Hedge and Price American Options with Static Hedging Portfolio Method under Stochastic Volatility

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Abstract

Under stochastic volatility (SV), despite the abundant literature on American option pricing, there is little work on American option hedging. This paper develops a feasible and excellent performing static hedging portfolio (SHP) method for hedging and pricing an American option under SV by constructing a portfolio of European options to match the payoff, delta, and vega of the target American option along its early exercise boundary. The novelty of the proposed SHP method is incorporating the expected variance conditional on the stock price into Chung and Shih's (2009) method and further improving their method by imposing the vega-matching condition. Our numerical analyses show the superiority of the proposed SHP method in effectively hedging and accurately pricing American options in the presence of SV, especially when the vega-matching condition is taken into account. For a large, randomly generated set of American option contracts, average pricing error (hedging risk, measured by 5% Value at Risk) of the proposed SHP method ranges from 0.11% to 0.14% (0.78% to 0.88%) of the average option value, and the 5% Value at Risk of the proposed SHP method is around 3% of that of the widely used dynamic delta-neutral hedging method.

Keywords: Static Hedging; American Options; Stochastic Volatility; Conditional Expected Variance; Vega Matching

JEL classification: G10; G13

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1. Introduction

The hedging and pricing of American options are classic and ongoing issues, and the additional consideration of stochastic volatility (SV) makes the hedging and pricing of American options even more complex and difficult. Besides the classic multidimensional tree models and finite different methods (FDM), many other methods price American options under the SV assumption. Most of these methods focus primarily on pricing issues, with little emphasis on hedging. Even when hedging is addressed, this is often through dynamic delta-neutral hedging (DDH). In contrast, this paper contributes to the literature by proposing a feasible and excellent performing static hedging portfolio (SHP) method to study the hedging and pricing of American options under SV.

The fundamental concept behind the SHP method is employing commonly traded European options (and/or other derivative assets) on the same underlying asset to create a portfolio whose value corresponds to the value of a target option at its boundary conditions by determining the maturity date, strike price, and investment amount of each European option in the portfolio. *Static hedging* refers to a situation where, throughout the entire duration of the target option, regardless of stock price fluctuations, the issuer (hedger) needs not make any adjustments to the SHP. Either at or before maturity, when it is time to pay the due compensation to the holder of the target option, the issuer (hedger) needs only liquidate the SHP to generate the corresponding cash flow to pay the option holder. Compared with the DDH method, the purpose of using the SHP method is to achieve more effective hedging or save hedging transaction costs. In addition, the SHP method may be less sensitive to model risk than the DDH method, since the pricing biases of the target option and the constructed SHP caused by model risk could partially offset each other. Last, the SHP's value today can also closely approximate the current value of the target option as long as the SHP's liquidation payoffs at the boundary conditions are close to those of the target option. Theoretically speaking, since any derivative asset on an underlying asset follows the same partial differential equation (PDE), if the target option and the constructed SHP have equal boundary-condition payoffs, their value today should be identical.

Static hedging was first introduced by Bowie and Carr (1994), Derman, Ergener, and Kani (1995), and Carr, Ellis, and Gupta (1998). It is used to hedge exotic options, such as European barrier options, under the assumption of the geometric Brownian motion (GBM) model. Contrary to the focus of most of the SHP literature, which predominantly examines barrier options or other exotic European options, Chung and Shih (2009) are the first to apply the SHP method to the valuation of American options. Building upon the value-matching condition proposed by Derman, Ergener, and Kani (1995), they introduce an additional smooth-pasting condition to better align with the boundary conditions of American options. Using backward induction to solve for the investment proportions of European options in the SHP, they concurrently derive the critical exercise price for American options, which serves as an approximate estimate of the true early exercise boundary. Under both the GBM and constant elasticity of variance (CEV) models, Chung and Shih (2009)'s SHP method accurately determines the price, delta, and gamma values of American options. Similarly, following this vein, Chung, Shih, and Tsai (2013a, 2013b) extend this methodology to hedge and evaluate American touch-in and touch-out options.

The literature on SHP methods under non-GBM models is sparse, but a summary of the key contributions is as follows. To our knowledge, Fink (2003) is the first to consider using an SHP to hedge and price European up-and-out call options under the Heston (1993) SV model.[1] To address the additional dimension of variance, Fink (2003) chooses one to four representative variances and includes more call options with strike prices higher than the barrier into the SHP to ensure value matching at the barrier boundaries under these representative variances. By introducing value matching at each time and variance node along the constant barrier, theoretically, the hedging performance of the constructed SHP is better the finer the segmentation in

[1] To account for the SV model, Allen and Padovani (2002) also conduct optimized value matching at time and variance nodes along the barrier boundary. Their method, purely a hedging model, combines static and dynamic hedging to form a quasi-static hedge, aiming for optimized hedging of long-term or exotic European options. Since their method is not an exact SHP method, theoretically, it cannot be used to evaluate the target European barrier option.

time and variance dimensions. Nalholm and Poulsen (2006) extend the framework of Fink (2003) within a stochastic volatility with log-normal jumps. They focus on static hedging for European up-and-out barrier options, considering the inclusion of additional European options in the SHP, whose strike price is the most possible stock price level after the stock price penetrates the barrier from below due to a jump occurring. Takahashi and Yamazaki (2009) implement static hedging for European path-independent options in an SV model. They consider a pricing process with local volatility, identical to the stock price distribution in the SV model, based upon which an SHP, consisting of risk-free assets, shorter-maturity forwards, European calls, and puts, can be determined. Tsai (2014) hedges and prices European barrier options under the CEV model. Unlike Derman, Ergener, and Kani (1995) or earlier works, he considers both value-matching and theta-matching conditions and introduces binary options into the SHP to significantly improve hedging effectiveness. Huh, Jeon, and Ma (2020) use an SHP to hedge and price European barrier options in a fast mean-reverting SV model. Leveraging the fast mean-reverting characteristic, they combine asymptotic expansion and perturbation theory, as suggested by Fouque, Papanicolaou, Sircar, and Sølna (2003), to transform the static hedging problem along the dimensions of both time and variance into two simpler static hedging problems along just the time dimension. Last, Guo and Chang (2020) evaluate European barrier options using an SHP in a generalized CEV model, where stock price volatility is an exponential function of the stock price, unrestricted in its exponent. Following Chung, Shih, and Tsai (2010, 2013a) and Tsai (2014), they consider not only value matching but also theta matching and incorporate binary options into the SHP to improve matching performance. Moreover, they validate the use of repeated Richardson extrapolation to significantly enhance the accuracy of the SHP in evaluating European barrier options.

To the best of our knowledge, despite the well-known advantages of SHP methods, no SHP method studies the hedge of American options under SV. We argue that this is because there are practical and extensional concerns to Fink's (2003) method, although his method seems theoretically sound under the SV model. The first concern is about the choice of representative variances. In the time dimension, based on the option's time to maturity, such as half a year, appropriate and dispersed time points for value matching within this range of period, such as monthly or weekly intervals, can be sufficient. However, the appropriate range of the stochastic variance at each time point is unknown. Fink's (2003) method is limited to arbitrarily selecting a few representative variance values, which remain constant throughout the duration of the target option, casting doubt on whether this adequately represents the entire variance dimension. The second is the quantity of option contracts in the SHP. When both variance and time dimensions are considered, the number of options included in the SHP grows exponentially, which gradually erodes the benefit of lower hedging transaction costs for the SHP method. Moreover, although theoretical research might assume the availability of vanilla European options with any expiration date and strike price, in practice, the possible expiration dates and (deeply out-of-the-money) strike prices for European options are limited. The third is the numerical issue when solving systems of equations. In Fink's method, out-of-the-money options with the same expiration date but different strike prices may have very small and similarly scaled values given different variance levels, possibly leading to very positive or negative results when determining the investment amounts of plain vanilla European options, as discussed by Nalholm and Poulsen (2006), Huh, Jeon, and Ma (2020), and Fink (2003). If large-scale trading for any single option is needed when constructing the SHP, this method's benefit of lower hedging transaction costs could be further undermined. Finally, Fink's (2003) method is designed for European barrier options, utilizing the known (constant) barrier level to determine the strike prices of options in an SHP. It may not be feasible to combine Fink's method with the SHP methods developed by Chung and Shih (2009) or Chung, Shih, and Tsai (2013a, 2013b) for hedging and evaluating American options under SV, since their methods concurrently construct the SHP and determine the unknown early exercise boundary for the target American options.[2] Furthermore, under SV, the early exercise boundary at a given

[2] This characteristic allows Chung and Shih's (2009) and Chung, Shih, and Tsai's (2013a, 2013b) methods to not only hedge but also price American options. This is because the early exercise boundary of American options is a free boundary problem: solving for this boundary also yields the present value of the American option. In other words, if another American option valuation model is used first to obtain the early exercise boundary, the present value of the American option is also obtained. Therefore, subsequently using Fink's SHP method to fit the obtained boundary conditions of the American option and achieving an approximate valuation would be somewhat redundant.

time point is a function of both stock price and variance, adding to the model complexity.

Rather than arbitrarily considering a few fixed representative variances, this paper introduces the concept of expected variance conditional on the stock price into Chung and Shih's (2009) SHP method for hedging and pricing American options. Since it is not possible to increase the number of representative variances in an unlimited manner, we consider only the most probable occurring variance along the early exercise boundary conditions, i.e., we consider the conditional expected variance given the stock price equal to the early exercise boundary. Note that with the use of conditional expected variance, the method proposed in this paper differs from the original concept of the SHP method, such as in Fink (2003), in accurately fitting boundary conditions for all examined time-variance nodes at the constant barrier. We believe that if the proposed SHP method has a greater likelihood of fitting well when touching the early exercise boundary of the target American option, it probabilistically should produce more accurate valuation results for the target American option, and the average hedging performance for the target American option should be better. Furthermore, since the value and early exercise boundaries of American options are sensitive to changes in variance, the SHP method proposed in this paper also considers the vega-matching condition, in addition to the value-matching condition and smooth-pasting condition proposed by Chung and Shih (2009).

We argue that there are several theoretical advantages of the proposed SHP method. First, Fink (2003) arbitrarily considers a few fixed representative volatilities for matching the boundary conditions at the barrier; for example, he chooses the initial volatility level as one such representative volatility. However, it is critical to note that the probability of these representative volatilities occurring at the boundary conditions might be very low. Taking an American put for example, the critical stock price on the early exercise boundary could be far lower than the prevailing stock price; moreover, due to the inverse relationship between the stock price and volatility processes in Heston's SV model, when the stock price moves toward the early exercise boundary, the accompanying volatility level could be substantially higher than its initial level. Even if Fink's (2003) method fits the boundary conditions under these low-probability representative volatilities, this does not necessarily enhance the hedging and valuation capabilities of the SHP method. Second, since this paper considers only a single variance value that is most likely to occur on the early exercise boundary, as shown in Section 3, it becomes feasible to incorporate our idea with Chung and Shih's (2009) method for hedging and pricing American options given SV. Third, under the SV model, the variance is stochastic as well as the stock price. A natural choice, therefore, is to consider the vega-matching condition on the early exercise surface in the time, stock price, and variance space. If the smooth-pasting condition ensures that the sensitivities of the target American option and the SHP to stock price changes on the early exercise boundary are consistent, then the vega-matching condition ensures that their sensitivities to variance changes on the early exercise boundary are consistent.

Indeed, the experiments in this paper demonstrate that the above theoretical advantages can be translated to the superior performance of the proposed SHP method for hedging and pricing American puts. Average pricing error (hedging risk, measured by 5% Value at Risk) of the proposed SHP method ranges from 0.11% to 0.14% (0.78% to 0.88%) of the average option value of a large, randomly generated set of American option contracts, and the 5% Value at Risk of the proposed SHP method is around 3% of that of the widely used DDH method. Therefore, for issuers or hedgers of American options, the proposed SHP method is preferred to the widely used DDH method to achieve more effective and lower cost hedging management.

The remainder of the paper is organized as follows. Section 2 begins by reviewing the construction of an SHP for pricing American options in Chung and Shih (2009) and Heston's SV option pricing formula for European options. Section 3 details how to implement our conditional expectation of the variance under the SV model and integrate it into the SHP method of Chung and Shih (2009) for hedging and pricing American options. Section 4 analyzes the pricing, convergence, and hedging performance of the proposed SHP method. Section 5 concludes the paper.

2. Review of Chung and Shih (2009) and Heston (1993)

2.1 Chung and Shih's (2009) SHP Method for Pricing American Options

Chung and Shih (2009) establish an SHP for American options under the assumption of the GBM model for the underlying stock price, i.e.,

$$\frac{dS(t)}{S(t)} = (r-q)dt + \sigma dW(t), \tag{1}$$

where $S(t)$ is the underlying stock price at t, σ is a constant volatility, r is the risk-free rate, q is the dividend yield, and $W(t)$ denotes a standard Wiener process in the risk-neutral probability measure. Denote F as the value of any derivative asset on the underlying stock S. Then F satisfies the following PDE:

$$\frac{1}{2}\sigma^2 S^2 F_{SS} + (r-q)SF_S + F_t = rF, \tag{2}$$

where F_S and F_t denote the partial differentiation of F with respect to S and t, respectively, and F_{SS} denotes the second-order partial differentiation of F with respect to S. Based on Equation (2), for all derivative assets on S, their different values today result from their different boundary conditions. This feature inspires the emergence of the SHP method: if one can construct a portfolio with more fundamental options on S (hedging position) by determining their strike prices, times to maturity, and investment amounts to match the boundary conditions of a more complicated target derivative on S (hedged position), one can obtain the equality of values between the hedging and hedged positions not only at the boundary but also the theoretical values today. Therefore, the SHP method serves both hedging and pricing purposes.

However, the early exercise boundary of American options is not known before conducting the SHP method; moreover, since it is a free boundary problem, the early exercise boundary of American options should be determined concurrently during the pricing process, i.e., during the process of constructing the SHP. To solve this problem, Chung and Shih (2009) impose the value-matching as well as smooth-pasting conditions between the target American put and the SHP hedging positions on the early exercise boundary. During the construction of the SHP from the maturity backward toward today, the early exercise boundary of the target American put is also determined simultaneously and the information of the early exercise boundary at later time points affects the early exercise boundary at earlier time points.

Chung and Shih (2009) begin the construction of the SHP with one unit of the counterpart European vanilla put, with its strike price (X) and maturity (T) corresponding to the target American put. Then, n evenly-spaced time points before maturity are selected, i.e., t_0, $t_1 = t_0 + \delta t$, ..., $t_{n-1} = T - \delta t$, where $\delta t = \frac{T-t_0}{n}$, assuming that the SHP also matches the boundary condition of the American put at these time points. To determine the unknown boundary condition B_i at t_i, one must add to the SHP w_i units of standard European puts with maturity at t_{i+1} and a strike price of B_i. Taking the time point t_{n-1} for example, the value-matching and smooth-pasting conditions on the early exercise boundary are employed to solve the two unknowns, B_{n-1} and w_{n-1}, as

$$X - B_{n-1} = p(B_{n-1}, X, T - t_{n-1}, \sigma^2) + w_{n-1} p(B_{n-1}, B_{n-1}, T - t_{n-1}, \sigma^2), \quad (3)$$

$$-1 = p_S(B_{n-1}, X, T - t_{n-1}, \sigma^2) + w_{n-1} p_S(B_{n-1}, B_{n-1}, T - t_{n-1}, \sigma^2), \quad (4)$$

where $p(S, X, TM, \sigma^2)$ and $p_S(S, X, TM, \sigma^2)$ are the Black–Scholes European put price and delta, respectively, with the inputs of the stock price (S), the strike price (X) and remaining time to maturity of the option (TM), and the stock return variance (σ^2). Since the risk-free interest rate (r) and dividend yield (q) are fixed as constants, we do not include them as the input parameters for simplicity. After acquiring the weight w_{n-1} and the early exercise boundary B_{n-1} at t_{n-1}, one should iteratively perform backward induction from t_{n-2} to t_0 to determine the weight and the boundary for the remaining time points. Finally, the value of the SHP given n time points for the American put, $P_n^{SHP}(t_0)$, is formulated as

$$P_n^{SHP}(t_0) = p(S(t_0), X, T - t_0, \sigma^2)$$

$$+ w_{n-1} p(S(t_0), B_{n-1}, T - t_0, \sigma^2)$$

$$+w_{n-2}p(S(t_0), B_{n-2}, t_{n-1} - t_0, \sigma^2)$$

$$+\cdots$$

$$+w_0 p(S(t_0), B_0, t_1 - t_0, \sigma^2). \quad (5)$$

Chung and Shih (2009) show that when n increases, $P_n^{SHP}(t_0)$ converges to the theoretical value of the target American put.

2.2 Heston SV Model for Pricing European Options

After the 1987 crash, the assumption of constant volatility became unrealistic. Among abundant SV literature, this paper focuses on the classical Heston (1993) model, which is an affine SV model and therefore prices European options efficiently with the analytic option formulas proposed by Heston (1993). Heston's assumptions of stochastic processes under the risk-neutral probability measure are

$$\frac{dS(t)}{S(t)} = (r - q)dt + \sqrt{v(t)}dW(t), \quad (6)$$

$$dv(t) = \kappa[\theta - v(t)]dt + \sigma_v\sqrt{v(t)}dZ(t), \quad (7)$$

$$dZ(t) = \rho dW(t) + \epsilon\sqrt{(1 - \rho^2)dt}, \quad (8)$$

where $v(t)$ is the variance process of stock return at time t, κ determines the reverting speed of the variance, θ is the long run mean of variance, σ_v is the volatility of the variance, $W(t)$ and $Z(t)$ are Wiener processes with correlation ρ, and ϵ is a standard normally distributed random variable which is independent of $dW(t)$. The value of any derivative asset F on the underlying stock S, such as European and American options, satisfies the following PDE:

$$\frac{1}{2}vS^2F_{SS} + \rho\sigma_v vSF_{Sv} + \frac{1}{2}\sigma_v^2 vF_{vv} + (r - q)SF_S + \kappa(\theta - v)F_v + F_t = rF, \quad (9)$$

where F_s and F_{ss} (F_v and F_{vv}) represent the first- and second-order partial derivatives of value F with respect to stock price S (the variance of stock return v) and F_{sv} denotes the cross partial derivative with respect to S and v, whereas F_t signifies the partial derivative with respect to time t.

Based on the PDE in Equation (9), Heston (1993) derives the analytic-form pricing formula for the European call option as

$$H^c\left(S(t_0), X, T-t_0, v(t_0)\right) = \frac{1}{2}J(T-t_0;-i) - \frac{K}{2}J(T-t_0;0)$$

$$-\frac{1}{\pi}\int_0^\infty \frac{Im[J(T-t_0;-i-u)\exp(iu\times\ln K)]}{u}du$$

$$+\frac{K}{\pi}\int_0^\infty \frac{Im[J(T-t_0;-u)\exp(iu\times\ln K)]}{u}du, \quad (10)$$

where

$$J(T-t_0;\phi) = S(t)^{i\phi}\exp[A(T-t_0;\phi) + B(T-t_0;\phi)v(t_0)],$$

$$A(T-t_0;\phi) = [\phi i(r-q) - r](T-t_0)$$

$$-\frac{\kappa\theta}{\sigma_v^2}\left[(\varepsilon - \rho\sigma_v\phi i + \kappa)(T-t_0) - 2\ln\left(1 - \frac{(\varepsilon+\rho\sigma_v\phi i-\kappa)(1-\exp(-\varepsilon(T-t_0)))}{2\varepsilon}\right)\right],$$

$$B(T-t_0;\phi) = \frac{i\phi(i\phi-1)(1-\exp(-\varepsilon(T-t_0)))}{2\varepsilon-(\varepsilon+i\phi\sigma_v\rho-\kappa)(1-\exp(\varepsilon(T-t_0)))},$$

$$\varepsilon = \sqrt{(\rho\sigma_v\phi i-\kappa)^2 - \sigma_v^2\phi i(\phi i-1)}.$$

The parameters in the function $H^c(S, X, TM, v)$ include the stock price, strike price and time to maturity of the examined options, and the initial variance level. We omit r, q, κ, and θ in the parameter list for simplicity. In addition, to obtain the value of European put options under the Heston model, we exploit the put-to-call parity equation, which is

$$H^p\left(S(t_0), X, T-t_0, v(t_0)\right) = H^c\left(S(t_0), X, T-t_0, v(t_0)\right)$$

$$-Se^{-q(T-t_0)} + Xe^{-r(T-t_0)}. \quad (11)$$

3. Our Model

3.1 Static Hedging Portfolio under SV

If the stock price follows Heston's SV model, it is not straightforward to hedge and price American options using the SHP method proposed in Chung and Shih (2009) due to the additional variance dimension. The major problem is because in the presence of SV, their SHP method must concurrently solve the early exercise surface (rather than the early exercise boundary line) in the space of (S, v, t) during the construction of the SHP. Even given a solution to the early exercise surface, the SHP method would be nearly infeasible to match exponentially growing time-variance nodes due to the limited number of options existing in markets. To address this problem, we examine matching conditions for only the expected variance conditional on the critical early exercise stock price, the most probable variance level when the stock price touches the early exercise boundary, rather than for the constant representative variances discussed in the literature. As we argue in the introduction, if our SHP method has a greater likelihood of fitting well when touching the early exercise boundary of the target American option, it should on average produce more accurate valuation results for the target American option, and the average hedging performance for the target American option should be satisfactory. In addition to the value-matching and smooth-pasting conditions, we impose the vega-matching condition. According to Equation (9), since both F_s and F_v have roles to play, it is natural to consider the vega-matching condition (the sensitivity of the SHP with respect to v) in addition to the smooth-pasting condition (the sensitivity of the SHP with respect to S).

Following Chung and Shih (2009), we initiate the process with one unit of standard European put options, where all parameters correspond with the target American put option. Next, we evenly divide the remaining time before maturity into n time points, t_0, $t_1 = t_0 + \delta t$, ..., $t_{n-1} = T - \delta t$, where $\delta t = \frac{T-t_0}{n}$. Third, we match the value-matching and smooth-pasting conditions by adding w_i (to be solved) units of standard European options with maturity at t_{i+1} and strike price at B_i (to be solved) into the SHP. However, in contrast to Chung and Shih (2009), we replace the constant variance σ^2 with the conditional expected variance, $E[v(t_i)|B_i]$, the most probable occurring variance on the early exercise boundary B_i. The calculation of $E[v(t_i)|B_i]$ will be discussed in the next two sections. To fulfill the additional vega-matching condition, we refer to Fink (2003) and add $\widehat{w}_i$ (to be solved) units of more deeply out-of-the-money European put options with maturity at t_{i+1} and strike price at $B_i - \gamma$, where $\gamma > 0$ is a given parameter. Our SHP is conducted using backward iteration, beginning at t_{n-1}, as

$$X - B_{n-1} = H^p(B_{n-1}, X, T - t_{n-1}, E[v(t_{n-1})|B_{n-1}])$$

$$+w_{n-1}H^p(B_{n-1}, B_{n-1}, T - t_{n-1}, E[v(t_{n-1})|B_{n-1}])$$

$$+\widehat{w}_{n-1}H^p(B_{n-1}, B_{n-1} - \gamma, T - t_{n-1}, E[v(t_{n-1})|B_{n-1}]), \tag{12}$$

$$-1 = \Delta^p(B_{n-1}, X, T - t_{n-1}, E[v(t_{n-1})|B_{n-1}])$$

$$+w_{n-1}\Delta^p(B_{n-1}, B_{n-1}, T - t_{n-1}, E[v(t_{n-1})|B_{n-1}])$$

$$+\widehat{w}_{n-1}\Delta^p(B_{n-1}, B_{n-1} - \gamma, T - t_{n-1}, E[v(t_{n-1})|B_{n-1}]), \tag{13}$$

$$0 = \upsilon^p(B_{n-1}, X, T - t_{n-1}, E[v(t_{n-1})|B_{n-1}])$$

$$+w_{n-1}\upsilon^p(B_{n-1}, B_{n-1}, T - t_{n-1}, E[v(t_{n-1})|B_{n-1}])$$

$$+\widehat{w}_{n-1}\upsilon^p(B_{n-1}, B_{n-1} - \gamma, T - t_{n-1}, E[v(t_{n-1})|B_{n-1}]), \tag{14}$$

where $H^p(S, X, TM, v)$, $\Delta^p(S, X, TM, v)$, and $\upsilon^p(S, X, TM, v)$ denote the price, delta, and vega (partial differentiation of $H^p(S, X, TM, v)$ with respect to the volatility, $\sqrt{v}$) of the European put under the Heston model; at the critical stock price B_{n-1} on the early exercise boundary, the payoff, delta, and vega of the target American put option are $X - B_{n-1}$, –1 , and 0, respectively. The vega is zero because if the American put is exercised, its payoff ($X - B_{n-1}$) is independent of the stock return variance. We solve the critical stock price B_{n-1} and the weights w_{n-1}, $\widehat{w}_{n-1}$ based on the above three equations. The first step involves expressing w_{n-1} and $\widehat{w}_{n-1}$ as functions of B_{n-1} based on Equations (13) and (14). The next step is to incorporate the obtained functions of w_{n-1} and $\widehat{w}_{n-1}$ into Equation (12). The final step entails determining the boundary root B_{n-1} through the application of the bisection method. In instances where the bisection method fails to solve a root, a brute force method is instead employed, which examines a sufficiently fine grid within the S-space to ascertain the solution.

For t_{n-2}, we add two more options that mature at t_{n-1} in the SHP to match the three required conditions; the process is repeated backwards for t_{n-3}, t_{n-4}, ..., t_0. Since we always apply the SHP method to hedge and price an American option that is not yet early exercised at t_0 but can only be early exercised after t_0, it is unnecessary to solve the critical stock price B_0 at t_0, which cannot be touched when we conduct the hedging analysis. Therefore, we instead match the early exercise boundary at a time point slightly later than t_0, denoted as $\tilde{t}_0 = t_0 + 0.0001$. The system of equations examined at $\tilde{t}_0$ is

$$\begin{aligned}
B_0 - X = {} & H^p(B_0, X, T - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]) \\
& + w_{n-1} H^p(B_0, B_{n-1}, T - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]) \\
& + \widehat{w}_{n-1} H^p(B_0, B_{n-1} - \gamma, T - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]) \\
& + w_{n-2} H^p(B_0, B_{n-2}, t_{n-1} - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]) \\
& + \widehat{w}_{n-2} H^p(B_0, B_{n-2} - \gamma, t_{n-1} - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]) \\
& + \cdots \\
& + w_1 H^p(B_0, B_1, t_2 - \widetilde{t_0}, E[v(\tilde{t}_0)|B_0]) \\
& + \widehat{w}_1 H^p(B_0, B_1 - \gamma, t_2 - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]) \\
& + w_0 H^p(B_0, B_0, t_1 - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]) \\
& + \widehat{w}_0 H^p(B_0, B_0 - \gamma, t_1 - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]), \qquad (15)
\end{aligned}$$

$$-1 = \Delta^p(B_0, X, T - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+w_{n-1}\Delta^p(B_0, B_{n-1}, T-\tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_{n-1}\Delta^p(B_0, B_{n-1}-\gamma, T-\tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+w_{n-2}\Delta^p(B_0, B_{n-2}, t_{n-1}-\tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_{n-2}\Delta^p(B_0, B_{n-2}-\gamma, t_{n-1}-\tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\cdots$$

$$+w_1\Delta^p(B_0, B_1, t_2-\widetilde{t_0}, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_1\Delta^p(B_0, B_1-\gamma, t_2-\tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+w_0\Delta^p(B_0, B_0, t_1-\tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_0\Delta^p(B_0, B_0-\gamma, t_1-\tilde{t}_0, E[v(\tilde{t}_0)|B_0]), \quad (16)$$

$$0 = v^p(B_0, X, T-\tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+w_{n-1}v^p(B_0, B_{n-1}, T-\tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_{n-1}v^p(B_0, B_{n-1}-\gamma, T-\tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+w_{n-2}v^p(B_0, B_{n-2}, t_{n-1}-\tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_{n-2}v^p(B_0, B_{n-2}-\gamma, t_{n-1}-\tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\cdots$$

$$+w_1v^p(B_0, B_1, t_2-\widetilde{t_0}, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_1v^p(B_0, B_1-\gamma, t_2-\tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+w_0v^p(B_0, B_0, t_1-\tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_0v^p(B_0, B_0-\gamma, t_1-\tilde{t}_0, E[v(\tilde{t}_0)|B_0]). \quad (17)$$

Here we continue to use the B_0 notation for convenience, but B_0 is actually the critical stock price on the early exercise boundary at $\tilde{t}_0$. The same logic should be applied to interpret w_0 and $\hat{w}_0$. The above backward iteration process to construct our SHP is illustrated in Table 1.

[Table 1 should be here]

After solving all the unknowns B_i, w_i, $\hat{w}_i$ at n different time points, the value of the SHP at t_0, $P_n^{SHP}(t_0)$ can be expressed as

$$\begin{aligned}
P_n^{SHP}(t_0) = {} & H^p(S(t_0), X, T - t_0, v(t_0)) \\
& + w_{n-1} H^p(S(t_0), B_{n-1}, T - t_0, v(t_0)) \\
& + \hat{w}_{n-1} H^p(S(t_0), B_{n-1} - \gamma, T - t_0, v(t_0)) \\
& + w_{n-2} H^p(S(t_0), B_{n-2}, t_{n-1} - t_0, v(t_0)) \\
& + \hat{w}_{n-2} H^p(S(t_0), B_{n-2} - \gamma, t_{n-1} - t_0, v(t_0)) \\
& + \cdots \\
& + w_1 H^p(S(t_0), B_1, t_2 - t_0, v(t_0)) \\
& + \hat{w}_1 H^p(S(t_0), B_1 - \gamma, t_0 - t_0, v(t_0)) \\
& + w_0 H^p(S(t_0), B_0, t_1 - t_0, v(t_0)) \\
& + \hat{w}_0 H^p(S(t_0), B_0 - \gamma, t_1 - t_0, v(t_0)).
\end{aligned} \tag{18}$$

3.2 Simulated Method for Conditional Expectation of Variance

Regarding the conditional expected variance, it is theoretically possible to determine the probability density function of the conditional variance if one knows the transition probability function $\phi(S(t), v(t) | S(t_0), v(t_0))$. This can be achieved through the following equation:

$$\phi(v(t)|S(t)) = \frac{\phi(S(t),v(t)|S(t_0),v(t_0))}{\phi(S(t)|S(t_0),v(t_0))} = \frac{\phi(S(t),v(t)|S(t_0),v(t_0))}{\int_0^\infty \phi(S(t),v(t)|S(t_0),v(t_0))dv(t)}. \quad (19)$$

Subsequently, the conditional expected variance can be derived as

$$E[v(t)|S(t)] = \int_0^\infty v(t)\phi(v(t)|S(t))dv(t). \quad (20)$$

Dragulescu and Yakovenko (2002) successfully combine the inverse Fourier and Laplace transformations to derive the transition probability function $\phi(S(t), v(t)|S(t_0), v(t_0))$ under the Heston (1993) model. Consequently, by numerically integrating the denominator of Equation (19) and Equation (20), one can obtain $E[v(t)|S(t)]$. However, preliminary tests reveal that this method is not only excessively complex in a methodological sense but also results in lengthy computation times for the backward induction described in the previous section.

One alternative way to estimate the most likely variance on the boundary at time t_i, $E[v(t_i)|B_i]$, is through simulation. We first simulate, for example, 15000 paths of stock price and variance with a time step of 5×10^{-5} based on the Heston model. Then we cluster the stock prices at time t_i according to a set of price intervals, where the stock prices are spaced, for example, by 5% of the initial stock price $S(t_0)$. After that, we calculate the average price and average corresponding variance in each cluster, i.e., $\bar{S}_j$ and $\bar{v}_j$ for $j = 1, 2, ...$, which are then employed to estimate the conditional expected variance specifically at $\bar{S}_j$ as $E\left[v(t_i)|\bar{S}_j\right] = \bar{v}_j$ for each cluster. Finally, we implement piecewise linear interpolation across all clusters (linear extrapolation based on the two outermost clusters at each end) to obtain the most likely variance conditional on any value of B_i.

However, this workaround has a few disadvantages. First, it is time-consuming to simulate so many stock-price and variance paths. Second, as the time draws nearer to t_0, the early exercise boundary of the American put (call) option falls (rises) deeply, but fewer price paths can cross the early exercise boundary. Therefore, the estimated conditional expected variance near the early exercise boundary close to t_0 is unreliable. In Panel A of Figure 1, since the price range for the simulated stock paths is wide enough at $t_{n-1} = 5/12$, we observe a substantial portion on which to apply piecewise interpolation to estimate $E[v(t_{n-1})|S(t_{n-1})]$. In contrast, in Panel B of Figure 1, since the price range for the simulated stock paths is narrow at $t_1 = 1/12$, we must use linear extrapolation to generate less reliable estimations when the stock price is relatively low (the case for solving the early exercise boundary of American puts) and high (the corresponding case for American calls).

[Figure 1 should be here]

3.3 Efficient Methods for Conditional Expectation of Variance

To avoid the drawbacks of the above-mentioned simulation method, in this section we propose two methods by which to efficiently approximate the conditional expectation of variance that can be combined with the proposed SHP method under the SV model. First, we apply the Ito's lemma to Equation (6) to obtain

$$d\ln S(t) = \left(r - q - \frac{v(t)}{2}\right)dt + \sqrt{v(t)}dW(t), \tag{21}$$

after which we consider the discrete-time counterparts of Equations (21), (7), and (8):

$$\Delta \ln S(t) = \left(r - q - \frac{v(t)}{2}\right)\Delta t + \sqrt{v(t)}\Delta W(t), \tag{22}$$

$$\Delta v(t) = \kappa[\theta - v(t)]\Delta t + \sigma_v\sqrt{v(t)}\Delta Z(t), \tag{23}$$

$$\Delta Z(t) = \rho\Delta W(t) + \epsilon\sqrt{(1-\rho^2)\Delta t}. \tag{24}$$

In the first efficient method, we utilize the Euler discretization to approximate the evolution of the stock price and variance processes, with the assumption that there is only one time step between t_0 and any time point t_i. Therefore, the conditional expectation of the variance at t_i can be expressed as

$$E[v(t_i)|S(t_i)] = E[v(t_0) + \Delta v(t_0)|S(t_i)] = v(t_0) + E[\Delta v(t_0)|S(t_i)]. \tag{25}$$

According to Equation (23), we further approximate $E[\Delta v(t_0)|S(t_i)]$ as

$$E[\Delta v(t_0)|S(t_i)] = E\left[\kappa[\theta - v(t_0)](t_i - t_0) + \sigma_v\sqrt{v(t_0)}(Z(t_i) - Z(t_0))|S(t_i)\right]$$

$$= \kappa[\theta - v(t_0)](t_i - t_0) + \sigma_v\sqrt{v(t_0)}E[Z(t_i) - Z(t_0)|S(t_i)]. \quad (26)$$

In addition, we rewrite Equation (24) as

$$Z(t_i) - Z(t_0) = \rho\big(W(t_i) - W(t_0)\big) + \epsilon\sqrt{(1-\rho^2)\Delta t};$$

given $S(t_i)$ is known, we approximate the spot innovation to be

$$W(t_i) - W(t_0) = \frac{\ln S(t_i) - \ln S(t_0) - \left(r - q - \frac{v(t_0)}{2}\right)(t_i - t_0)}{\sqrt{v(t_0)}} \quad (27)$$

according to Equation (22). Combining everything into Equation (26) yields

$$E[\Delta v(t_0)|S(t_i)] = \kappa[\theta - v(t_0)](t_i - t_0)$$

$$+\sigma_v\sqrt{v(t_0)}E\left[\rho\big(W(t_i) - W(t_0)\big) + \epsilon\sqrt{(1-\rho^2)\Delta t}|S(t_i)\right]$$

$$= \kappa[\theta - v(t_0)](t_i - t_0)$$

$$+\sigma_v\sqrt{v(t_0)}\rho\frac{\ln S(t_i) - \ln S(t_0) - \left(r - q - \frac{v(t_0)}{2}\right)(t_i - t_0)}{\sqrt{v(t_0)}} \quad (28)$$

due to the independence of ϵ from $\Delta W(t)$; thus $E[\epsilon|S(t_i)] = 0$.

Hence we have the approximated conditional expectation of the variance at t_i:

$$E[v(t_i)|S(t_i)] = v(t_0) + \kappa[\theta - v(t_0)](t_i - t_0)$$

$$+\rho\sigma_v\left[\ln\frac{S(t_i)}{S(t_0)} - \left(r - q - \frac{v(t_0)}{2}\right)(t_i - t_0)\right]. \quad (29)$$

The conditional expected variance estimation method outlined in Equation (29) is straightforward and convenient. In terms of computational efficiency for calculating $E[v(t_i)|S(t_i)]$, the method described in equation (29) significantly outperforms Monte Carlo simulation combined with piecewise linear interpolation and extrapolation. However, it inherently carries bias due to its neglect of the path-dependent nature of the SV model within the time interval between t_0 and any time point t_i. This error accumulates as t_i grows.

Although the accuracy of the Euler discretization might raise concerns, in this paper we propose a second efficient method to enhance the approximation of the conditional expectation of variance by attempting to consider the time-dependent property of the variance to some extent under the Heston's SV process. To accomplish this, we employ the concept of drift interpolation proposed in van Haastrecht and Pelsser (2010) to approximate $\int_{t_0}^{t_i} v(\tau)d\tau$ as $\frac{v(t_i)+v(t_0)}{2}(t_i - t_0)$. In addition, we further fix $v(\tau)$ as $v(t_0)$ when evaluating $\int_{t_0}^{t_i} \sqrt{v(\tau)}dW(\tau)$ and $\int_{t_0}^{t_i} \sqrt{v(\tau)}dZ(\tau)$. Consequently, integrating Equation (21) over time, we have

$$\ln\frac{S(t_i)}{S(t_0)} = \int_{t_0}^{t_i} \left(r - q - \frac{v(\tau)}{2}\right) d\tau + \int_{t_0}^{t_i} \sqrt{v(\tau)}dW(\tau)$$

$$= (r-q)(t_i - t_0) - \frac{1}{2}\int_{t_0}^{t_i} v(\tau)d\tau + \int_{t_0}^{t_i} \sqrt{v(\tau)}dW(\tau)$$

$$= (r-q)(t_i - t_0) - \frac{1}{2}\left(\frac{v(t_i)+v(t_0)}{2}\right)(t_i - t_0) + \sqrt{v(t_0)}\int_{t_0}^{t_i} dW(\tau)$$

$$= (r-q)(t_i - t_0) - \frac{1}{2}\left(\frac{v(t_i)+v(t_0)}{2}\right)(t_i - t_0) + \sqrt{v(t_0)}\big(W(t_i) - W(t_0)\big). \quad (30)$$

Similarly, integrating Equation (7) over time yields

$$v(t_i) - v(t_0) = \int_{t_0}^{t_i} \kappa[\theta - v(\tau)]d\tau + \int_{t_0}^{t_i} \sigma_v\sqrt{v(\tau)}dZ(\tau)$$

$$= \kappa\theta(t_i - t_0) - \kappa\int_{t_0}^{t_i} v(\tau)d\tau + \sigma_v\int_{t_0}^{t_i} \sqrt{v(\tau)}dZ(\tau)$$

$$= \kappa\theta(t_i - t_0) - \kappa\left(\frac{v(t_i)+v(t_0)}{2}\right)(t_i - t_0) + \sigma_v\sqrt{v(t_0)}\int_{t_0}^{t_i} dZ(\tau)$$

$$= \kappa\theta(t_i - t_0) - \kappa\left(\frac{v(t_i)+v(t_0)}{2}\right)(t_i - t_0) + \sigma_v\sqrt{v(t_0)}\big(Z(t_i) - Z(t_0)\big). \quad (31)$$

Since Equation (30) implies

$$W(t_i) - W(t_0) = \frac{\ln\frac{S(t_i)}{S(t_0)} - \left[r - q - \frac{1}{2}\left(\frac{v(t_i)+v(t_0)}{2}\right)\right](t_i - t_0)}{\sqrt{v(t_0)}}, \tag{32}$$

we rewrite Equation (31) as

$$v(t_i) - v(t_0) = \kappa\theta(t_i - t_0) - \kappa\left(\frac{v(t_i)+v(t_0)}{2}\right)(t_i - t_0)$$

$$+\sigma_v\sqrt{v(t_0)}\left(\rho\left(W(t_i) - W(t_0)\right) + \epsilon\sqrt{(1-\rho^2)(t_i - t_0)}\right)$$

$$= \kappa\theta(t_i - t_0) - \kappa\left(\frac{v(t_i)+v(t_0)}{2}\right)(t_i - t_0)$$

$$+\sigma_v\sqrt{v(t_0)}\left\{\rho\frac{\ln\frac{S(t_i)}{S(t_0)} - \left[r - q - \frac{1}{2}\left(\frac{v(t_i)+v(t_0)}{2}\right)\right](t_i - t_0)}{\sqrt{v(t_0)}} + \epsilon\sqrt{(1-\rho^2)(t_i - t_0)}\right\}. \tag{33}$$

Therefore,

$$v(t_i)\left[1 + (\frac{\kappa}{2} - \frac{\rho\sigma_v}{4})(t_i - t_0)\right] = v(t_0) + \kappa\theta(t_i - t_0) - \kappa\frac{v(t_0)}{2}(t_i - t_0)$$

$$+\rho\sigma_v\left[\ln\frac{S(t_i)}{S(t_0)} - \left(r - q - \frac{v(t_0)}{4}\right)(t_i - t_0)\right]$$

$$+\epsilon\sigma_v\sqrt{v(t_0)}\sqrt{(1-\rho^2)(t_i - t_0)}. \tag{34}$$

Finally, by taking the expectation conditional on $S(t_i)$ on both sides of the above equation, we obtain the second efficient approximations for the conditional expectation of the variance:

$$E[v(t_i)|S(t_i)] = \frac{v(0) + \left(\kappa\theta - \kappa\frac{v(t_0)}{2}\right)(t_i - t_0) + \rho\sigma_v\left[\ln\frac{S(t_i)}{S(t_0)} - \left(r - q - \frac{v(t_0)}{4}\right)(t_i - t_0)\right]}{1 + (\frac{\kappa}{2} - \frac{\rho\sigma_v}{4})(t_i - t_0)}. \tag{35}$$

4. Numerical Results

This section analyzes the pricing and hedging performance of the three SHP methods for calculating the conditional expected variance. Moreover, we focus on the degree of performance improvement caused by introducing the vega-matching condition. Finally, as a robustness test, we also investigate the impact of constructing the SHP using only European options with standard strike prices rather than arbitrary strike prices. Although this paper majorly focuses on the hedging performance of the proposed SHP methods for American options under SV, we think that it is still necessary to examine the pricing performance of the proposed SHP methods. Accurate pricing results imply that the proposed SHP methods match the boundary conditions of American options satisfactorily, which can be regarded as a premise of the excellent hedging performance of the proposed SHP models. In contrast, if the proposed SHP methods generate inaccurate pricing results, how could it be convinced for the proposed SHP methods to deliver excellent hedging performance for American options?

4.1 Pricing Performance Analysis

We first compare the pricing performance of the SHP method with the Euler-discretized conditional expectation variance (Method 1^*), drift-interpolated conditional expected variance (Method 2^*), and simulated conditional expected variance (Method 3^*). The counterpart no-vega-matching models—Method 1, Method 2 and Method 3—are also included for comparison to determine whether the vega-match condition is worth considering. When there is no vega-matching condition, it is not necessary to add $\hat{w}_i$ units of European puts with the strike price to $B_i - \gamma$ into the SHP; thus our SHP method degenerates to Chung and Shih's (2009) SHP method except that the constant variance used in Chung and Shih's (2009) SHP method is replaced with the conditional expected variance.

Two sets of American puts are examined, where the size of the first set is small, appropriate for detailed analyses, and the other set of American put contracts is large and randomly generated, used to measure the average performance of our SHP method when it is used in the real world. Finally, the FDM is employed to calculate the benchmark, where the grid sizes in time, log stock price, and variance are 5×10^{-5}, $\sqrt{1.5 \times 5 \times 10^{-5}}$, and 0.005, respectively.

The parameter values of American puts in Set 1 are $S(t_0) = 100$, $r = 0.05$, $T = 0.5$, $\sigma_v = 0.3$, $\kappa = 1$, $\theta = 0.09$, $\rho = -0.7$,[3] $X = \{90,100,110\}$, $q \in \{0.02,0.05,0.08\}$, and $v(t_0) \in \{0.04,0.09,0.16\}$. The detailed pricing results of the methods are presented in Table 2. When implementing the proposed SHP methods, $\gamma = 2.5$ (2.5%$\times$ $S(t_0)$) and $n = 6$. The pricing results for the 27 contracts in Set 1 are presented in Table 2. Appendix A further takes the first contract in Table 2 as an example to show the detailed compositions of the SHPs in Methods 1^*, 2^*, and 3^*. When $n = 6$, Methods 1^*, 2^*, and 3^* (Methods 1, 2, and 3) employ 13 (=$1 + 2n$) (7 (=$1 + n$)) European puts to construct SHPs as one needs consider only the conditional expected variance on the early exercise boundary. In contrast, suppose that when attempting to apply Fink's (2003) method with four representative variances given the early exercise boundary of the target American put able to be known in advance, 25 (=$1 + 4n$) European puts may be needed to construct the SHP for implementing the value-matching condition on the early exercise boundary. If also implementing the smooth-pasting and vega-matching conditions, 73 (=$1 + 12n$) European puts may be needed to construct the SHP. In practice, it is unreasonable to hedge a single target option with 73 more fundamental options; it is also infeasible since there may not exist so many out-of-the-money European puts with different strike prices and times to maturity in the option market.

Table 3 shows the root mean square error (RMSE) for the pricing results in Table 2. Panel A of Table 3 shows that simulation (Methods 3^* and 3) performs the best, followed by drift interpolation (Methods 2^* and 2), followed by the Euler discretization (Methods 1^* and 1). In addition, implementation of the vega-matching condition yields a substantial reduction in the RMSE. The reductions are quantified by 66.8%, 77.4%, and 73.3% respectively, for Methods 1^*, 2^*, and 3^*, highlighting the importance of considering the vega-matching condition. Although Methods 1^* and

[3] We argue that $\rho = -0.7$ is not an extreme assumption. According to Hung, Ko, and Wang (2023), ρ ranged in [-0.84, -0.74] for the S&P 500 index from 1996 to 2017.

2^*, which approximate the conditional expected variance efficiently, do not perform as well as Method 3^*, the pricing performance of Methods 1^* and 2^* is still impressive. The RMSEs of Methods 1^* and 2^* are 0.0115 and 0.0061, both of which are extremely small, representing only 0.13% and 0.07% of the average option value of Set 1 (8.8041), respectively. In addition, Panel B of Table 3 presents the RMSEs under $r \geq q$ and $r < q$. Note that in instances where $r \geq q$, we still observe a substantial reduction in RMSE for Methods 1^*, 2^*, and 3^* (vs. Methods 1, 2, and 3), quantified by 66.7%, 77.6%, and 74.0% respectively; however, when $r < q$, there is no substantial difference in RMSE regardless of whether the vega-matching condition is used or not. It is well-known that the early exercise boundary for $r \geq q$ is higher than that for $r < q$, all else being equal. The results in Panel B of Table 3 suggest that when $r \geq q$, where the current stock price is closer to the early exercise boundary and thus it is likely to early exercise an American put, the proposed SHP methods perform satisfactorily, and the vega-matching condition is more effective in this scenario. In fact, when $r < q$, where the current stock price is further away from the early exercise boundary, the probability of early exercise is smaller, and the American put is more likely to resemble its European counterpart with the same strike price and time to maturity. As a result, it is of little use to consider the vega-matching condition on the early exercise boundary or even the proposed SHP methods: the issuer (hedger) can achieve still satisfactory but cheaper hedging by using only the counterpart European put, since for the contracts with $r < q$ in Set 1, the RMSE between the target American puts and the counterpart European puts is merely 0.0029. In Panel C of Table 3, we conduct a subsample analysis in terms of $v(t_0)$. Although Method 3^* generally performs the best, Method 2^* outperforms Method 3^* by a very small difference when $v(t_0)$ is 0.09 and 0.16; perhaps due to the uncertainty associated with the simulation approach, the RMSE of Method 3^* (with vega matching) is higher than that of Method 3 (without vega matching) when $v(t_0) = 0.16$.

[Table 2 should be here]

[Table 3 should be here]

The basic idea of the SHP method implies that when the hedging time points n increases, the portfolio value should converge to the theoretical value of the target option. However, we modify the SHP by introducing the approximated conditional expected variance under the Heston's SV model. Therefore, we are interested in whether convergence occurs when n increases. Here we analyze the RMSPE (root mean square percentage error) among 27 contracts in Set 1 given different values of n to examine the convergence property of Methods 1^* vs. 1 and Methods 2^* vs. 2. For each panel in Figure 2, given a different n, in addition to reporting the RMSPE among 27 contracts in Set 1 and marking it with a solid circle, the top of the vertical line, the top of the vertical bar, the bottom of the vertical bar, and the bottom of the vertical line represent the maximum, 75% quantile, 25% quantile, and minimum absolute percentage error among the 27 contracts in Set 1, respectively. Regardless of whether the vega-matching condition is considered or not, the proposed SHP methods converge quickly when $n = 4$. However, non-vega-matching SHP methods are less reliable as their SHP option values begin to diverge from the FDM benchmark after $n = 4$. In addition, comparing the vertical lines and bars for Methods 1^* vs. 1 (Methods 2^* vs. 2), we find that the variation ranges of the absolute percentage errors among 27 contracts in Set 1 based on the SHP methods without vega matching (Panels B and D) are significantly higher and tend to increase with n. The results in Figure 2 demonstrate that if the vega-matching condition is imposed, even though the proposed SHP methods only examine one variance value—the conditional expected variance—the convergence pattern with respect to n is largely unchanged.

[Figure 2 should be here]

Note that in practice, strike prices of tradable European option contracts are not continuous. Therefore, European options with strike prices B_i in the SHP may not exist in the market because B_i are not standard strike prices. We seek to test the proposed model by considering the more realistic constraint of trading only European options with standard strike prices. Therefore, in contrast to the above analyses, we assume that the existing strike prices are multiples of, say, 5% of the initial stock price and then examine the pricing and hedging performance of the demonstrated option contracts in Set 1. These experiments better reflect how well the SHP replicates real-world American puts.

To determine a standard-strike critical boundary ($\ddot{B}_i$), we consider only strike prices that are divisible by 5 (=5%× S_0) and the set of $\Theta = \{40,45,\ldots,95,100\}$ to represent standard strike prices. We iteratively search from high to low for the appropriate critical boundary at time t_i. During this optimization process, we further impose two additional constraints. At time t_{n-1}, the optimization problem is formulated as

$$\min_{\ddot{B}_{n-1}, w_{n-1}, \widehat{w}_{n-1}} (\text{error of value-matching condition at } t_{n-1})^2$$

$$+(\text{error of smooth-matching condition at } t_{n-1})^2$$

$$+(\text{error of vega-matching condition at } t_{n-1})^2$$

$$\text{s.t.} \quad \ddot{B}_{n-1} \in \Theta, \ddot{B}_{n-1} \leq \min(X, \frac{r}{q}X), \tag{36}$$

where $\min(X, \frac{r}{q}X)$ is the theoretical early exercise boundary of an American put when the time is the maturity date T, and the errors of value-matching, smooth-pasting, and vega-matching conditions correspond to the mismatches between the left-hand- and right-hand-side of Equations (12), (13), and (14), respectively. For time t_i other than t_{n-1}, the optimization problem is formulated as

$$\min_{\ddot{B}_i, w_i, \widehat{w}_i} (\text{error of value-matching condition at } t_i)^2$$

$$+(\text{error of smooth-matching condition at } t_i)^2$$

$$+(\text{error of vega-matching condition at } t_i)^2$$

$$\text{s.t.} \quad \ddot{B}_i \in \Theta, \ \ddot{B}_i \leq \ddot{B}_{i+1}. \tag{37}$$

Moreover, when analyzing the constraints of using only standard strike prices, the parameter γ is set to the minimal interval between the standard strike prices, i.e., $\gamma = 5$ in our experiments. The pricing results and corresponding analyses are presented in Tables 4 and 5, respectively. By comparing Table 5 with Table 3, it is inevitable that the RMSEs increase since the standard strike prices for the European options in the SHP are not the exact solution when solving the value-matching, smooth-pasting, and vega-matching conditions. Additionally, when using only standard strike prices, the methods with vega-matching remain superior: the vega-matching condition reduces RMSEs by 38.4%, 77.1%, and 21.5%, respectively for Methods 1^*, 2^*, and 3^*, and this phenomenon is consistent in the scenarios where $r \geq q$. Moreover, the performance of Method 3^* (the simulated method to approximate the conditional expected variance) seems to degrade more than Methods 1^* and 2^*. In general, Method 2^* performs the best, but Method 1^* (slightly) outperforms Method 2^* when $v(t_0)$ is 0.09 (0.16).

[Table 4 should be here]

[Table 5 should be here]

Since the superior performance for the small group of contracts in Set 1 may not reflect the true ability of the proposed SHP methods, we randomly generate 600 parameter combinations to test their pricing and hedging performance to demonstrate the robustness of the proposed SHP methods. To differentiate the 27 option contracts examined in Table 2, these 600 option contracts are referred to as Set 2.[4] We intend to examine the proposed SHP methods under a sufficient number of possibilities and measure the average pricing and hedging performance of the proposed SHP method to reflect realistic conditions for issuers who continuously adopt the proposed SHP methods to price and hedge American puts. We argue that it may not be proper to conduct an empirical study by examining several American put contracts traded in option markets. First, we focus on the hedging performance of the proposed SHP

[4] We exclude one contract from Set 2 because it is early exercised at the current time point t_0, the details of which are $X = 110$, $T = 1/12$, $r = 0.05$, $q = 0.014426$, $v(t_0) = 0.036352$, $\kappa = 2.12351$, $\theta = 0.13$, $\sigma_v = 0.341381$, and $\rho = -0.61616$. This is because the proposed SHP method is only applicable to American options that currently have yet to be early exercised.

methods, since many models have been proven to price American puts accurately. However, it is difficult to test the average or the distribution of hedging errors in practice, since there is only one time series for the underlying stock price during the option life. Second, it is always possible to select a small group of contracts that suit the proposed SHP methods to exhibit satisfactorily theoretical performance, but which may not translate to the real-world performance of the proposed SHP methods.

The examined parameter values in Set 2 are summarized in Table 6. For simplicity and also for the purposes of further analyses, we fix $S(t_0) = 100$, $r = 0.05$, and $\theta = 0.13$ and examine 30 combinations of (X, T), where $X \in \{90,95,100,105,110\}$ and $T \in \left\{\frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}\right\}$. We randomly draw 20 sets of $(q, v(t_0), \kappa, \sigma_v, \rho)$ from each parameter's individual uniform distribution within a reasonable range. Each combination of (X, T) is combined with the simulated 20 sets of $(q, v(t_0), \kappa, \sigma_v, \rho)$ to form 20 examined contracts. Table 7 (Table 8) shows the pricing errors of different methods (using European puts with standard strike prices) for American puts in Set 2.

[Table 6 should be here]

[Table 7 should be here]

[Table 8 should be here]

In Panel A of Table 7, Method 2^* is the best-performing method, the RMSE of which is 0.0072, representing only 0.09% of the average option value of Set 2 (7.7062). The SHP methods with vega matching continue to be more accurate for pricing American put options. Compared with Table 3, Method 3^* performs worse for Set 2. We attribute this result to the theoretical drawback of the simulation approach in estimating the conditional expected variance at time points near today. By grouping and analyzing American puts with different maturities, we find that Method 3^* exhibits poor pricing capability for the shortest maturity; i.e., when maturity = $\frac{1}{12}$ in Set 2, it yields an RMSE of 0.0143, significantly larger than the RMSE of 0.0069

(0.0055) based on Method 1^* (Method 2^*). Similarly, in Panel B of Table 7, when $r \geq q$, the pricing errors of the SHP methods with the vega-matching condition (Methods 1^*, 2^*, and 3^*) are substantially lower than the SHP methods without the vega-matching conditions (Methods 1, 2, and 3), but the effect from the vega-matching condition is minor when $r < q$. Last, Panel C of Table 7 attests the superior performance of Method 2^* given different levels of the initial variance, although for the medium levels of the initial variance, the pricing accuracy of Methods 2^* and 3^* is nearly equal.

In Table 8, due to the constraint of using only European puts with standard strike prices to construct the SHPs, the pricing errors are clearly higher than their counterparts in Table 7. However, the pricing errors are still small, compared to the average option value. Taking Method 2^*, the best performer, for example, its RMSE is 0.0820, representing only 1.1% of the average option value of Set 2 (7.7062). The relative patterns among different methods and subsample analyses in Table 8 are similar to those in Table 7.

4.2 Hedging Performance Analysis

In this section, we discuss the core contribution of this paper: the hedging performance of the SHP methods from the viewpoint of American put issuers. The experiment design follows Chung, Huang, Shih, and Wang (2019). For each contract, we simulate 1000 stock price-variance paths based on Equations (22)-(24) with $\Delta t = 5 \times 10^{-5}$ and then calculate the cumulative hedging error of each path either when the American put is early exercised or matured at T. Suppose that as long as a stock price-variance path hits the early exercise surface generated from the benchmark FDM, the American put is exercised by its holder.

For each simulation path, when the issuer sells an American put option at $P(t_0)$ (generated by the benchmarked FDM method), he simultaneously uses the sales proceeds to construct the static hedging portfolio $P_n^{SHP}(t_0)$ in Equation (18), after which he deposits (borrows) the margin (lost) into (from) the bank account to earn (pay) the interest rate r, i.e.,

$$\eta_0 = P(t_0) - P_n^{SHP}(t_0), \tag{38}$$

where η_0 represents the initial balance of the bank account. With the passage of time, suppose that the American put does not exercise before $t \leq t_1$, the balance of the bank account grows at the risk-free interest rate:

$$\eta_t = \eta_0 e^{r(t-t_0)}. \tag{39}$$

The issuer then earns the payoffs from the European puts expired at t_1 in the SHP into the bank account, i.e.,

$$\eta_1 = \eta_0 e^{r(t_1-t_0)} + w_0(B_0 - S(t_1))^+ + \widehat{w}_0(B_0 - \gamma - S(t_1))^+. \tag{40}$$

Based on the balance of η_1 and remaining SHP, the process is then continually updated in this manner with the passage of time until the American put option is exercised or matured. If the American put does not exercise before $t \leq t_2$, the balance of the bank account grows to $\eta_t = \eta_1 e^{r(t-t_1)}$, the issuer earns the payoffs from the European puts expired at t_2 in the SHP into the bank account as $\eta_2 = \eta_1 e^{r(t_2-t_1)} + w_1(B_1 - S(t_2))^+ + \widehat{w}_1(B_1 - \gamma - S(t_2))^+$, and then the process is continued based on the bank account balance of η_2 and remaining SHP. In contrast, if the American put is terminated at the time point τ, the issuer liquidates the remaining SHP for $P_n^{SHP}(\tau)$ and realizes the hedging error defined as

$$HE = e^{-r(\tau-t_0)}[P_n^{SHP}(\tau) + \eta_\tau - (X - S(\tau))^+], \tag{41}$$

where $(X - S(\tau))^+$ is the amount paid out by the issuer.

If $P_n^{SHP}(\tau) + \eta_\tau$ is larger than $(X - S(\tau))^+$, the issuer has positive hedging error (HE); otherwise, the issuer has negative HE. To evaluate the extreme loss of the hedging risk, we adopt four measures suggested by Siven and Poulsen (2009). The first risk measure is the 5% Value at Risk, defined as $VaR_{0.05} = -inf\{z \in R; Pr(HE \leq z) \leq 0.05\}$. The second risk is the expected shortfall, defined as $ES_{0.05} = -E[HE|HE \leq VaR_{0.05}]$. The third risk measure is the expected squared hedging error, defined as $ESHE = E[HE^2]$. The last risk measure is the expected loss, defined as $EL = -E[HE|HE \leq 0]$. For all four risk measures, smaller results indicate better hedging performance. The risk measurements based on the DDH method with daily rebalance are also reported for comparison. Appendix B explains the detailed process of the DDH method. While the DDH method is probably the most commonly used method to hedge derivatives, we note that in contrast to our SHP method, which considers both the smooth-pasting (delta-matching) and vega-matching conditions along the early exercise boundary, the DDH method matches only delta values (but not also vega values) at rebalance time points during the hedge. We think it would be valuable for future work to compare our SHP method with, for example, a dynamic delta-vega-neutral hedge method, which, although more complex, ensures both delta- and vega-neutrality at rebalance time points during the hedge.

We first analyze the hedging performance of the proposed SHP methods using European puts with either non-standard or standard strike prices for the 27 contracts in Set 1. The corresponding results are reported in Panels A and B, respectively, in Table 9. Panel A in Table 9 shows that the hedging ability of the proposed SHP methods is in general excellent, particularly for Methods 2^* and 3^*. Compared to the average option value in Set 1 (8.8041), the $VaR_{0.05}$ of Methods 1^*, 2^*, and 3^* are only 0.0834, 0.0752, and 0.0731, which represent 0.95%, 0.85%, and 0.83% of the average option value, respectively. The hedging performance is strong, which means that the probability for the maximum hedging loss for an American put higher than 0.83%–0.95% of its initial sales proceeds is less than 5%. Moreover, incorporating the vega-matching condition into the proposed SHP methods further reduces the hedging risk, compared to the counterpart non-vega-matching SHP methods. In contrast, for the $VaR_{0.05}$ of the DDH method, the probability for the maximum hedging loss for an American put higher than 27.89% (=2.4549/8.8041) of the initial sales proceeds is less than 5%. Perhaps one reason for the inferior hedging performance of the DDH method is that it does not manage the vega risk. However, note that for Methods 1, 2, and 3, which merely consider the value-matching and smooth-pasting conditions (without considering the vega-matching condition) along the early exercise boundary, their hedging performance is still significantly superior to that of the DDH method. Even excluding the benefit of introducing the vega-matching condition, our SHP methods, based on the novel idea of conditional expected variance, still exhibit more satisfactory hedging performance for American options than the widely used DDH method.

As expected, due to the non-optimality of considering only standard strike prices as potential critical stock prices that solve the value-matching, smooth-pasting, or vega-matching condition, the hedging performance is weakened in Panel B in Table 9. However, the hedging performance of the proposed SHP methods is still satisfactory. For example, the $VaR_{0.05}$ of Methods 1^*, 2^*, and 3^* are 0.1252, 0.1698, and 0.1511, which represent 1.42%, 1.93%, and 1.72% of the average option value, respectively. Furthermore, even the hedging results of the proposed SHP methods in Panel B are much smaller than those associated with the DDH method, demonstrating again the superior hedging performance of the proposed SHP methods in the real world.

[Table 9 should be here]

Last, we examine the hedging performance of the proposed SHP methods for Set 2 in Tables 10 and 11. Panel A of Table 10 shows that the proposed SHP methods perform excellently in hedging American puts. For instance, compared to the average option value in Set 2 (7.7062), the $VaR_{0.05}$ of Methods 1^*, 2^*, and 3^* are only 0.0677, 0.0607, and 0.0601, which represent 0.88%, 0.79%, and 0.78% of the average option value, respectively. Methods 2^* and 3^* perform almost equally well in terms of the four risk measurements. In addition, the SHP methods with vega matching are more capable than those counterparts without vega matching in hedging American puts. For example, the $VaR_{0.05}$ decreases by 21.76%, 25.25%, and 23.83% and $ES_{0.05}$ decreases by 6.37%, 9.44%, and 7.90% due to the inclusion of the vega-matching condition in Methods 1^*, 2^*, and 3^* (vs. Methods 1, 2, and 3). Even without the vega-matching condition, the hedging performance of Methods 1, 2, and 3 still dominate that of the DDH method. When using merely European puts with standard strike prices to construct SHPs, Method 2^* exhibits the best hedging performance in Panel B of Table 10; for example, its $VaR_{0.05}$ represents 2.08% (=0.1604/7.7062) of the average option value. Furthermore, for all the proposed SHP methods and whether using European puts with nonstandard or standard strike prices, their hedging losses are much smaller than those of the DDH method in terms of the four risk measurements, demonstrating the merit of the proposed SHP methods in hedging American puts.

Table 11 reports the hedging performance analysis for all early exercised paths of the 600 contracts in Set 2. We conduct this analysis because the issuers or hedgers who employ our SHP methods are most concerned about hedging risk when the target American puts are early exercised. For early exercise paths, the performance of Methods 2^* and 3^* is excellent and the differences between them are minor in Panel A of Table 11. For example, the $VaR_{0.05}$ of Method 2^* (3^*) is 0.0419 (0.0412), which represents only 0.54% (0.53%) of the average option value of Set 2. In contrast, the $VaR_{0.05}$ of the DDH methods is 1.6167, roughly equal to 20.98% of the average option value of Set 2. In addition, the four risk measurements of Methods 1^*, 2^*, and 3^* are smaller than those of Methods 1, 2, and 3 due to the inclusion of vega matching. For example, the $VaR_{0.05}$ decreases by 28.15%, 32.85%, and 31.56% and $ES_{0.05}$ decreases by 22.32%, 17.71%, and 51.09% for Methods 1^*, 2^*, and 3^* (vs. Methods 1, 2, and 3). When using merely European puts with standard strike prices in the proposed SHP methods, Method 2^* dominates other methods in hedging American puts. Even the four risk measurements of any proposed SHP method in Panel B remain smaller than those of the DDH method. Thus the results in Table 11 demonstrate the superior performance of Methods 1^*, 2^*, and 3^* when hedging American puts that are early exercised.

[Table 10 should be here]

[Table 11 should be here]

5. Conclusion

In this paper, we propose a feasible and excellent performing method to construct an SHP under Heston's SV model to hedge American puts. First, to extend Chung and Shih's (2009) method into the SV framework, we replace the constant variance with the conditional expected variance in their method. Furthermore, we examine three methods to approximate the conditional expected variance: the Euler discretization, drift interpolation, and simulation methods. In addition, since the purpose of the smooth-pasting condition is to align the sensitivity with respect to the stock price change between the SHP and the target option on the early exercise boundary, this paper is the first to incorporate the vega-matching condition in the SHP method, which aligns the sensitivity with respect to the variance change between the SHP and the target option on the early exercise boundary. To implement the vega-matching condition, we borrow Fink's (2003) idea to involve a further out-of-the-money European put at each examined time point.

Among 600 randomly generated American put contracts, our numerical results indicate that the proposed SHP methods show impressive pricing accuracy. Average pricing errors of the proposed SHP methods range from 0.11% to 0.14% of the average option value of the 600 American puts. Even when trading only European puts with standard strike prices when implementing the proposed SHP methods, pricing errors range from 1.06% to 1.16% of the average option value. Introducing the vega-matching condition reduces the pricing error by 68.09%, 64.53%, and 36.50% (23.89%, 25.79%, and 19.29%) for the Euler discretization, drift interpolation, and simulation method, respectively (when only European puts with standard strike prices are allowed to trade). The hedging performance of the proposed SHP methods is also strong. Take $VaR_{0.05}$ for example. Without (with) the constraint of trading only European puts with standard strike prices, the values of $VaR_{0.05}$ range from 0.78% to 0.88% (2.08% to 2.14%) of the average option value. In contrast, the $VaR_{0.05}$ of the traditional dynamic delta-neutral hedging method is 23.58% of the average option value. The advantage of introducing the vega-matching condition is also pronounced in reducing hedging risks. The hedging risk is significantly smaller for the vega-matching SHP methods, resulting in the reduction of $VaR_{0.05}$ by 21.73%, 25.25%, and 23.83% (34.72%, 36.32%, and 35.93%) for the Euler discretization, drift interpolation, and simulation method, respectively (when only European puts with standard strike prices are allowed to trade).

In summary, when hedging American puts under Heston's SV model, introducing the concept of conditional expected variance into the SHP method not only makes this method feasible when one can trade only a limited number of out-of-money options with different strike prices and times to maturity but also yields excellent hedging and pricing performance for American puts. For issuers or hedgers who focus on short-term (less than half a year) American puts and care about computational efficiency, we recommend using the drift interpolation method to approximate the conditional expected variance (Method 2^*). For issuers or hedgers who focus on long-term (longer than half a year) American puts, using simulation method to approximate the conditional expected variance (Method 3^*) may be more reliable but at the cost of more computation time.

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Appendix A: Examples of Detailed Compositions of Our SHPs.

To facilitate understanding the compositions of the SHPs based on Methods 1^*, 2^*, and 3^*, we take the first American put contract in Table 2 for example to show our computation details. The weights, maturities, and strike prices for the component European puts for Methods 1^*, 2^*, and 3^* are shown in Tables A1, A2, and A3, respectively. In addition, at each examined time point t_i, we also compare the conditional expected variance $E[v(t_i)|B_i]$ with the corresponding variance for $S(t_i) = B_i$ on the early exercise boundary in the FDM. Although there is no theory stating the relationship between these two quantities, we believe that the differences between them should not be too volatile across all examined time points.

Comparing Table A3 with Tables A1 and A2, for the examined time points distant from today, e.g., $t_2, \dots, t_5$, Method 3^* is theoretically more accurate than Methods 1^* and 2^* because it has little discretization error in the time dimension. However, for the examined time points near today, e.g., $\tilde{t}_0$ and t_1, since the price range for the simulated stock paths is not wide enough, Method 3^* seems to suffer significant interpolation errors when calculating the conditional expected variances. Especially at $\tilde{t}_0 = 0.0001$, even though B_0 is as low as 45.144, the conditional expected variance $E[v(\tilde{t}_0)|B_0]$ does not rise according to the negative relationship between the stock price and its variance, but is equal to the initial variance $v(t_0) = 0.04$, because almost all the simulated stock prices concentrate around $v(t_0) = 0.04$ at $\tilde{t}_0$. In contrast, Methods 1^* and 2^* perform more reasonably for the examined time points near today. When B_0 is around 60.2, the corresponding $E[v(\tilde{t}_0)|B_0]$ equals 0.147, which is, as expected, significantly higher than the initial variance $v(t_0) = 0.04$. Moreover, for the examined time points distant from today, the results of Method 2^* are closer to those of Method 3^*, indicating that Method 2^* is less affected by the discretization error in the time dimension than Method 1^*.

Contract 1 in Table 2 (Method 1^*)			
Weight	Option contract		Conditional expected variance vs. corresponding variance for $S(t_i) = B_i$ on early exercise boundary in FDM
	Maturity	Strike price	
1	$t_6 = 0.5$	$X = 90$	
$w_5 = 0.437$	$t_6 = 0.5$	$B_5 = 72.048$	$E[v(t_5)\|B_5] = 0.131$.
$\widehat{w}_5 = -0.568$	$t_6 = 0.5$	$B_5 - \gamma = 69.548$	On early exercise boundary in FDM at t_5, if $S(t_5) = 72.048$, corresponding $v(t_5)$ is 0.151.
$w_4 = 0.443$	$t_5 = 0.5 \times \frac{5}{6}$	$B_4 = 67.488$	$E[v(t_4)\|B_4] = 0.140$.
$\widehat{w}_4 = -0.466$	$t_5 = 0.5 \times \frac{5}{6}$	$B_4 - \gamma = 64.988$	On early exercise boundary in FDM at t_4, if $S(t_4) = 67.488$, corresponding $v(t_4)$ is 0.170.
$w_3 = 0.417$	$t_4 = 0.5 \times \frac{4}{6}$	$B_3 = 64.508$	$E[v(t_3)\|B_3] = 0.145$.
$\widehat{w}_3 = -0.454$	$t_4 = 0.5 \times \frac{4}{6}$	$B_3 - \gamma = 62.008$	On early exercise boundary in FDM at t_3, if $S(t_3) = 64.508$, corresponding $v(t_3)$ is 0.172.
$w_2 = 0.430$	$t_3 = 0.5 \times \frac{3}{6}$	$B_2 = 62.491$	$E[v(t_2)\|B_2] = 0.147$.
$\widehat{w}_2 = -0.481$	$t_3 = 0.5 \times \frac{3}{6}$	$B_2 - \gamma = 59.991$	On early exercise boundary in FDM at t_2, if $S(t_2) = 62.491$, corresponding $v(t_2)$ is 0.179.
$w_1 = 0.453$	$t_2 = 0.5 \times \frac{2}{6}$	$B_1 = 61.123$	$E[v(t_1)\|B_1] = 0.148$.
$\widehat{w}_1 = -0.513$	$t_2 = 0.5 \times \frac{2}{6}$	$B_1 - \gamma = 58.623$	On early exercise boundary in FDM at t_1, if $S(t_1) = 61.123$, corresponding $v(t_1)$ is 0.171.
$w_0 = 0.470$	$t_1 = 0.5 \times \frac{1}{6}$	$B_0 = 60.192$	$E[v(\tilde{t}_0)\|B_0] = 0.147$.
$\widehat{w}_0 = -0.536$	$t_1 = 0.5 \times \frac{1}{6}$	$B_0 - \gamma = 57.692$	On early exercise boundary in FDM at $\tilde{t}_0$, if $S(\tilde{t}_0) = 60.192$, corresponding $v(\tilde{t}_0)$ is 0.172.

Table A1. Detailed computation results for Contract 1 in Table 2 based on Method 1^*.

Contract 1 in Set 1 (Method 2^*)			
Weight	Option contract		Conditional expected variance vs. variance corresponding to B_i on early exercise boundary in FDM
	Maturity	Strike price	
1	$t_6 = 0.5$	$X = 90$	
$w_5 = 0.441$	$t_6 = 0.5$	$B_5 = 73.619$	$E[v(t_5)\|B_5] = 0.110$.
$\widehat{w}_5 = -0.584$	$t_6 = 0.5$	$B_5 - \gamma = 71.119$	On early exercise boundary in FDM at t_5, if $S(t_5) = 73.619$, corresponding $v(t_5)$ is 0.133.
$w_4 = 0.452$	$t_5 = 0.5 \times \frac{5}{6}$	$B_4 = 69.261$	$E[v(t_4)\|B_4] = 0.120$.
$\widehat{w}_4 = -0.474$	$t_5 = 0.5 \times \frac{5}{6}$	$B_4 - \gamma = 66.761$	On early exercise boundary in FDM at t_4, if $S(t_4) = 69.261$, corresponding $v(t_4)$ is 0.143.
$w_3 = 0.418$	$t_4 = 0.5 \times \frac{4}{6}$	$B_3 = 66.135$	$E[v(t_3)\|B_3] = 0.128$.
$\widehat{w}_3 = -0.447$	$t_4 = 0.5 \times \frac{4}{6}$	$B_3 - \gamma = 63.635$	On early exercise boundary in FDM at t_3, if $S(t_3) = 66.135$, corresponding $v(t_3)$ is 0.146.
$w_2 = 0.425$	$t_3 = 0.5 \times \frac{3}{6}$	$B_2 = 63.745$	$E[v(t_2)\|B_2] = 0.135$.
$\widehat{w}_2 = -0.463$	$t_3 = 0.5 \times \frac{3}{6}$	$B_2 - \gamma = 61.245$	On early exercise boundary in FDM at t_2, if $S(t_2) = 63.745$, corresponding $v(t_2)$ is 0.155.
$w_1 = 0.432$	$t_2 = 0.5 \times \frac{2}{6}$	$B_1 = 61.826$	$E[v(t_1)\|B_1] = 0.141$.
$\widehat{w}_1 = -0.471$	$t_2 = 0.5 \times \frac{2}{6}$	$B_1 - \gamma = 59.326$	On early exercise boundary in FDM at t_1, if $S(t_1) = 61.826$, corresponding $v(t_1)$ is 0.167.
$w_0 = 0.435$	$t_1 = 0.5 \times \frac{1}{6}$	$B_0 = 60.220$	$E[v(\tilde{t}_0)\|B_0] = 0.147$.
$\widehat{w}_0 = -0.471$	$t_1 = 0.5 \times \frac{1}{6}$	$B_0 - \gamma = 57.720$	On early exercise boundary in FDM at $\tilde{t}_0$, if $S(\tilde{t}_0) = 60.220$, corresponding $v(\tilde{t}_0)$ is 0.172.

Table A2. Detailed computation results for Contract 1 in Table 2 based on Method 2^*.

Contract 1 in Set 1 (Method 3^*)			
Weight	Option contract		Conditional expected variance vs. variance corresponding to B_i on early exercise boundary in FDM
	Maturity	Strike price	
1	$t_6 = 0.5$	$X = 90$	
$w_5 = 0.441$	$t_6 = 0.5$	$B_5 = 73.773$	$E[v(t_5)\|B_5] = 0.108$.
$\hat{w}_5 = -0.585$	$t_6 = 0.5$	$B_5 - \gamma = 71.273$	On early exercise boundary in FDM at t_5, if $S(t_5) = 73.773$, corresponding $v(t_5)$ is 0.134.
$w_4 = 0.485$	$t_5 = 0.5 \times \frac{5}{6}$	$B_4 = 68.527$	$E[v(t_4)\|B_4] = 0.129$.
$\hat{w}_4 = -0.494$	$t_5 = 0.5 \times \frac{5}{6}$	$B_4 - \gamma = 66.027$	On early exercise boundary in FDM at t_4, if $S(t_4) = 68.527$, corresponding $v(t_4)$ is 0.152.
$w_3 = 0.412$	$t_4 = 0.5 \times \frac{4}{6}$	$B_3 = 65.487$	$E[v(t_3)\|B_3] = 0.134$.
$\hat{w}_3 = -0.449$	$t_4 = 0.5 \times \frac{4}{6}$	$B_3 - \gamma = 62.987$	On early exercise boundary in FDM at t_3, if $S(t_3) = 65.487$, corresponding $v(t_3)$ is 0.156.
$w_2 = 0.435$	$t_3 = 0.5 \times \frac{3}{6}$	$B_2 = 63.594$	$E[v(t_2)\|B_2] = 0.135$.
$\hat{w}_2 = -0.495$	$t_3 = 0.5 \times \frac{3}{6}$	$B_2 - \gamma = 61.094$	On early exercise boundary in FDM at t_2, if $S(t_2) = 63.594$, corresponding $v(t_2)$ is 0.164.
$w_1 = 2.243$	$t_2 = 0.5 \times \frac{2}{6}$	$B_1 = 49.094$	$E[v(t_1)\|B_1] = 0.379251$.
$\hat{w}_1 = -2.442$	$t_2 = 0.5 \times \frac{2}{6}$	$B_1 - \gamma = 46.594$	On early exercise boundary in FDM at t_1, if $S(t_1) = 49.094$, corresponding $v(t_1)$ is 0.379253.
$w_0 = -2.536$	$t_1 = 0.5 \times \frac{1}{6}$	$B_0 = 45.144$	$E[v(\tilde{t}_0)\|B_0] = 0.040$.
$\hat{w}_0 = 6.464$	$t_1 = 0.5 \times \frac{1}{6}$	$B_0 - \gamma = 42.644$	On early exercise boundary in FDM at $\tilde{t}_0$, if $S(\tilde{t}_0) = 45.144$, corresponding $v(\tilde{t}_0)$ is 0.431.

Table A3. Detailed computation results for Contract 1 in Table 2 based on Method 3^*.

Appendix B: Hedging Errors of Dynamic Delta-Neutral Hedging (DDH) Method Based on Finite Difference Method (FDM)

In practice, it is common to use the dynamic delta-neutral hedging (DDH) strategy with daily rebalance for hedging derivatives. Therefore, we implement the daily rebalancing DDH method for comparison. Given a simulated stock price-variance path, the DDH method for hedging American puts can be described as the following three-step method.

Step 1. Suppose an issuer sells an American put at the benchmarked option value $P(t_0)$ and uses the sales proceeds to invest $\Delta_0 S(t_0)$ in the underlying stock, where $\Delta_0 = \Delta_0(S(t_0), v(t_0))$ is the delta value of the American put at time t_0 calculated by the FDM. The remaining fund (could be negative), i.e., $\eta_0 = P(t_0) - \Delta_0 S(t_0)$, is invested to earn the risk-free interest rate r.

Step. 2. Just before any rebalance day $\tau = t_0 + h,\ t_0 + 2h,\ t_0 + 3h, \ldots$, where h is one day, the DDH portfolio value evolves to be $\Pi_\tau = \Delta_{\tau-h} S(\tau) e^{qh} + \eta_{\tau-h} e^{rh}$. If the stock price-variance path hits the early exercise surface generated from the FDM, the present value of the hedging error is calculated as follows and the hedging process is terminated.

$$HE = e^{-r(\tau - t_0)} \left[\Pi_\tau - \left(X - S(\tau) \right)^+ \right].$$

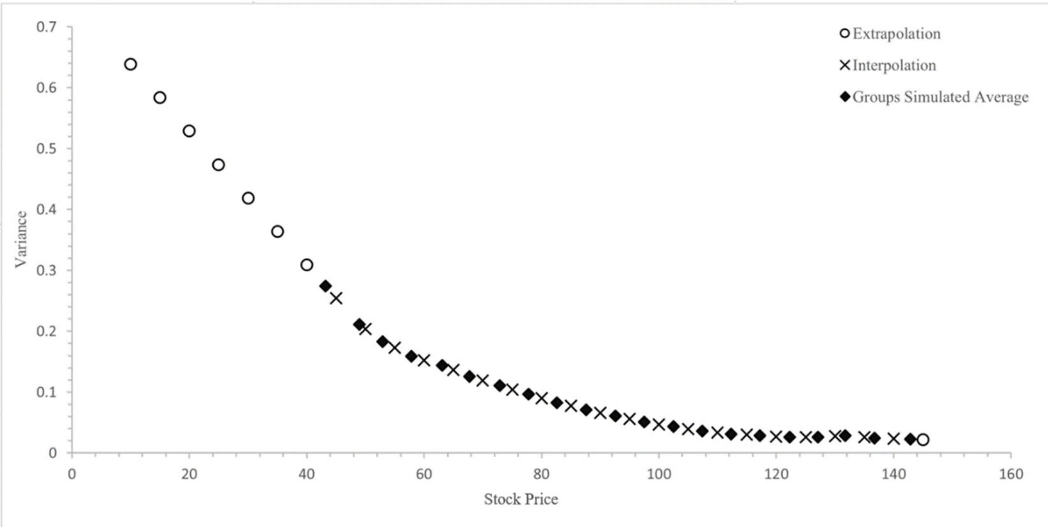
Otherwise, calculate the new delta value of the American put given the simulated $S(\tau)$ and $v(\tau)$ at τ, i.e., $\Delta_\tau = \Delta_\tau(S(\tau), v(\tau))$, using the FDM. Invest $\Delta_\tau S(\tau)$ in the underlying stock and the remaining fund $\eta_\tau = \Pi_\tau - \Delta_\tau S(\tau)$ in the risk-free asset, respectively.

Step 3. Repeating Step 2 until the maturity date T if possible, the DDH portfolio value evolves to be $\Pi_T = \Delta_{T-h} S(T) e^{qh} + \eta_{T-h} e^{rh}$ and the present value of the hedging error at T is

$$HE = e^{-r(T - t_0)} \left[\Pi_T - \left(X - S(T) \right)^+ \right].$$

Note that when implementing the DDH method, we simply assume that the parameter values are known and thus do not re-calibrate model parameters such as $v(\tau)$ for calculating $\Delta_\tau = \Delta_\tau(S(\tau), v(\tau))$. This is because our experiments focus on evaluating model superiority and attempt to exclude unexpected errors from other factors. This point highlights another advantage of the proposed SHP methods in practice, which must estimate/calibrate model parameters only once at the beginning of the hedging period, whereas the DDH method may need to re-calibrate model parameters at rebalance time points during the hedging period.

Panel A: Simulated variance conditional on stock price at $t = \frac{5}{12}$



Panel B: Simulated variance conditional on stock price at $t = \frac{1}{12}$

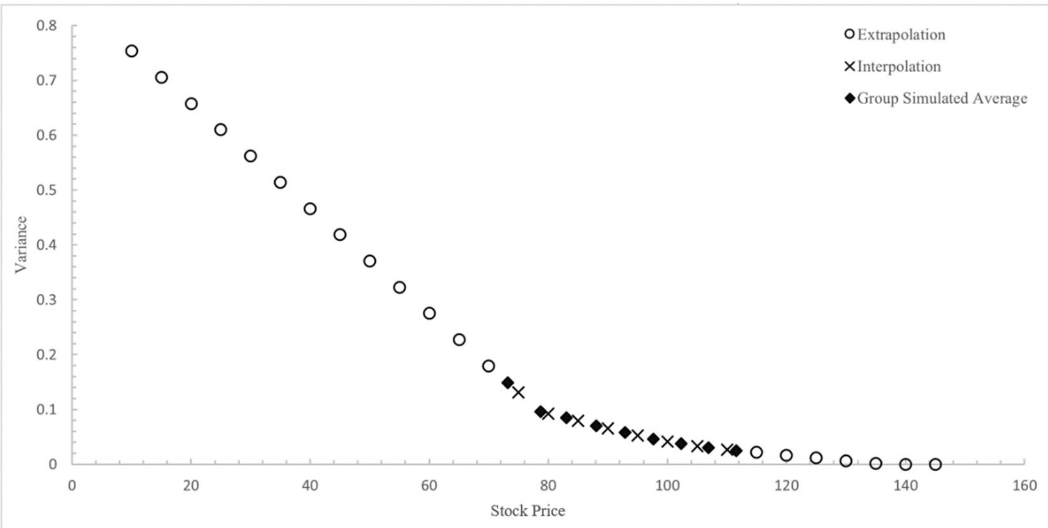
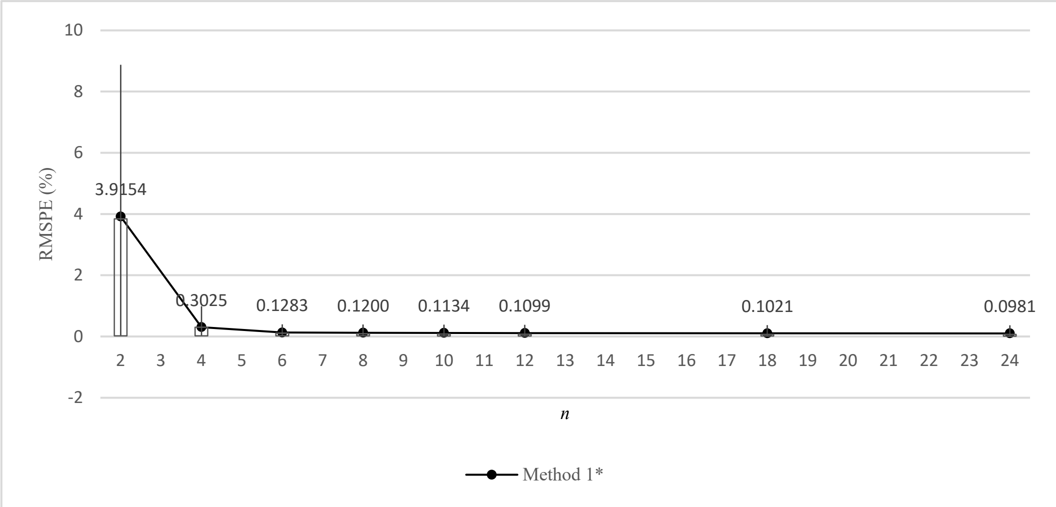
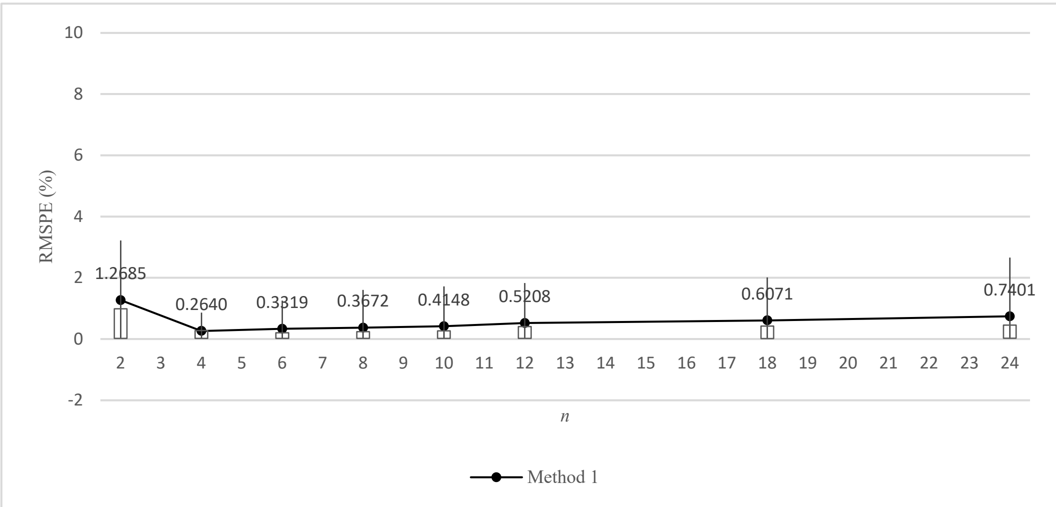


Figure 1. Simulated conditional expected variance. We conduct 15,000 simulated paths with a time step of 5×10^{-5} based on Equations (22)-(24) and then group the price paths into fixed price intervals at each examined time t_i. The parameters are $S(t_0) = 100$, $X = 90$, $r = 0.05$, $q = 0.02$, $T = 0.5$, $v(t_0) = 0.04$, $\sigma_v = 0.3$, $\kappa = 1$, $\theta = 0.09$, $\rho = -0.7$, and $n = 6$. In the figure, the diamonds represent the average variance at t_i across all corresponding variance paths within each price interval, with the interval's average stock price serving as the representative stock value. The crosses indicate the variance values piecewise interpolated from the simulated average variance and representative stock value. The circles represent the stock prices outside the bounds of the stock price paths, necessitating the estimation of the conditional expected variance with linear extrapolation based on the two outermost diamond points.

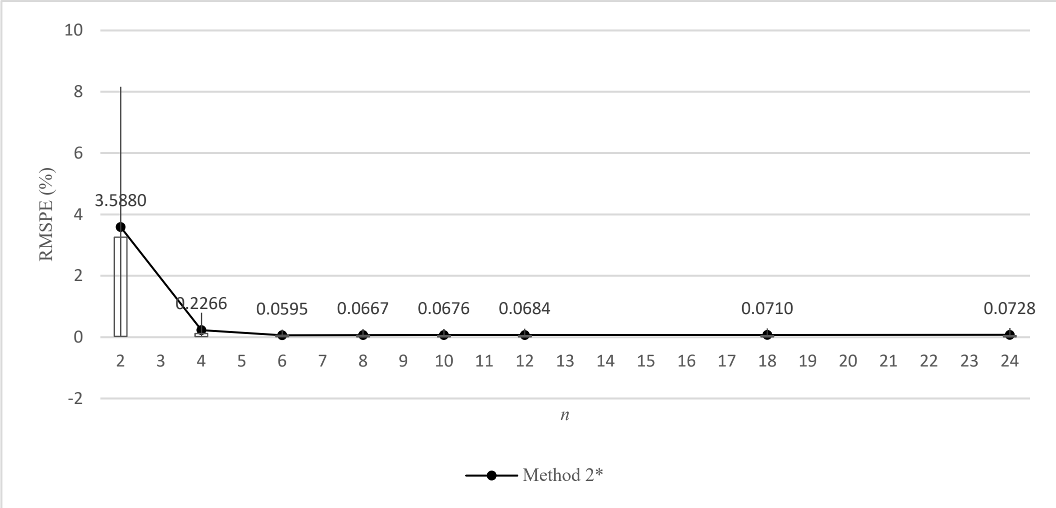
Panel A: Method 1^* convergence analysis of Set 1



Panel B: Method 1 convergence analysis of Set 1



Panel C: Method 2^* convergence analysis of Set 1



Panel D: Method 2 convergence analysis of Set 1

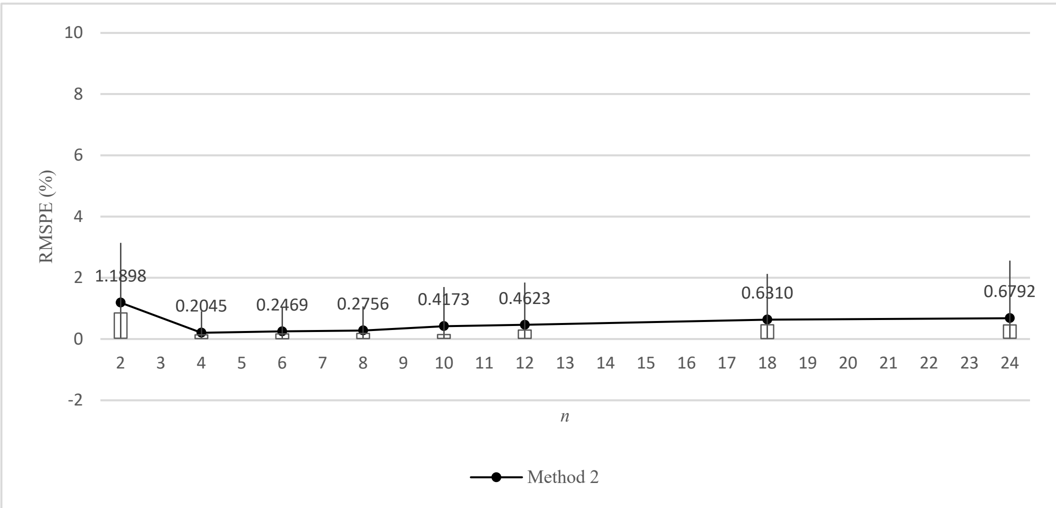


Figure 2. Convergence analyses with respect to n. RMSPE indicates the root mean square percentage error. The figures and solid circles illustrate the RMSPEs of SHP Method 1^*, 1, 2^*, and 2 for Set 1 given different n. For a given n, the top of the vertical line, the top of the vertical bar, the bottom of the vertical bar, and the bottom of the vertical line represent the maximum, 75th percentile, 25th percentile, and minimum absolute percentage error among the 27 contracts in Set 1, respectively. If there is no vega-matching condition, the pricing results clearly do not improve with n: not only do the RMSPEs of the SHP methods clearly increase with n, but the variation ranges of the absolute percentage errors among Set 1 also increase with n.

<table>
<tr><th rowspan="2">Weight</th><th colspan="2">Option contract</th><th colspan="7" rowspan="2">Steps of backward process to construct SHP for American put given $n = 6$</th></tr>
<tr><th>Maturity</th><th>Strike price</th></tr>
<tr><td>1</td><td>t_6</td><td>X</td><td>1. Match maturity payoff at $T = t_6$ with counterpart European option.</td><td rowspan="3">2. Determine w_5, $\widehat{w}_5$, and B_5 by matching three boundary conditions at t_5.</td><td rowspan="5">3. Determine w_4, $\widehat{w}_4$, and B_4 by matching three boundary conditions at t_4.</td><td rowspan="7">4. Determine w_3, $\widehat{w}_3$, and B_3 by matching three boundary conditions at t_3.</td><td rowspan="9">5. Determine w_2, $\widehat{w}_2$, and B_2 by matching three boundary conditions at t_2.</td><td rowspan="11">6. Determine w_1, $\widehat{w}_1$, and B_1 by matching three boundary conditions at t_1.</td><td rowspan="13">7. Determine w_0, $\widehat{w}_0$, and B_0 by matching three boundary conditions at $\tilde{t}_0$.</td></tr>
<tr><td>w_5</td><td>t_6</td><td>B_5</td><td rowspan="2"></td></tr>
<tr><td>$\widehat{w}_5$</td><td>t_6</td><td>$B_5 - \gamma$</td></tr>
<tr><td>w_4</td><td>t_5</td><td>B_4</td><td rowspan="2" colspan="2"></td></tr>
<tr><td>$\widehat{w}_4$</td><td>t_5</td><td>$B_4 - \gamma$</td></tr>
<tr><td>w_3</td><td>t_4</td><td>B_3</td><td rowspan="2" colspan="3"></td></tr>
<tr><td>$\widehat{w}_3$</td><td>t_4</td><td>$B_3 - \gamma$</td></tr>
<tr><td>w_2</td><td>t_3</td><td>B_2</td><td rowspan="2" colspan="4"></td></tr>
<tr><td>$\widehat{w}_2$</td><td>t_3</td><td>$B_2 - \gamma$</td></tr>
<tr><td>w_1</td><td>t_2</td><td>B_1</td><td rowspan="2" colspan="5"></td></tr>
<tr><td>$\widehat{w}_1$</td><td>t_2</td><td>$B_1 - \gamma$</td></tr>
<tr><td>w_0</td><td>t_1</td><td>B_0</td><td rowspan="2" colspan="6"></td></tr>
<tr><td>$\widehat{w}_0$</td><td>t_1</td><td>$B_0 - \gamma$</td></tr>
</table>

Table 1. Steps of backward process to construct our SHP for American options given $n = 6$. The three boundary conditions indicate the value-matching, smooth-pasting, and vega-matching conditions given $S(t_i) = B_i$ at t_i. When examining each time point t_i in a backward sequence, i.e., for $i = 5, 4, \ldots, 0$, two more European puts with the strike prices of B_i and $B_i - \gamma$ and matured at t_{i+1} are included into our SHP. This results in a total of 13 (=$1 + 2n$) European puts in the SHP given $n = 6$.

X	$v(t_0)$	q	FDM	Method 1*	Method 2*	Method 3*	Method 1	Method 2	Method 3
90	0.04	0.02	2.2485	2.2430	2.2472	2.2465	2.2396	2.2446	2.2470
90	0.04	0.05	2.5017	2.5005	2.5016	2.5012	2.5003	2.5010	2.5013
90	0.04	0.08	2.7997	2.8000	2.8000	2.7999	2.8000	2.8000	2.8000
90	0.09	0.02	3.7897	3.7835	3.7876	3.7868	3.7787	3.7833	3.7872
90	0.09	0.05	4.0961	4.0945	4.0958	4.0954	4.0942	4.0958	4.0957
90	0.09	0.08	4.4480	4.4484	4.4485	4.4484	4.4484	4.4484	4.4484
90	0.16	0.02	5.5979	5.5920	5.5953	5.5947	5.5875	5.5900	5.5945
90	0.16	0.05	5.9396	5.9458	5.9438	5.9428	5.9375	5.9390	5.9390
90	0.16	0.08	6.3234	6.3239	6.3240	6.3239	6.3239	6.3240	6.3239
100	0.04	0.02	5.5089	5.4931	5.5042	5.5060	5.4742	5.4876	5.5085
100	0.04	0.05	6.0045	6.0026	6.0054	6.0057	5.9990	6.002	6.0066
100	0.04	0.08	6.6083	6.6107	6.6108	6.6107	6.6107	6.6107	6.6107
100	0.09	0.02	7.5786	7.5666	7.5742	7.5748	7.5510	7.5581	7.5764
100	0.09	0.05	8.0734	8.0712	8.0736	8.0735	8.0705	8.0736	8.0742
100	0.09	0.08	8.6580	8.6598	8.6599	8.6599	8.6598	8.6599	8.6599
100	0.16	0.02	9.7996	9.7914	9.7959	9.7958	9.7826	9.7938	9.7935
100	0.16	0.05	10.2929	10.2973	10.2968	10.2935	10.2902	10.2925	10.2926
100	0.16	0.08	10.8579	10.8595	10.8596	10.8596	10.8593	10.8595	10.8595
110	0.04	0.02	11.3155	11.2690	11.2893	11.3006	11.1735	11.1982	11.2259
110	0.04	0.05	11.9836	11.9752	11.9809	11.9818	11.9582	11.9643	11.9864
110	0.04	0.08	12.9184	12.9210	12.9211	12.9210	12.9209	12.9211	12.9210
110	0.09	0.02	13.2721	13.2495	13.2597	13.2585	13.1797	13.2140	13.2591
110	0.09	0.05	13.9124	13.9068	13.9106	13.9109	13.9055	13.9102	13.9068
110	0.09	0.08	14.7239	14.7259	14.7261	14.7260	14.7258	14.7260	14.7259
110	0.16	0.02	15.4979	15.4862	15.4914	15.4894	15.4808	15.4657	15.4942
110	0.16	0.05	16.1134	16.1118	16.1136	16.1125	16.1090	16.1118	16.1140
110	0.16	0.08	16.8457	16.8475	16.8478	16.8476	16.8472	16.8476	16.8474

Table 2. Pricing results for Set 1. The parameters are $S(t_0) = 100$, $r = 0.05$, $T = 0.5$, $\sigma_v = 0.3$, $\kappa = 1$, $\theta = 0.09$, $\rho = -0.7$, $X \in \{90,100,110\}$, $q \in \{0.02,0.05,0.08\}$, $v(t_0) \in \{0.04,0.09,0.16\}$, $\gamma = 2.5$, and $n = 6$.

Panel A: Pricing errors for Set 1						
	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
RMSE	0.0115	0.0061	0.0047	0.0346	0.0270	0.0176
Panel B: RMSE given $r \geq q$ or $r < q$						
	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
$r \geq q$	0.0141	0.0074	0.0056	0.0423	0.0330	0.0215
$r < q$	0.0017	0.0018	0.0018	0.0016	0.0018	0.0017
Panel C: RMSE given different $v(t_0)$						
	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
$v(t_0) = 0.04$	0.0168	0.0090	0.0053	0.0496	0.0403	0.0299
$v(t_0) = 0.09$	0.0091	0.0046	0.0049	0.0325	0.0201	0.0050
$v(t_0) = 0.16$	0.0058	0.0034	0.0036	0.0090	0.0113	0.0028

Table 3. Pricing error analyses for Set 1. RMSE indicates the root mean square error. For reference, the average option value of the American puts in Set 1 is 8.8041 based on the benchmarked FDM method; when implementing the SHP methods, $\gamma = 2.5$ and $n = 6$. Panel A shows that the simulation method (Method 3*) performs the best, followed by the drift interpolation method (Method 2*), followed by the Euler discretization method (Method 1*); under the same method for estimating conditional expected variance, the SHP methods with vega matching yield much smaller pricing errors than the non-vega-matching SHP methods. Moreover, the RMSEs of Method 1^*, 2^*, and 3^* are extremely small, representing only 0.13%, 0.07%, and 0.05% of the average option value. In Panel B, under $r \geq q$, corresponding to a higher early exercise boundary and consequently greater probability of early exercise, Method 3* is still the best performer; there is a significant reduction in errors for the SHP methods with vega matching. Panel C presents the subsample analysis with respect to $v(t_0)$. The performance of Method 3* is consistently good given different values of $v(t_0)$, but Method 2* generates slightly smaller RMSEs than Method 3* when $v(t_0)$ is 0.09 and 0.16.

X	$v(t_0)$	q	FDM	Method 1*	Method 2*	Method 3*	Method 1	Method 2	Method 3
90	0.04	0.02	2.2485	2.2182	2.2187	2.2574	2.2180	2.2180	2.2180
90	0.04	0.05	2.5017	2.4973	2.5015	2.5021	2.4946	2.4952	2.4952
90	0.04	0.08	2.7997	2.8000	2.8000	2.8000	2.7999	2.7999	2.7999
90	0.09	0.02	3.7897	3.7868	3.7777	3.7781	3.7392	3.7396	3.7396
90	0.09	0.05	4.0961	4.0897	4.0905	4.0905	4.0825	4.0825	4.0825
90	0.09	0.08	4.4480	4.4486	4.4485	4.4486	4.4482	4.4482	4.4482
90	0.16	0.02	5.5979	5.5785	5.5289	5.5296	5.5286	5.5286	5.5286
90	0.16	0.05	5.9396	5.9270	5.9347	5.9274	5.9168	5.9167	5.9167
90	0.16	0.08	6.3234	6.3238	6.3247	6.3233	6.3232	6.3232	6.3232
100	0.04	0.02	5.5089	5.3955	5.3981	5.4937	5.3946	5.3946	5.3946
100	0.04	0.05	6.0045	5.9945	6.0055	6.0042	5.9804	5.9826	5.9826
100	0.04	0.08	6.6083	6.6109	6.6108	6.6108	6.6105	6.6105	6.6105
100	0.09	0.02	7.5786	7.5846	7.5562	7.5527	7.4412	7.4412	7.4412
100	0.09	0.05	8.0734	8.0767	8.0680	8.0676	8.0356	8.0383	8.0382
100	0.09	0.08	8.6580	8.6599	8.6598	8.6599	8.6589	8.6589	8.6589
100	0.16	0.02	9.7996	9.7915	9.7795	9.6483	9.6457	9.6457	9.6457
100	0.16	0.05	10.2929	10.3105	10.3092	10.3067	10.2442	10.2441	10.2441
100	0.16	0.08	10.8579	10.8607	10.8603	10.8595	10.8569	10.8569	10.8569
110	0.04	0.02	11.3155	10.9250	11.2869	10.9327	10.9214	10.9214	10.9214
110	0.04	0.05	11.9836	11.9906	11.9685	11.9573	11.9030	11.9093	11.9090
110	0.04	0.08	12.9184	12.9213	12.9211	12.9210	12.9201	12.9210	12.9201
110	0.09	0.02	13.2721	13.2774	13.2371	12.9454	12.9386	12.9385	12.9385
110	0.09	0.05	13.9124	13.9308	13.9135	13.9081	13.8210	13.8279	13.8276
110	0.09	0.08	14.7239	14.7268	14.7265	14.7265	14.7229	14.7228	14.7228
110	0.16	0.02	15.4979	15.5602	15.5376	15.5206	15.1916	15.1915	15.1915
110	0.16	0.05	16.1134	16.0851	16.1338	16.1004	16.0106	16.0180	16.0178
110	0.16	0.08	16.8457	16.8482	16.8476	16.8475	16.8405	16.8405	16.8405

Table 4. Pricing results for Set 1 when using only European puts with standard strike prices. The parameters are $S(t_0) = 100$, $r = 0.05$, $T = 0.5$, $\sigma_v = 0.3$, $\kappa = 1$, $\theta = 0.09$, $\rho = -0.7$, $X \in \{90,100,110\}$, $q \in \{0.02,0.05,0.08\}$, $v(t_0) \in \{0.04,0.09,0.16\}$, $\gamma = 5$, and $n = 6$. In the experiment, the critical early exercise boundary ($\ddot{B}_i$), serving as the strike price of the European option in the SHP, is restricted to being in the set of $\Theta = \{40,45,\ldots,95,100\}$.

Panel A: Pricing errors for Set 1 using options with standard strike prices

	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
RMSE	0.0799	0.0296	0.1025	0.1298	0.1292	0.1293

Panel B: RMSE using options with standard strike prices given $r \geq q$ or $r < q$

	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
$r \geq q$	0.0979	0.0362	0.1255	0.1590	0.1583	0.1583
$r < q$	0.0022	0.0020	0.0018	0.0020	0.0022	0.0021

Panel C: RMSE using options with standard strike prices given different $v(t_0)$

	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
$v(t_0) = 0.04$	0.1360	0.0398	0.1280	0.1400	0.1396	0.1396
$v(t_0) = 0.09$	0.0073	0.0147	0.1094	0.1259	0.1253	0.1253
$v(t_0) = 0.16$	0.0250	0.0288	0.0564	0.1228	0.1222	0.1222

Table 5. Pricing error analyses for Set 1 when using only European puts with standard strike prices. RMSE indicates the root mean square error. For reference, the average option value of the American puts in Set 1 is 8.8041 based on the benchmarked FDM method; when implementing the SHP methods, $\gamma = 5$, $n = 6$, and the critical early exercise boundary ($\ddot{B}_i$), serving as the strike price of the European put in the SHP, is restricted to being in the set of $\Theta = \{40,45,\dots,95,100\}$. Upon comparison with the results presented in Table 3, it is evident that the restriction of considering only standard strike prices results in increasing, but still acceptable pricing errors, e.g., the RMSE of Method 2*, the best-performing method here, is 0.0296, representing only 0.3% of the average option value. Under the same method for estimating conditional expected variance, the SHP methods with vega matching yield smaller errors than the non-vega-matching SHP methods. In Panel B, under $r \geq q$, corresponding to a higher early exercise boundary and consequently greater probability of early exercise, Method 2* is still the best performer; there is an obvious reduction in errors for the SHP methods with vega matching. Panel C presents the subsample analysis with respect to $v(t_0)$. The performance of Method 2* is the best when $v(t_0) = 0.04$, but Method 1* generates the smallest RMSEs when $v(t_0)$ is 0.09 and 0.16.

Parameters	Examined values
X	{90, 95, 100, 105, 110}
T	$\{\frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}\}$
$S(t_0)$	100
r	0.05
q	Randomly drawing from unif(0, 0.08)
$v(t_0)$	Randomly drawing from unif(0.01, 0.25)
κ	Randomly drawing from unif(0.1, 5)
θ	0.13
σ_v	Randomly drawing from unif(0.1, 0.5)
ρ	Randomly drawing from unif(-0.9, -0.5)

Table 6. Parameter values for Set 2. To facilitate further comparison, we randomly drew 20 sets of data for q, $v(t_0)$, κ, σ_v, and ρ, where unif(a,b) is defined as the uniform distribution between a and b, and then each combination of X and T, for example, $(X,T) = (90,3/12)$, was combined with the simulated 20 sets of q, $v(t_0)$, κ, σ_v, and ρ to form 20 examined contracts. We generate a total of 600 option contracts for Set 2.

Panel A: Pricing errors for Set 2						
	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
RMSE	0.0105	0.0072	0.0087	0.0329	0.0203	0.0137

Panel B: RMSE given $r \geq q$ or $r < q$						
	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
$r \geq q$	0.0110	0.0075	0.0091	0.0347	0.0214	0.0144
$r < q$	0.0037	0.0038	0.0038	0.0040	0.0036	0.0038

Panel C: RMSE given different $v(t_0)$						
	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
Small $v(t_0)$	0.0053	0.0040	0.0082	0.0099	0.0060	0.0053
Medium $v(t_0)$	0.0089	0.0074	0.0073	0.0215	0.0101	0.0082
Large $v(t_0)$	0.0149	0.0092	0.0104	0.0517	0.0331	0.0216

Table 7. Pricing results for Set 2. RMSE indicates the root mean square error. For reference, the average option value of the American puts in Set 2 is 7.7062 based on the benchmarked FDM method; when implementing the SHP methods, $\gamma = 2.5$ and $n = 6$. In Panel A, Method 2^* demonstrates the best pricing performance; there is significant benefit in including the vega-matching condition for all examined methods. Moreover, the RMSEs of Method 1^*, 2^*, and 3^* are extremely small, representing only 0.14%, 0.09%, and 0.11% of the average option value. In Panel B, under $r \geq q$, corresponding to a higher early exercise boundary and consequently greater probability of early exercise, Method 2* is still the best performer; there is a significant reduction in errors for the SHP methods with vega matching. Panel C presents the subsample analysis with respect to $v(t_0)$. The performance of Method 2* is consistently good given different levels of $v(t_0)$, but Method 3* generates slightly smaller RMSEs than Method 2* for the medium $v(t_0)$ values.

Panel A: Pricing errors for Set 2						
	Method 1*	Method 2*	Method 3*	Method 1	Method 2	Method 3
RMSE	0.0841	0.0820	0.0891	0.1105	0.1105	0.1104
Panel B: RMSE given $r \geq q$ or $r < q$						
	Method 1*	Method 2*	Method 3*	Method 1	Method 2	Method 3
$r \geq q$	0.0887	0.0865	0.0915	0.1165	0.1165	0.1164
$r < q$	0.0026	0.0022	0.0643	0.0093	0.0082	0.0070
Panel C: RMSE given different $v(t_0)$						
	Method 1*	Method 2*	Method 3*	Method 1	Method 2	Method 3
Small $v(t_0)$	0.0989	0.0916	0.1147	0.1132	0.1130	0.1129
Medium $v(t_0)$	0.0821	0.0802	0.0750	0.1117	0.1115	0.1114
Large $v(t_0)$	0.0686	0.0732	0.0711	0.1055	0.1070	0.1069

Table 8. Pricing error analyses for Set 2 when using only European puts with standard strike prices. RMSE indicates the root mean square error. For reference, the average option value of the American puts in Set 1 is 7.7062 based on the benchmarked FDM method; when implementing the SHP methods, $\gamma = 5$, $n = 6$, and the critical early exercise boundary ($\ddot{B}_i$), serving as the strike price of the European put in the SHP, is restricted to being in the set of $\Theta = \{40,45,\ldots,95,100\}$. Upon comparison with the results presented in Table 7, it is evident that the restriction of considering only standard strike prices results in increasing, but still acceptable pricing errors, e.g., the RMSE of Method 2*, the best-performing method here, is 0.0820, representing only 1.1% of the average option value. Under the same method for estimating conditional expected variance, the SHP methods with vega matching yield smaller errors than the non-vega-matching SHP methods. In Panel B, under $r \geq q$, corresponding to a higher early exercise boundary and consequently greater probability of early exercise, Method 2* is still the best performer and the SHP methods with vega matching still show better pricing ability. Panel C presents the subsample analysis with respect to $v(t_0)$. The performance of Method 2* is the best for small $v(t_0)$ values, but Method 3* (1*) generates the smallest RMSEs for the medium (large) $v(t_0)$ values.

Panel A: Hedging risk measurements for Set 1							
	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3	DDH
$VaR_{0.05}$	0.0834	0.0752	0.0731	0.1090	0.1002	0.0934	2.4549
$ES_{0.05}$	0.1158	0.1085	0.1069	0.1413	0.1319	0.1248	3.0893
$ESHE$	0.0432	0.0430	0.0426	0.0457	0.0449	0.0449	2.1873
EL	0.0411	0.0376	0.0369	0.0551	0.0505	0.0465	1.2258

Panel B: Hedging risk measurements for Set 1 using European puts with standard strike prices						
	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
$VaR_{0.05}$	0.1252	0.1698	0.1511	0.2856	0.2705	0.2818
$ES_{0.05}$	0.2013	0.2113	0.2245	0.3703	0.3298	0.3660
$ESHE$	0.0569	0.0578	0.0623	0.0776	0.0716	0.0772
EL	0.0672	0.0903	0.0839	0.1581	0.1476	0.1559

Table 9. Hedging risk measurements for Set 1. For reference, the average option value of the American puts in Set 1 is 8.8041 based on the benchmarked FDM method; when implementing the SHP methods, $\gamma = 2.5$ (5) and $n = 6$ for Panel A (B). In addition, the critical early exercise boundary ($\ddot{B}_i$), serving as the strike price of the European option in the SHP, is restricted to be in the set of $\Theta = \{40,45,\dots,95,100\}$ in Panel B. First, both panels consistently show that the proposed SHP methods exhibit smaller hedging risk than the traditional DDH (dynamic delta-neutral hedging) method, and the SHP methods with vega matching outperform the counterparts without vega matching. Taking Method 2^* for example, its $VaR_{0.05}$ is 0.0752 (0.1698) without (with) the constraint of using European puts with standard strike prices. These two $VaR_{0.05}$ values represent 0.85% and 1.93% of the average American put value, respectively. In contrast, the $VaR_{0.05}$ of the DDH method is 2.4549, representing 27.88% of the average American put value. Nevertheless, constrained to considering only standard strike prices, it is inevitable that hedging risk increases in Panel B. However, the hedging performance of SHP methods in Panel B is still satisfactory and significantly superior to that of the DDH method.

Panel A: Hedging risk measurements for Set 2

	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3	DDH
$VaR_{0.05}$	0.0677	0.0607	0.0601	0.0865	0.0812	0.0789	1.8172
$ES_{0.05}$	0.1412	0.1324	0.1329	0.1508	0.1462	0.1443	2.3848
$ESHE$	0.0071	0.0066	0.0065	0.0083	0.0073	0.0072	1.3618
EL	0.0365	0.0314	0.0319	0.0460	0.0428	0.0405	0.8863

Panel B: Hedging risk measurements for Set 2 using European puts with standard strike prices

	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
$VaR_{0.05}$	0.1651	0.1604	0.1612	0.2529	0.2519	0.2516
$ES_{0.05}$	0.2279	0.2238	0.2419	0.3212	0.3203	0.3200
$ESHE$	0.0180	0.0175	0.0361	0.0265	0.0264	0.0264
EL	0.0913	0.0893	0.0914	0.1388	0.1382	0.1380

Table 10. Hedging risk measurements for Set 2. For reference, the average option value of the American puts in Set 1 is 7.7062 based on the benchmarked FDM method; when implementing the SHP methods, $\gamma = 2.5$ (5) and $n = 6$ for Panel A (B). In addition, the critical early exercise boundary ($\ddot{B}_i$), serving as the strike price of the European option in the SHP, is restricted to be in the set of $\Theta = \{40,45,\dots,95,100\}$ in Panel B. First, both panels consistently show that the proposed SHP methods exhibit smaller hedging risk than the traditional DDH (dynamic delta-neutral hedging) method, and the SHP methods with vega matching outperform the counterpart without vega matching. Taking Method 2^* for example, its $VaR_{0.05}$ is 0.0607 (0.1604) without (with) the constraint of using European puts with standard strike prices. These two $VaR_{0.05}$ values represent 0.79% and 2.08% of the average American put value, respectively. In contrast, the $VaR_{0.05}$ of the DDH method is 1.8172, representing 23.58% of the average American put value. Nevertheless, constrained to considering only standard strike prices, it is inevitable that hedging risk increases in Panel B. However, the hedging performance of SHP methods in Panel B is still satisfactory and significantly superior to that of the DDH method.

Panel A: Hedging risk measurements for early exercised paths of Set 2							
	Method 1*	Method 2*	Method 3*	Method 1	Method 2	Method 3	DDH
$VaR_{0.05}$	0.0490	0.0419	0.0412	0.0682	0.0624	0.0602	1.6167
$ES_{0.05}$	0.0877	0.0776	0.0782	0.0999	0.0943	0.0921	2.1825
$ESHE$	0.0051	0.0043	0.0041	0.0063	0.0050	0.0050	1.4466
EL	0.0320	0.0290	0.0290	0.0400	0.0364	0.0352	0.9907

Panel B: Hedging risk measurements for early exercised paths of Set 2 using European puts with standard strike prices						
	Method 1*	Method 2*	Method 3*	Method 1	Method 2	Method 3
$VaR_{0.05}$	0.1480	0.1429	0.1448	0.2362	0.2351	0.2348
$ES_{0.05}$	0.1848	0.1801	0.1988	0.2875	0.2862	0.2858
$ESHE$	0.0197	0.0193	0.0558	0.0311	0.0310	0.0309
EL	0.0870	0.0843	0.0875	0.1325	0.1320	0.1317

Table 11. Hedging risk measurements for early exercised paths of Set 2. For reference, the average option value of the American puts in Set 1 is 7.7062 based on the benchmarked FDM method; when implementing the SHP methods, $\gamma = 2.5\ (5)$ and $n = 6$ for Panel A (B). In addition, the critical early exercise boundary ($\ddot{B}_i$), serving as the strike price of the European option in the SHP, is restricted to be in the set of $\Theta = \{40,45,\ldots,95,100\}$ in Panel B. First, both panels consistently show that the proposed SHP methods exhibit smaller hedging risk than the traditional DDH (dynamic delta-neutral hedging) method, and the SHP methods with vega matching outperform the counterparts without vega matching. Taking Method 2* for example, its $VaR_{0.05}$ is 0.0419 (0.1429) without (with) the constraint of using European puts with standard strike prices. These two $VaR_{0.05}$ values represent 0.54% and 1.85% of the average American put value, respectively. In contrast, the $VaR_{0.05}$ of the DDH method is 1.6167, representing 20.98% of the average American put value. Nevertheless, constrained to considering only standard strike prices, it is inevitable that hedging risk increases in Panel B. However, the hedging performance of SHP methods in Panel B is still satisfactory and significantly superior to that of the DDH method.