = 100$. LB1 and UB1 (LB2 and UB2) are gammas calculated from Broadie and Detemple's (our) lower and upper bounds. LBA1 and LUBA1 (LBA2 and LUBA2) are gammas based on the approximations of Broadie and Detemple's (our) bounds. The "true value" column is computed from the extended tree method described in Pelsser and Vorst (1994) using the Binomial Black and Scholes method where the length of each step is 0.0001 years.

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<th>Option parameter</th>
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<td>0.32093</td>
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<td>0.78131</td>
<td>0.78033</td>
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<td>0.78081</td>
<td>0.77756</td>
<td>0.77993</td>
<td>0.77976</td>
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</tr>
</tbody>
</table>

$\text{RMS}$ exclude the 5th option 1.2248% 0.1343% 0.3397% 0.0339% 0.8714% 0.0843% 0.4094% 0.0893% 0.0322%

$\Delta_1(S_t) = \lim_{\Delta S \to 0} (C_t(S_t, K, T - t) - C_t(S_t, S_t, \Delta S) - \Delta S)$ \(\frac{\partial C_t}{\partial S} - \Delta S) \, \Delta S.

$\Delta_2(S_t) = e^{-qT} N(d_2(S_t, L, K, T - t)) + \int_t^T \left[ q e^{-q(T-s)} N(d_2(S_t, L, K, T - t), s) - r Ke^{-r(T-s)} \right] \frac{\partial C_t}{\partial S} \, ds.

where $N_{(.)}$ is the probability density function of a standard normal variable.

We can also obtain the hedge ratios of Broadie and Detemple's bounds in the same way. For comparison, we also calculate hedge ratios of LBA1, LBA2, LUBA1, and LUBA2. We apply the extended tree method described in Pelsser and Vorst (1994) to compute benchmark values of $A$ and $T$ using the Binomial Black and Scholes method with the length of each time step equaling 0.0001 years.

Tables 5 and 6 present the deltas of American call options considered in Tables 1 and 2, respectively. The results suggest that the deltas based on bounds and the related approximations are generally accurate. The RMS relative errors range from 0.0203% (UB2) to 0.4715% (LB1) for short term options (maturity $T = 0.5$ years) and from 0.0225% (LBA2) to 0.4109% (LB1) for long term options (maturity $T = 3$ years). Moreover, we also find that the deltas of our bounds can enhance the accuracy of the deltas of Broadie and Detemple's (1996) lower and upper bounds by 81.9% and 88.1%, respectively.

When the fifth option in Table 7 is excluded, the RMS relative error is defined as RMS = \sqrt{\frac{1}{m} \sum_{i=1}^{m} (C_i - \hat{C}_i)^2 / \hat{C}_i^2}, where \( C_i \) is the "true" option price estimated by the BBSR method with the length of each time step to be 0.0001 years, \( \hat{C}_i \) is the estimated option price. Speed is measured in option prices calculated per second on a Pentium 4 3.4 GHz PC. The BBSR method is based on the length of each time step to be 0.1, 0.05, 0.025, and 0.01 years. Preferred methods are in the upper-left corner.

Tables 7 and 8 illustrate the estimations of gammas based on bounds and their related approximations. The accuracy of gamma estimates is similar to that of price and delta estimates except for the fifth option, a deep in-the-money option, in Table 7. When the fifth option in Table 7 is excluded, the RMS relative errors of gamma estimates based on our LB2, UB2, LUBA2, and LUBA2 are 0.1372%, 0.0512%, 0.0982%, and 0.0741%, respectively. Overall, the results in Tables 5–8 demonstrate that the proposed method can generate very accurate delta and gamma estimates.

4.3. Comparing the numerical efficiency of pricing American options

To make a comprehensive analysis of numerical efficiency, we compare our bounds and approximations with those of Broadie and Detemple (1996). Following Broadie and Detemple (1996), the comparison is on the basis of the computational speed and the accuracy of the results over a wide range of option parameters.

We apply the same methodology of Broadie and Detemple (1996) to choose 2500 options to test the results. Volatility (\( \sigma \)) is distributed uniformly between 0.1 and 0.6. Time to maturity is, with probability 0.75, uniform between 1.0 and 1.5 years, and, with probability 0.25, uniform between 1.0 and 5.0 years. The strike price (\( K \)) is fixed at 100. The dividend yield (\( q \)) is uniform between 0.0 and 0.1. The risk-free rate (\( r \)) is, with probability 0.8, uniform between 0.0 and 0.1, and, with probability 0.2, equal to 0.0. Each parameter is drawn independently of the others. Following Broadie and Detemple (1996), 500 sets of \( q, r, T \) are generated, and for each parameter set, five initial stock prices (\( S_0 \)) are examined from the uniform distribution between 70 and 130.

The accuracy measure used in this paper is the root mean squared (RMS) relative error. RMS relative error is defined as

\[
\text{RMS} = \sqrt{\frac{1}{m} \sum_{i=1}^{m} (C_i - \hat{C}_i)^2 / \hat{C}_i^2}
\]

where \( C_i \) and \( \hat{C}_i \) are option prices at the ith observation, and \( m \) is the number of observations.

Fig. 2. Speed-accuracy trade-off for all observations with option price \( S \) > 0.5. RMS relative error is defined as RMS = \( \sqrt{\frac{1}{m} \sum_{i=1}^{m} (C_i - \hat{C}_i)^2 / \hat{C}_i^2} \), where \( C_i \) is the "true" option price estimated by the BBSR method with the length of each time step to be 0.0001 years, \( \hat{C}_i \) is the estimated option price. Speed is measured in option prices calculated per second on a Pentium 4 3.4 GHz PC. The BBSR method is based on the length of each time step to be 0.1, 0.05, 0.025, and 0.01 years. Preferred methods are in the upper-left corner.
RMS = \sqrt{\frac{1}{m} \sum_{i=1}^{m} e_i^2},

where \( e_i = (C_i - \hat{C}_i)/C_i \) is the relative error, \( C_i \) is the “true” option price (estimated by the BBSR method with the length of each time step equaling 0.0001 years), \( \hat{C}_i \) is the estimated option price. Same as in Broadie and Detemple (1996), the summation is taken over options in the data set satisfying \( C_i \geq 0.5 \). Out of the 2500 options, 2285 satisfy this criterion.

Fig. 2 indicates that the approximations based on our bounds and Broadie and Detemple’s (1996) bounds are more efficient than the BBSR method. Our LBA2 and LUBA2 can improve the accuracy of Broadie and Detemple’s (1996) LBA1 and LUBA1 by 74.7% and 9.2%, respectively. Our LUBA2 is the most accurate one in comparison to the other approximations although it takes more computational time.

5. Conclusion

The optimal exercise boundary is vital for pricing American options. Thus a better approximation of the early exercise boundary yields a better lower bound for the American option price. In contrast to the constant exercise barrier assumed by Broadie and Detemple (1996), this paper uses an exponential function to approximate the early exercise boundary and obtains tight lower bounds for both the American option value and the optimal exercise boundary. Moreover, the tight lower bound of the optimal exercise boundary allows us to derive a tight upper bound of the American option price using the premium integral of Kim (1990), Jacka (1991), and Carr et al. (1992).

The numerical results can be summarized as follows: first of all, the American option prices are bounded tightly between our lower and upper bounds. The average difference between our upper bound and lower bound is only 0.49 cents and the maximum difference is just 1.34 cents for short term options. Secondly, our bounds can improve the pricing errors of the lower bound and upper bound of Broadie and Detemple (1996) by 83.0% and 87.5%, respectively. Moreover, the approximations (LBA2 and LUBA2) based on our bounds are also more accurate than the approximations (LBA1 and LUBA1) based on their bounds. Thirdly, the accuracy of our upper bounds for pricing American is analogous to that of the best accurate/efficient methods in the literature, e.g. Ju (1998) and Ibáñez (2003). Moreover, our LBA2 and LUBA2 are generally more accurate than Ju (1998) and Ibáñez (2003) for pricing American put options. Finally, our pricing bounds and approximations also provide accurate hedge ratios except for deep in-the-money options. The approximation errors of our pricing bounds and approximations for estimating deltas and gammas range from 0.0235% to 0.1398%. The small approximation errors show the superiority of our method to estimate deltas and gammas for American options.

Acknowledgement

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Appendix A

Proposition 1. Suppose \( L e^{\sigma (T-t)} \geq \max(S_t, K) \). Then \( \partial C_{l}^T (S_t, L, a)/\partial a \) can be written as

\[
\partial C_{l}^T (S_t, L, a)/\partial a = -K e^{\gamma (T-t)} \left\{ \left[ \frac{2 (T-t) n \left( d_1 \left( L e^{\sigma (T-t)} \right) \right)}{\sqrt{(T-t) \sigma}} \right]^{2 \gamma + 1} + \left[ \frac{2 (T-t) n \left( d_1 \left( L e^{\sigma (T-t)} \right) \right)}{(T-t) \sigma} \right] \left[ -\left( \gamma + 1 \right) (T-t) - \frac{2 \ln \left( \gamma \right)}{\sigma^2} \right] \right\} \\
- K \left\{ \frac{1}{\sqrt{(T-t) \sigma}} \left[ n \left( d_0 \right) - (T-t) \left( \frac{b}{\beta} - 1 \right) \right]^{2 \gamma + 1} + n \left( d_0 \right) \left[ -\left( T-t \right) \left( \frac{b}{\beta} - 1 \right) + \frac{1}{\sigma} \left( \frac{b}{\beta} - 1 \right) \right] \ln \left( \gamma \right) \right\}^{2 \gamma + 1} \\
+ n \left( d_0 + 2 \beta \sigma \sqrt{T-t} \right) - \sqrt{\frac{T-t}{\sigma}} \left( \frac{b}{\beta} + 1 \right) ^{2 \gamma + 1} + n \left( d_0 + 2 \beta \sigma \sqrt{T-t} \right) \left[ -\left( T-t \right) \left( \frac{b}{\beta} - 1 \right) - \frac{1}{\sigma^2} \left( \frac{b}{\beta} + 1 \right) \right] \ln \left( \gamma \right) \right\}^{2 \gamma + 1} \\
+ L e^{\sigma (T-t)} \left\{ \left[ \frac{2 (T-t) n \left( d_1 \left( L e^{\sigma (T-t)} \right) \right)}{\sqrt{(T-t) \sigma}} \right] \left( 1 + \frac{1 - \beta}{\sigma} \right) ^{2 \gamma + 1} + n \left( d_0 + 2 \beta \sigma \sqrt{T-t} \right) \left[ -\left( T-t \right) \left( \frac{b}{\beta} - 1 \right) - \frac{1}{\sigma^2} \left( 1 + \frac{1 - \beta}{\sigma} \right) \right] \ln \left( \gamma \right) \right\}^{2 \gamma + 1} \\
+ L e^{\sigma (T-t)} \left\{ \left[ \frac{2 (T-t) n \left( d_1 \left( L e^{\sigma (T-t)} \right) \right)}{\sqrt{(T-t) \sigma}} \right] \left( 1 + \frac{1 - \beta}{\sigma} \right) ^{2 \gamma + 1} + n \left( d_0 + 2 \beta \sigma \sqrt{T-t} \right) \left[ -\left( T-t \right) \left( \frac{b}{\beta} - 1 \right) - \frac{1}{\sigma^2} \left( 1 + \frac{1 - \beta}{\sigma} \right) \right] \ln \left( \gamma \right) \right\}^{2 \gamma + 1} \\
+ S_0 e^{-\gamma (T-t)} \left\{ \left[ \frac{2 (T-t) n \left( d_1 \left( L e^{\sigma (T-t)} \right) - \sigma \sqrt{T-t} \right)}{\sqrt{(T-t) \sigma}} \right] \left( \gamma - 1 \right) (T-t) - \frac{2 \ln \left( \gamma \right)}{\sigma^2} \right\} ^{2 \gamma + 1} \\
+ n \left( d_0 + 2 \beta \sigma \sqrt{T-t} \right) - \sqrt{\frac{T-t}{\sigma}} \left( \frac{b}{\beta} + 1 \right) ^{2 \gamma + 1} + n \left( d_0 + 2 \beta \sigma \sqrt{T-t} \right) \left[ -\left( T-t \right) \left( \frac{b}{\beta} - 1 \right) - \frac{1}{\sigma^2} \left( 1 + \frac{1 - \beta}{\sigma} \right) \right] \ln \left( \gamma \right) \right\}^{2 \gamma + 1},
\]

Moreover, our results (not reported here) show that LB2 and UB2 can improve the accuracy of Broadie and Detemple’s (1996) LB1 and UB1 by 82.4% and 78.3%, respectively.
\[ \partial C_t^f(S_t, L_t, a_t) / \partial L_t \] can be written as

\[ \left. \begin{array}{l}
\partial C_t^f(S_t, L_t, a_t) / \partial L_t \\
= e^{\theta(T-t)} \left[ \frac{n(d_t^1)(Le^{\theta(T-t)})}{L\sqrt{(T-t)\sigma}} + \frac{2n(d_t^1)(Le^{\theta(T-t)})}{L\sqrt{(T-t)\sigma}} \right] + \left( \frac{\hat{\beta} + \hat{\beta}}{\sqrt{\frac{(T-t)\sigma}{\sigma}}} \right) \right] \\
\end{array} \right] \\
\]
Appendix B

B.1. LBA2: Approximation based on our lower bound

In this paper, a process similar to that in Broadie and Detemple (1996) is adopted to convert our lower bound to the approximate American option value LBA2. The LBA2 is assumed to follow

\[ \text{LBA2} = \lambda_1 C^*_1(S_t) \]

where the adjusting parameter \( \lambda_1 = \lambda_1(S_t, K, T, r, q) \) is a function of \( S_t, K, T, r, \) and \( q \).

The details of deciding \( \lambda_1(S_t, K, T, r, q) \) are as follows. First, we define \( x_1 = 1, x_2 = T, x_3 = \sqrt{T}, x_4 = S_t/K, x_5 = r, x_6 = q, x_7 = \min \left( \tau, \max (q, 10^{-5}) \right), x_8 = x_3 - x_5, x_9 = (C^*_1(S_t) - C^*_1(S_t))/K, x_{10} = x_2, x_{11} = C^*_1(S_t)/C_1(S_t), x_{12} = S_t/\bar{x}_1, \) and \( x_{13} = a_1(S_t) \). Then derive an intermediate variable \( y_i \) via the following equation.

\[
y_i = 1.002E+00x_1 + 1.647E-04x_2 + 8.245E-05x_3 - 1.336E-03x_4 - 3.679E-03x_5 + 1.933E-02x_6 + 1.399E-02x_7 - 6.357E-04x_8 - 1.035E-02x_9 + 1.292E-02x_{10} - 2.726E-04x_{11} + 3.976E-04x_{12} - 4.452E-04x_{13}.
\]

Finally, \( \lambda_1(S_t, K, T, r, q) \) can be derived by

\[
\lambda_1(S_t, K, T, r, q) = \begin{cases} 1 & \text{if } C^*_1(S_t) = c(S_t) \text{ or } C^*_1(S_t) \leq S_t - K, \\ \max(\min(y_i, 1.008), 1) & \text{otherwise} \end{cases}
\]

where the maximum value of \( \lambda_1(S_t, K, T, r, q) \) is assumed to be 1.008 because our lower bounds are always within 0.78% of the true option values. The coefficients for \( y_i \) are derived from a regression based on the randomly generated 2500 option contracts in Section 4.3.22

B.2. LUBA2: Approximation based on our lower and upper bounds

The process to derive the approximate option value LUBA2 based on the information of \( C^*_1(S_t) \) and \( C^*_1(S_t) \) is elaborated as follows. First, consider a linearly weighted average relation between the LUBA2 and the lower and upper bounds \( C^*_1(S_t) \) and \( C^*_1(S_t) \). LUBA2 is a function of \( S_t, K, T, r, \) and \( q \).

\[
\text{LUBA2} = \lambda_2 C^*_1(S_t) + (1 - \lambda_2) C^*_1(S_t)
\]

where the weighted average parameter \( \lambda_2 = \lambda_2(S_t, K, T, r, q) \) is a function of \( S_t, K, T, r, \) and \( q \).

To determine the function \( \lambda_2(S_t, K, T, r, q) \), we first define \( x_1 = 1, x_2 = T, x_3 = \sqrt{T}, x_4 = r, x_5 = q, x_6 = \min \left( \tau, \max (q, 10^{-5}) \right), x_7 = x_3 - x_5, x_8 = (C^*_1(S_t) - C^*_1(S_t))/K, x_{10} = x_3 - x_5, x_{12} = (C^*_1(S_t) - C^*_1(S_t))/K, x_{13} = (C^*_1(S_t) - C^*_1(S_t))/K, x_{15} = S_t/\bar{x}_1, \) and \( x_{20} = a_1(S_t) \). Then we calculate an intermediate variable \( y_i \) via the following equation.

\[
y_i = 2.329E-01x_1 - 2.384E-02x_2 + 1.457E-01x_3 + 3.718E-02x_4 + 1.849E-01x_5 - 3.111E-01x_6 + 2.447E-01x_7 - 1.887E-01x_8 + 3.801E-01x_9 + 3.556E-01x_{10} - 6.465E-01x_{11} + 4.622E-02x_{12} + 6.454E-02x_{13} - 2.170E-01x_{14} + 8.079E-02x_{15} + 2.020E-01x_{16} + 6.245E-01x_{17} - 2.970E-01x_{18} - 4.320E-01x_{19} + 2.964E-01x_{20}.
\]

Finally, \( \lambda_2(S_t, K, T, r, q) \) can be derived according to the following rule:

\[
\lambda_2(S_t, K, T, r, q) = \begin{cases} 1 & \text{if } C^*_1(S_t) = \bar{c}(S_t) \text{ or } C^*_1(S_t) \leq S_t - K, \\ \max(\min(y_i, 1.00), 1) & \text{otherwise} \end{cases}
\]

Based on the above framework for determining \( \lambda_2(S_t, K, T, r, q) \), a weighted regression is employed to find the coefficients in the formula of \( y_i \). More specifically, the target function is

\[
\min \sum \left( C^*_i - \bar{C}_i \right) \left( \lambda_2 C^*_1 + (1 - \lambda_2) C^*_1 - \bar{C}_i \right)^2.
\]

where \( C^*_1, \bar{C}_1, \) and \( C_1 \) are the lower and upper bounds and the true value for the ith option contract. The intuition behind the weighted regression is that if the lower and upper bounds are very tight, the value of \( \lambda_2 \) become less important in predicting \( \bar{C}_i \). Via performing this weighted regression on the randomly generated 2500 option contracts, the coefficients in the equation of \( y_i \) can be determined.

References

Ingersoll Jr., J.E., 1998. Approximating American options and other financial contracts using the RMD-speed results in Fig. 2. For the set to determine \( \lambda_1 \) and \( \lambda_2 \), the number of qualified option contracts (i.e. option price \( \geq 0.5 \)) is 2277.
In this paper, two separately random sets of option contracts are generated following the same rule described in Section 4.3. One is used to estimate \( \lambda_1 \) and \( \lambda_2 \), and the other is used to compute the RMD-speed results in Fig. 2. For the set to determine \( \lambda_1 \) and \( \lambda_2 \), the number of qualified option contracts (i.e. option price \( \geq 0.5 \)) is 2277.
In this paper, the term \( x = dC^*_1(S_t)/dS_1 \) is approximated by using a numerical differentiation with respect to \( S_t \) given the same exponential exercise barrier \( x(S_t)e^{\bar{x}_1t} \).