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RESEARCH ARTICLE

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A stochastic-volatility equity-price tree for pricing convertible bonds with endogenous firm values and default risks determined by the first-passage default model

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Abstract

This paper proposes a novel equity-price-tree-based convertible bond (CB) pricing model based on the first-passage default model under stochastic interest rates. By regarding equity values as down-and-out call options on firm values (FVs), at each tree node, we solve the implied FV and equity-price volatility (EPV), and then endogenously settle the default probability (DP) and also the dilution effect subject to CB conversions with the implied FV and capital structure. Our model captures the stylized negative (positive) relationships between the stochastically evolving DP and FV or EP (EPV) that cannot be fully achieved by existing CB pricing models.

KEYWORDS

convertible bond, dilution effect, first-passage default model, stochastic interest rate, stochastic volatility

1 | INTRODUCTION

A convertible bond (CB) is a kind of corporate bond that allows bondholders to share the profits and growth of the issuer by converting the bond into a predetermined number of the issuer's equity shares at certain stipulated time points before maturity. With the upside potential of the embedded conversion option, investors buy CBs even if they are issued at higher prices or carry lower yields. On the other hand, a CB issuer can raise debt capital with less funding costs at the expense of possible equity dilutions due to CB conversions. CBs are widely issued and frequently traded in financial markets, but it is difficult to price CBs precisely since there are many factors (which can be intercorrelated or interactive with each other) that influence CB prices. Most CBs grant issuers the right to call back the bonds at prespecified call prices; this embedded call option can either save interest expenses due to occurrences of unexpected interest rate drops or force CB holders to convert their CBs into equity shares. Since both conversion and call-back provisions embedded in CBs are analogous to the early exercise provision of American options, optimal conversion and call-back boundaries are simultaneously evaluated when pricing CBs. Tree models are widely used to determine the path dependencies of the equity and short rate processes for touching the optimal conversion and call-back boundaries. Moreover, a CB is subject to the issuer's default risk, which is theoretically time varying and negatively (positively) related to the stochastically evolving issuing firm value (FV)[1] and equity price (equity-price volatility [EPV]).

However, past structural and reduced-form CB pricing models[2] fail to fully capture the relationships among factors that affect CB values, including the default risk, issuing FV, equity value (defined as the product of the equity price and the number of outstanding shares), EPV, and dilution effect. A CB pricing model that can neither take all those factors into account nor formulate proper relationships among them is unlikely to generate economically reasonable pricing results. The contribution of this paper is to address this problem by proposing an equity-price-tree-based CB pricing model[3] based on Black and Cox's (1976) first-passage default model. As a result, the proposed model properly formulates the stylized relationships among the default probability (DP), issuing FV, stochastic-volatility equity price, and dilution effect under a stochastic interest rate environment when pricing CBs.

CB pricing models can be categorized by how default risks are modeled.[4] The first approach naively models default probabilities by decomposing a CB value into the equity and the debt components (e.g., Cheung & Nelken, 1994; McConnell & Schwartz, 1986; Tsiveriotis & Fernandes, 1998; Yigitbasioglu, 2002). The equity component is evaluated under the risk-neutral valuation method that discounts future cash flows with the risk-free rate, and the debt component is evaluated by discounting its cash flow with the risky rate to reflect the default risk. Many empirical pricing studies adopt decomposition CB pricing models due to their lower data requirements. However, the risky discount rates employed by this category of models are independent of the stochastic evolving firm and equity value, or are even simply constant, and those models actually take neither default and recovery rates nor the dilution effect due to CB conversions into consideration since the issuer's capital structure is not modeled. In addition, a CB is in essence a hybrid security of equity and debt and subject to the default risk as a whole. Consequently, it is difficult to estimate the pricing error from decomposition CB pricing models.

To price CBs while simultaneously modeling the aforementioned capital structures, default rates, and dilution effects, the second category of structural default models can be adopted, where the evolution of an issuer's capital structure is simulated and different conditions leading to defaults are considered (e.g., Ballotta & Kyriakou, 2015; Brennan & Schwartz, 1977; Chen et al., 2013; Ingersoll, 1977; Sarkar, 2003). For example, Ingersoll (1977) and Brennan and Schwartz (1977) model the evolution of the issuing FV process and develop no-arbitrage arguments applied to derive PDEs for pricing CBs. Ingersoll (1977) focuses only on special cases that can be analytically solved, whereas Brennan and Schwartz (1977) numerically solve the PDE using the finite difference method to solve the free boundary problem caused by the embedded convertible and callable options. Brennan and Schwartz (1980) extend their work to model the stochastic interest rate by incorporating the Vasicek (1977) interest rate model into their pricing method. Sarkar (2003) extends the models in Ingersoll (1977) and Brennan and Schwartz (1977) by taking corporate taxes and premature defaults into account. Chen et al. (2013) price callable CBs with a nonzero-sum-game framework under the structural default model; Ballotta and Kyriakou (2015) numerically evaluate CBs by modeling the issuing FV as an exponential jump-diffusion process with correlated stochastic interest rates. Explicitly modeling the issuer FV process allows these structural default approaches to endogenously determine capital structures, recovery rates, and dilution effects due to CB conversions. The main criticism of the CB pricing models based on structural default models is that the FV of the issuer cannot be directly observed and traded, which limits the practicability of this approach. In addition, failing to calibrate equity prices[5] that can be directly observed in markets complicates accurate evaluations of CBs and embedded conversion options.

[1]In this paper, the term *firm value* refers to the market value of a firm's total assets.

[2]To simplify the wording, throughout this paper we refer to CB pricing models based on structural default models (reduced-form default models) as *structural (reduced-form) CB pricing models*.

[3]In the literature, it is also possible to solve the partial differential equation (PDE) of derivatives (either with respect to the FV or the equity price) with the finite difference or element methods or apply the least-squares Monte Carlo simulation in Longstaff and Schwartz (2001) to evaluate CBs. To name but a few, Brennan and Schwartz (1977, 1980), Tsiveriotis and Fernandes (1998), Takahashi et al. (2001), Yigitbasioglu (2002), Ayache et al. (2003), and Lau and Kwok (2004) price CBs with the finite difference method, and Lvov et al. (2004), Wilde and Kind (2005), Kimura and Shinohara (2006), Yang et al. (2010), and Batten et al. (2018) employ the Monte Carlo simulation to evaluate CBs.

[4]Batten et al. (2014) survey various provisions in CB contracts and different CB pricing models.

[5]Structural default models, such as Vassalou and Xing (2004), Duffie et al. (2007), and Bharath and Shumway (2008), propose a more robust approach to infer the firm asset value today and estimate the FV volatility by calibrating historical equity prices. To our knowledge, this stream of approaches has not been adopted in prior CB pricing models.

The third stream of CB pricing approaches is based on reduced-form default models in which default events are simulated by jump-to-default processes and corresponding default probabilities can be calibrated by matching credit spreads of an issuer's outstanding bonds (e.g., Chambers & Lu, 2007; Coonjobeharry et al., 2016; Finnerty, 2015; Hung & Wang, 2002; Jarrow & Turnbull, 1995; Kimura & Shinohara, 2006; Wang & Dai, 2017; Yang et al., 2010).[6] However, existing reduced-form CB pricing approaches focus on simulating the equity-price process but lack a clear theoretical link between default events and the equity price (or the FV). Take, for example, the tree-based approaches suggested by Hung and Wang (2002) and Chambers and Lu (2007). They model the evolution of equity prices with a CRR (Cox, Ross, and Rubinstein's, 1979) binomial tree and the issuer's default risk by introducing the jump-to-default process of Jarrow and Turnbull (1995). The fraction of the bond face value received by the holder as the issuer defaults (i.e., the recovery rate) is exogenously determined. The DP is modeled to calibrate the credit spread without considering the magnitudes of equity and debt values of the issuing firm. However, a higher equity price (and thus a higher equity value), ceteris paribus, generally implies that the bond issuer is in better financial shape and has a lower default risk, and vice versa. Improperly assigning a CB issuer's DP without considering its financial status may mislead corresponding call-back and conversion strategies. In addition, dilution effects due to CB conversions also cannot be modeled due to the failure to take into account issuers' capital structures. These disadvantages render this stream of approaches less intuitive.

Different from prior structural CB pricing models, our CB pricing model, somewhat similar to prior reduced-form CB pricing models, is constructed based on an equity-price tree. At the same time, our CB pricing model exploits the first-passage default model that treats the issuer's equity value as a down-and-out call option on the issuing FV to calibrate the implied FV for each node of the proposed equity-price tree. As a result, in contrast to the time-varying default probabilities used in prior reduced-form CB pricing models, our CB pricing model takes stochastic default probabilities into account by determining the DP of each node with the calibrated implied FV. More specifically, we first follow Black and Cox (1976) in assuming that the FV follows a geometric Brownian motion process. To estimate the FV process, we apply Vassalou and Xing's (2004) approach (but base it on the first-passage default model rather than Merton's (1974) default model) to determine the FV today and estimate the constant FV volatility by calibrating a historical time series of equity prices. Second, for each node in our equity-price tree, the implied issuing FV and EPV are endogenously solved by applying the down-and-out call option pricing formula to calibrate the equity value at that node. We thus determine the DP and also the dilution effect due to CB conversions by taking advantage of the implied issuing FV and capital structure at each node. Finally, to model the resulting stochastic drift and volatility of the equity price, our tree is constructed by modifying and combining the generalized autoregressive conditional heteroskedasticity (GARCH) trinomial tree proposed in Ritchken and Trevor (1999) and the mean-tracking method in Dai (2009).[7] Numerical experiments suggest that our CB pricing tree model captures the stylized negative (positive) relationships between the stochastically evolving equity price (EPV) and the DP that cannot be captured by past CB pricing models.

To consider the interest rate risk under our core idea for pricing CBs, we construct a two-factor (the equity price and the short-term interest rate) tree by elegantly combining the aforementioned equity-price tree, Hull and While (1994) stochastic interest rate tree (for implementing Vasicek's (1977) interest rate model), and the first-passage default model with stochastic interest rates. Here the relationship between the equity value and the issuing FV is modeled by the semiclosed-form formulas for pricing down-and-out call options under the Vasicek model in Bernard et al. (2008); in addition, the first-passage DP is solved by the formula provided in Collin-Dufresne and Goldstein (2001). We address the correlation between the equity price and the interest rate by using the orthogonal method suggested in Wang and Dai (2017) to adjust the branching probabilities.

Several sensitivity analyses are employed to illustrate how the proposed CB pricing tree can reasonably examine the impacts of several important parameters on CB values. We also price a real CB contract issued by Danaher Corporation (DHR)—the empirical case studied in Wang and Dai (2017)—to demonstrate how the proposed tree produces reasonable CB values.

[6]Although Unal et al. (2003), Carr and Linetsky (2006), Linetsky (2006), Duffie et al. (2007), Carr and Wu (2009), Das and Hanouna (2009), and Mendoza-Arriaga et al. (2010) extend the reduced-form default model to consider the relations between default risks and equity price (volatility) or other relevant economic variables, to our knowledge, such sophisticated designs have not been employed to price CBs.

[7]Note that extremely high default or risk-free interest rates result in unexpectedly high drift terms for the equity-price process which cause the classical binomial tree model in Cox et al. (1979) to produce invalid, negative branching probabilities, as mentioned in Lyuu and Wang (2011) and Chambers and Lu (2007). The mean-tracking method proposed by Dai (2009) and Dai and Lyuu (2010), which produces valid branching probabilities by allowing the trinomial branch structure to adjust with the stochastic drift term of the equity-price process, is adopted to solve this negative probability problem.

The rest of the paper is organized as follows. Section 2 shows how to model the equity value as an option on the issuing FV and then describes the corresponding equity-price process used to develop the tree model proposed later. Section 3 describes the construction of our stochastic-volatility equity-price tree that endogenously settles the default risk, early redemption, and dilution effect with the issuer's equity price and implied FV at each tree node. To highlight the advantages of the proposed model, we also illustrate the differences and similarities of the tree structures and schemes for modeling default events among structural, reduced-form, and our equity-price-tree-based CB pricing models in Section 3. Section 4 formally proposes our CB pricing model by incorporating the Vasicek interest rate model into the tree proposed in Section 3. Section 5 examines the reasonability and the robustness of our tree via sensitivity analyses and results on an empirical case. Section 6 concludes the paper.

2 | MODELING THE ISSUER'S EQUITY VALUE AND EQUITY-PRICE PROCESS

According to the first-passage model (see Black & Cox, 1976), an arbitrary firm, say a CB issuer, may default before maturity if its stochastically evolving FV falls below a default boundary V_B. Then the equity value of the issuer can be treated as a down-and-out call option on its value with the payoff function as

$$E_T = \begin{cases} (V_T - D)^+ & \text{if } V_{\min} > V_B, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where E_t denotes the equity value at time t, V_t denotes the issuing FV at time t, D denotes total debt amount of the issuer due at the CB maturity date T, and $V_{\min}$ is the realized minimal issuing FV from now to the maturity T. The (prior-maturity) default occurs once V_t reaches the default boundary V_B (i.e., $V_{\min} \leq V_B$). We follow Longstaff and Schwartz (1995) by setting the default boundary $V_B = xD$, where $x \in [0,1]$, to reflect acceleration bond covenants . Since our model is complicated and uses many parameters, all definitions and notations of the parameters used in this paper are summarized in Table A1 in Appendix A for quick reference.

Note that the equity price can be applied to infer the FV and the corresponding default risk as well as the conversion value that is useful for evaluating CBs. In addition, unlike the unobservable issuing FV, the equity price S_t (which equals the equity value E_t divided by the number of outstanding shares) and its volatility can be directly observed and estimated from the market. Thus, instead of the FV which is nonobservable and nontradable, the equity price is employed to develop our tree model that discretely simulates the price at time 0, Δt, $2\Delta t$, ... (i.e., S_0, $S_{\Delta t}$, $S_{2\Delta t}$, ...). The equity value at each node of our tree can be used to infer the corresponding FV while the DP and the stochastic EPV associated with the inferred FV in turn are used to infer the drift and the volatility of the equity-price process for the subsequent tree construction. To accommodate the prior-maturity default of the first-passage model and the stochastic volatility and interest rate, we model the one-step CB issuer's equity-price evolution under the risk-neutral measure as the following general form:

$$S_{t+\Delta t} = \begin{cases} e^{\ln S_t + \left(r_t - q + \vartheta_t(S_t, V_B) - \frac{\sigma_{S,t}^2}{2}\right)\Delta t + \sigma_{S,t}\Delta Z_{S,t}} & \text{with prob. } 1 - \epsilon_t(S_t, V_B), \\ 0 & \text{with prob. } \epsilon_t(S_t, V_B), \end{cases} \tag{2}$$

where r_t denotes the risk-free short rate, q denotes the dividend yield,[8] $\epsilon_t(S_t, V_B)$ denotes the DP (or the probability for the lognormal process V_t to reach the default boundary V_B) within the time span $(t, t + \Delta t]$, $\vartheta_t(S_t, V_B)$ denotes the drift adjustment term, $\sigma_{S,t}$ denotes the equity-price volatility, and $\Delta Z_{S,t}$ is the discrete counterpart of $dZ_{S,t}$, the standard Wiener process for the equity price. The drift adjustment, $\vartheta_t(S_t, V_B)$, makes the discounted defaultable equity-price

[8]The dividend could be an important driver of CB prices. This paper adopts the continuous dividend yield setting since it is widely adopted in the literature and it keeps the overall model derivation simple. Besides, it can be easily estimated from the financial reports. The constant cash divided setting may be modeled by the stair tree model proposed in Dai (2009), which is also used later in this paper to develop the equity-price tree. In addition, Dai (2009) can be extended to handle the case in which the dividend is defined as a function of the prevailing equity price. On the other hand, complex and stochastic dividend settings might be more proper for long-term CBs, although it is not easy to calibrate the associated parameters given infrequent dividend data. More proper settings for dividends might be an interesting future research issue.

process a martingale process. Finally, the conditional expectation and the variance of the log return of the equity price from time t to $t + \Delta t$ can be expressed as

$$E[\ln S_{t+\Delta t}|S_t, r_t] = \ln S_t + \left(r_t - q + \vartheta_t(S_t, V_B) - \frac{\sigma_{S,t}^2}{2}\right)\Delta t \tag{3}$$

and

$$\mathrm{var}(\ln S_{t+\Delta t}|S_t, r_t) = \sigma_{S,t}^2 \Delta t, \tag{4}$$

respectively.

3 | PRICING CONVERTIBLE BONDS WITH A ONE-FACTOR (EQUITY-PRICE) TREE MODEL

To clearly convey our main idea, we first construct a stochastic-volatility equity-price tree without interest rate risk as illustrated in Figure 1a that will be detailed in Section 3.1. The short rate r_t in Equations (2)–(4) is set as a constant r in this section for ease of description. For simplicity, we also shorten the notations of $E[\ln S_{t+\Delta t}|S_t, r_t]$ and $\mathrm{var}(\ln S_{t+\Delta t}|S_t, r_t)$ in Equations (3) and (4) to be $E[\ln S_{t+\Delta t}|S_t]$ and $\mathrm{var}(\ln S_{t+\Delta t}|S_t)$, respectively, in Section 3. This constraint will be relaxed in Section 4.

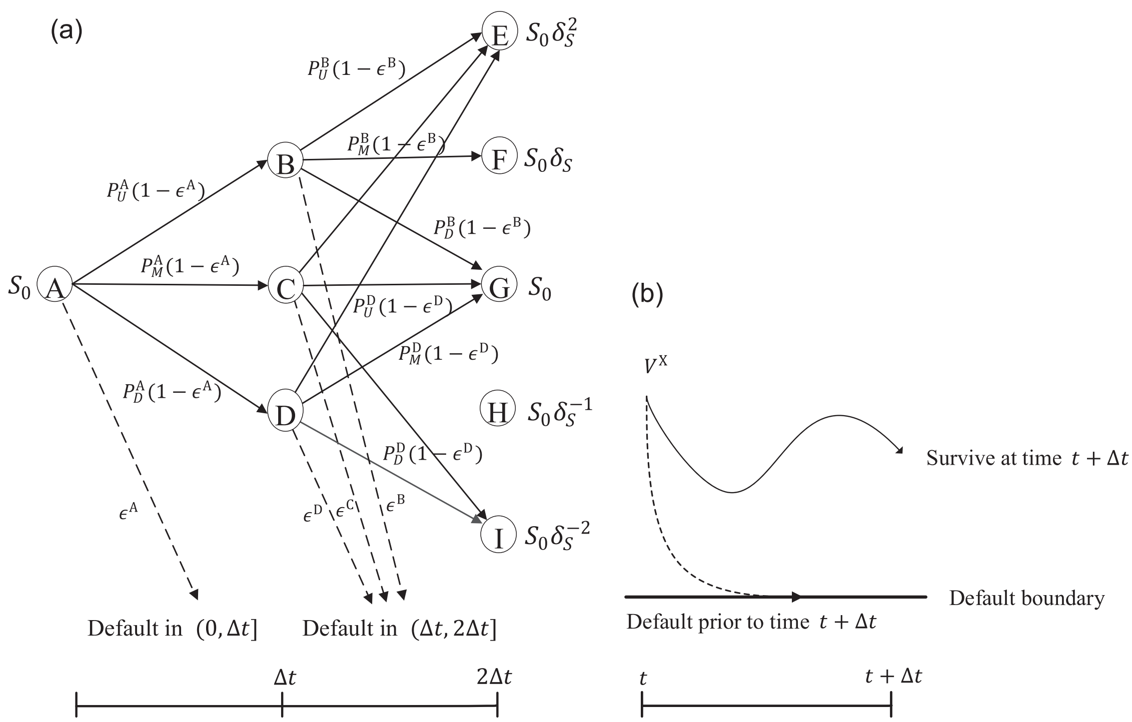

FIGURE 1 Trinomial equity-price tree with prior-to-maturity default branches. In Panel (a), the probability that an arbitrary node X defaults within a time step as illustrated by the dashed curve in Panel (b) is ε^X. The probabilities for the upward, middle, and downward moving branches emitting from node X to its successor nodes are P_U^X, P_M^X, and P_D^X, respectively; these branching probabilities are assumed to be independent to ε^X. The equity price for each node at time $2\Delta t$ is listed next to that node. By combining Dai (2009) and Ritchken and Trevor (1999) to account for stochastic drift and the volatility of the equity price, the three nondefault descendant nodes at the next time step may not be simply one grid tick higher than, equal to, and lower than the equity price of the examined node on the log-price plane. Panel (b) illustrates the default case (with probability ε^X) and the survival case defined in the first-passage default model by the dashed and solid curves, respectively, where V^X denotes the firm value of node X.

Suppose that a CB issuing FV follows a log-normal process $\frac{dV_t}{V_t} = (r - \phi)dt + \sigma_V dZ_{V,t}$ under the risk-neutral measure, where ϕ is the cash payment yield from the FV,[9] σ_V is a constant volatility for the FV, and $dZ_{V,t}$ is the standard Wiener process for the issuing FV. Given $V_t > V_B$ (i.e., the issuer survives at time t), the equity value at time t can be expressed as the value of a down-and-out call option on the issuing FV as

$$E_t = V_t\left[N(x) - \left(\frac{V_B}{V_t}\right)^{\left[\frac{2(r-\phi)}{\sigma_V^2}\right]+1} N(y)\right] - De^{-r(T-t)}\left[N\left(x - \sigma_V\sqrt{T-t}\right) - \left(\frac{V_B}{V_t}\right)^{\left[\frac{2(r-\phi)}{\sigma_V^2}\right]-1} N\left(y - \sigma_V\sqrt{T-t}\right)\right], \quad (5)$$

where $N(\cdot)$ denotes the cumulative distribution function of standard normal random variables, $x = \frac{\ln(\frac{V_t}{D})}{\sigma_V\sqrt{T-t}} + \Omega\sigma_V\sqrt{T-t}$, $y = \frac{\ln(\frac{V_B^2}{V_t D})}{\sigma_V\sqrt{T-t}} + \Omega\sigma_V\sqrt{T-t}$, and $\Omega = \frac{r-\phi+\frac{\sigma_V^2}{2}}{\sigma_V^2}$.

To solve the stochastically evolving FV V_t and the EPV $\sigma_{S,t}$ at each tree node located at a time point t, we additionally take advantage of the relation among the issuing FV, its volatility, the equity value, and the EPV proposed in Merton (1974) as[10]

$$\sigma_{S,t}E_t = \frac{\partial E_t}{\partial V_t}\sigma_V V_t. \quad (6)$$

Specifically, V_t and $\sigma_{S,t}$ are solved by substituting the prevailing equity value E_t (derived by S_t) at each tree node and the constant σ_V into Equations (5) and (6). The conditional DP over the time period $(t, s]$ can then be calculated by taking advantage of the reflection principle in Shreve (2004) as

$$p_t(\tau \le s) = N\left(\frac{\ln\left(\frac{V_B}{V_t}\right) - \left(r - \phi - 0.5\sigma_V^2\right)(s-t)}{\sigma_V\sqrt{s-t}}\right) + (V_B/V_t)^{\left[2\left(\frac{r-\phi-0.5\sigma_V^2}{\sigma_V^2}\right)\right]} \times N\left(\frac{\ln\left(\frac{V_B}{V_t}\right) + \left(r - \phi - 0.5\sigma_V^2\right)\left(s-t\right)}{\sigma_V\sqrt{s-t}}\right), \quad (7)$$

where $\tau = \inf\{u \ge t : V_u \le V_B\}$ denotes the first time that the log-normal FV process falls to V_B.

3.1 | Employing the equity price as the main factor

Our CB pricing model employs the equity price (rather than the FV) as the main factor and simulates its dynamics with a defaultable equity-price tree introduced later for the following reasons. First, the conversion option in CBs directly depends on the equity price, only indirectly linking to the FV. Second, the FV is nontradable as argued in Chen et al. (2013); thus it is difficult for market participants to utilize the Greek letters generated by firm-value-based CB pricing models for risk management. Third, even though one can still obtain the corresponding equity price of each node via Equation (5) in a firm-value-based CB pricing tree to proceed with the CB pricing procedure, in the context of the tree structure it is almost impossible to accommodate the stochastic volatility of the equity price that significantly influences the conversion decision, default risk, and thus CB value. Given these concerns, we propose a defaultable stochastic-volatility equity-price-tree-based CB pricing model. Even though our CB pricing model considers the equity price as the main factor, it possesses the advantages of structural default models, such as the modeling of dilution effects and default probabilities, by inferring the FV with the first-passage default model.

[9]This is defined as the sum of the dividend payments and interest expenses divided by the market value of the firm assets.
[10]Note that in the first-passage default model, the only source of uncertainty is the Wiener process for issuing FV. As a result, Merton's (1974) derivation associated with Equation (6) also implies that the essence underlying the Wiener process for the equity price ($dZ_{S,t}$) in Equation (2) should be the Wiener process for the issuing FV ($dZ_{V,t}$).

3.1.1 | Structure of defaultable equity-price tree

Our main innovation is to construct an equity-price tree like that depicted in Figure 1a that endogenously determines the changing issuer's capital structure to theoretically model default risks, dilution effects, and call/conversion decisions. A tree divides the time span from now (time 0) to the CB's maturity (time T) into n equal time steps. Then the length of each time step Δt is T/n and the ith time step is mapped to time $i\Delta t$. The equity price represented by a node, say A at time 0, moves to any of nodes B, C, or D at time Δt, or becomes 0 due to default. For an arbitrary node at time t, the issuing FV V_t and the EPV $\sigma_{S,t}$ are derived by solving the system of Equations (5) and (6) given the simulated equity price at that node and the constant issuing FV volatility σ_V. As illustrated in Figure 1b, the probability for that node (with aforementioned calculated V_t) to default within a time step, $\epsilon_t(S_t, V_\mathrm{B})$, is calculated by substituting $t + \Delta t$ for s in Equation (7) to obtain $p_t(\tau \leq t + \Delta t)$. It is worth noting that we transform the touch-to-DP of the first-passage default model in the FV space to the jump-to-DP in the equity-price space. Finally, to compensate for the deduction in the conditional expected equity price at $t + \Delta t$ due to the possibility of default, the drift adjustment $\vartheta_t(S_t, V_\mathrm{B})$ of that node is derived as

$$\vartheta_t(S_t, V_\mathrm{B}) = \frac{-\ln(1 - \epsilon_t(S_t, V_\mathrm{B}))}{\Delta t} \tag{8}$$

to make the equity-price process a martingale process under the risk-neutral measure.

3.1.2 | Mean-tracking and GARCH methods for stochastic default and volatility

The time-varying characteristic of $\vartheta_t(S_t, V_\mathrm{B})$ and $\sigma_{S,t}$ causes the conditional expectation and the variance of the log return of the equity price, $E[\ln S_{t+\Delta t}|S_t]$ and $\mathrm{var}(\ln S_{t+\Delta t}|S_t)$ in Equations (3) and (4), respectively, to be stochastic. To construct an equity-price tree with stochastic drift and volatility, we develop a feasible trinomial tree model by combining the mean-tracking method in Dai (2009) and the GARCH option pricing tree in Ritchken and Trevor (1999) described as follows.

First, the trinomial tree is constructed on the log-price plane, with a unit interval in the grid determined by the constant issuer-firm-value volatility $\delta_S = \sigma_V\sqrt{\Delta t}$. Specifically, the possible levels of $\ln S_t$ are selected among $\ln(S_0)$, $\ln(S_0) \pm \delta_S$, $\ln(S_0) \pm 2\delta_S$, ... Next, the middle descendant node of an arbitrary node with log-equity-price $\ln S_t$ at time t is selected by the mean-tracking method proposed in Dai (2009). Specifically, the middle descendant node at time $t + \Delta t$ is found by determining the integer k that makes the middle node's log-equity-price $\ln S_0 + k\delta_S$ the closest one to $E[\ln S_{t+\Delta t}|S_t]$.

To model the stochastic-volatility feature, we follow Ritchken and Trevor (1999) by setting the distance between the log prices of the upper and middle descendant nodes (or the middle or lower descendant nodes) as $\eta\delta_S$, where η is the smallest positive integer satisfying

$$\frac{\eta}{2} \leq \frac{\sigma_{S,t}\sqrt{\Delta t}}{\delta_S} \leq \sqrt{\eta^2 - 1}\,. \tag{9}$$

The proposed tree-building method is illustrated in Figure 2. Finally, the trinomial branching probabilities can be solved by calibrating the first and second moments (Equations (10) and (11)) of the log-equity-price process and by ensuring that the sum of the branch probabilities equals 1 (Equation (12)) as follows:

$$P_\mathrm{U}\alpha + P_\mathrm{M}\beta + P_\mathrm{D}\gamma = 0, \tag{10}$$

$$P_\mathrm{U}\alpha^2 + P_\mathrm{M}\beta^2 + P_\mathrm{D}\gamma^2 = \mathrm{var}(\ln S_{t+\Delta t}|S_t) = \sigma_{S,t}^2\Delta t, \tag{11}$$

$$P_\mathrm{U} + P_\mathrm{M} + P_\mathrm{D} = 1, \tag{12}$$

where α $(= \beta + \eta\delta_S)$, β, and γ $(= \beta - \eta\delta_S)$ denote the relative log-price differences between the upper, middle, and lower descendant nodes and $E[\ln S_{t+\Delta t}|S_t]$, respectively. Then P_U, P_M, and P_D are solved as

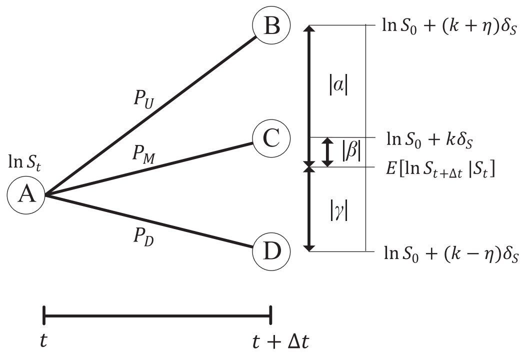

FIGURE 2 Proposed trinomial tree structure. We combine Dai (2009) and Ritchken and Trevor (1999) to develop this trinomial tree structure. Descendant nodes B, C, and D are connected by outgoing trinomial branches emitting from node A. The middle descendant node is chosen in the grid of equity prices to make its corresponding logarithmic equity price the closest to $E[\ln S_{t+\Delta t}|S_t]$. We further define α, β, and γ as the differences between the log-equity prices of the upper, middle, and lower descendant nodes and $E[\ln S_{t+\Delta t}|S_t]$, respectively. Adopting the mean-tracking property in Dai (2009) ensures that $\beta \in (-\delta_S, \delta_S)$. The difference between nodes B and C (or nodes C and D) is determined by the method in Ritchken and Trevor (1999) as $\eta\delta_S$, where η is the smallest positive integer satisfying $\frac{\eta}{2} \le \frac{\sigma_{S,t}\sqrt{\Delta t}}{\delta_S} \le \sqrt{\eta^2-1}$.

$$P_{\mathrm{U}} = \frac{\beta^2 - \beta\eta\delta_S + \sigma_{S,t}^2\Delta t}{2\eta^2\delta_S^2},$$

$$P_{\mathrm{M}} = -\frac{\beta^2 - \eta^2\delta_S^2 + \sigma_{S,t}^2\Delta t}{\eta^2\delta_S^2},$$

$$P_{\mathrm{D}} = \frac{\beta^2 + \beta\eta\delta_S + \sigma_{S,t}^2\Delta t}{2\eta^2\delta_S^2}.$$

The proposed method guarantees that P_{U}, P_{M}, and P_{D} always fall within [0,1] regardless of stochastic variations from r_t, $\vartheta_t(S_t, D)$, $\sigma_{S,t}$, and changes of Δt as proved in Appendix B.

In Figure 1a, we illustrate three possible branching scenarios constructed by the aforementioned approach. First, for node B, suppose the drift and volatility terms of the equity price of those nodes do not cause negative branch probabilities, so the equity prices of three descendant nodes are one grid tick higher than, equal to, and one grid tick lower than $S_0\delta_S$, respectively, on the log-price plane (i.e., they are $S_0\delta_S^2$, $S_0\delta_S$, and S_0). Second, for nodes C and D, the equity volatility may be so high that one positive and negative grid tick on the log-equity price is not wide enough to generate positive branch probabilities; thus the descendant nodes that are, for example, two grid ticks higher and lower than the middle descendant node are employed to overcome this problem, as suggested by Ritchken and Trevor (1999). Take node C, for example: its three descendant nodes are nodes E, G, and I, corresponding to the equity prices of $S_0\delta_S^2$, S_0, and $S_0\delta_S^{-2}$, respectively. Third, for node D, it may possess high equity drift in addition to high volatility, so the mean-tracking method in Dai (2009) suggests that the middle descendant node should be, for example, tilted up by one grid tick to avoid negative branch probabilities. Together with using Ritchken and Trevor's (1999) method to deal with high equity volatility, the three descendant nodes of node D are therefore nodes E, G, and I.

3.2 | Pricing CBs with our equity-price tree

Equipped with the aforementioned trinomial tree that models the relationship among the stochastic equity price, the corresponding capital structure, and the DP for each node of the tree, the CB price and the corresponding call-back, put-back, and conversion strategies can be evaluated by standard backward induction as described below.

At the last time step (i.e., the maturity date), given the issuer is not defaulting, CB holders decide whether to convert their CBs into θ equity shares. If not, the issuer redeems the CB at its par value F (normalized to 100 dollars in this paper) plus the coupon payment if any. The conversion value at maturity is evaluated as the product of the conversion ratio (θ) and the diluted equity price S_T^{AC}, which will be further discussed in Section 3.3. Thus the CB value for an arbitrary node X at maturity can be expressed as

$$CV^{\mathrm{X}} = \max\left(\left(1 + \frac{c_{\mathrm{B}}}{2}\right)F, \theta S_T^{\mathrm{AC}}\right), \tag{13}$$

where CV^{X} denotes the value of the CB at node X, and c_{B} denotes the annual coupon rate of the CB. For an arbitrary node Y located at a time t other than the maturity date, the issuer calls back the CB if the "holding value" HV^{Y}—the value of the CB provided that it is not converted or called back yet at node Y—is larger than the call price. Note that CB holders may exercise their conversion options to maximize their benefits or put back the CB to the issuer even when the call-back option is initiated; thus the CB value at node Y is

$$CV^{\mathrm{Y}} = \max\left(\min(HV^{\mathrm{Y}}, CP_t), \theta S_t^{\mathrm{AC}}, PP_t\right), \tag{14}$$

where CP_t and PP_t denote the call and put price at time t, respectively.

In our tree model, the holding value can be calculated as the discounted expected future CB values of the three descendant nodes. For example, the holding value for node A in Figure 1 is evaluated as

$$HV^{\mathrm{A}} = e^{-r\Delta t}\left(\epsilon^{\mathrm{A}}\omega x F + (1 - \epsilon^{\mathrm{A}})\left(P_{\mathrm{U}}^{\mathrm{A}} CV^{\mathrm{B}} + P_{\mathrm{M}}^{\mathrm{A}} CV^{\mathrm{C}} + P_{\mathrm{D}}^{\mathrm{A}} CV^{\mathrm{D}}\right)\right), \tag{15}$$

where $\omega \in [0, 1]$ represents the effective recovery rate, and $P_{\mathrm{U}}^{\mathrm{A}}$, $P_{\mathrm{M}}^{\mathrm{A}}$, and $P_{\mathrm{D}}^{\mathrm{A}}$ denote the probabilities of the upper, middle, and lower branches, respectively, of node A. In addition, if the examined node (e.g., node A in Equation 15) is on a coupon-payment date, the resulting holding value in Equation (15) is adjusted upward by $\frac{c_{\mathrm{B}}}{2}F$ to reflect the coupon income received by the CB holders.

3.3 | Dilution effect

Converting a CB into equity shares increases the number of outstanding shares and dilutes the equity price. Ignoring such dilution effects overprices conversion values and hence CBs. Nevertheless, dilution effects are difficult to incorporate into reduced-form-based pricing models, such as Hung and Wang (2002) and Chambers and Lu (2007), as the FV and the capital structure of the issuer are not explicitly modeled. The structural model embedded in our method, however, explores the inner relationships among the prevailing equity value, the FV, and the outstanding debt value to model changes in the firm's capital structure and the wealth transfer among different claim holders due to CB conversions.

Specifically, we adopt Brennan and Schwartz's (1980) model to account for the dilution effect. Their settings assume that the firm's asset value consists of the following three securities: straight bonds, CBs, and equity shares. Before CB conversion, the FV at time t is

$$V_t = N_{\mathrm{B}} B_t + N_{\mathrm{C}} CV_t + N_{\mathrm{O}} S_t^{\mathrm{BC}}, \tag{16}$$

where N_{B} and N_{C} denote the number of outstanding straight bonds (with a coupon rate c_{S} paid semiannually) and CBs, respectively, N_{O} denotes the number of outstanding equity shares, B_t denotes the value of straight bonds (given a normalized face value $F = 100$) at time t, and S_t^{BC} denotes the equity price before CB conversion. The bond price process (B_t) at each time step can be evaluated simultaneously through our proposed tree by applying standard backward induction for pricing coupon-bearing bonds.[11] Although the conversion of CBs does not change the FV, it does change the capital structure to consist of outstanding straight bonds and equity shares (including original shares

[11]The straight-bond values for terminal nodes are first set as $\left(1 + \frac{c_{\mathrm{S}}}{2}\right)F$, and then backward induction similar to Equation (15) is conducted for all other nodes by replacing the CB and holding values with the bond values. In addition, if the examined node is on a coupon-payment date, the resulting bond value after backward induction is adjusted upward by $\frac{c_{\mathrm{S}}}{2}F$ to reflect the coupon income received by straight bondholders.

and newly converted shares). Then the dilution effect from the viewpoint of the firm can be considered by formulating the conversion value as $\theta N_C S_t^{AC}$ as

$$V_t = N_B B_t + \left(N_O + \theta N_C\right) S_t^{AC}, \tag{17}$$

where S_t^{AC} denotes the equity price after conversion. At each tree node, since the FV V_t (inferred by solving Equations 5 and 6) and the value of straight bonds B_t (derived based on standard backward induction) are known, Equation (17) yields the corresponding diluted equity price S_T^{AC} as

$$S_t^{AC} = \frac{V_t - N_B B_t}{N_O + \theta N_C}. \tag{18}$$

3.4 | Determination of σ_V and V_0

One special feature of our equity-price-tree-based CB pricing model is to integrate the relationship between the equity and FVs in the equity-price tree. Therefore, we must determine the parameters of the FV volatility σ_V and the initial market value of the firm V_0 before constructing the equity-price tree. The FV volatility σ_V is estimated by adapting the iterative method proposed in Vassalou and Xing (2004). Roughly speaking, we make an initial guess at σ_V as the volatility of the FV and then solve the FV on each trading day for a past period of time by substituting the corresponding equity prices observed from the market into Equation (5). Then the guess for σ_V is updated by the annualized standard derivation of the time series of the obtained FVs. This procedure is repeated until the guess converges. This σ_V is assumed to be a constant over the life of the CB. By substituting the market-observable equity value E_0 (evaluated as the product of S_0 and the number of outstanding shares) at time 0 and σ_V into Equation (5), we obtain the issuing FV, V_0.

3.5 | Illustrative example

Here we price an example of a hypothetical 3-year CB by a three-time-step tree illustrated in Figure 3. As detailed in the legend, we make assumptions on the variables that can be observed or accessed from the market on the issuance date (at time 0), such as the numbers of outstanding equities, straight bonds, and CBs, and the equity price $S_0 = \$30$ as well as the volatility $\sigma_{S,0} = 0.3$. The call (put) price is constant (zero) during the CB life, that is, $CP_t = CP = 113$ and $PP_t = 0$. Then the FV $V_0 = \$730.77$ thousand dollars and the FV volatility $\sigma_V = 0.1220$ are derived via Equations (5) and (6).[12] Equation (7) is next applied to obtain the DP of 0.06% for the root node. Then we use the method introduced in Section 3.1 and $\sigma_{S,0}$ to determine the equity prices \$43.2623, \$30.0000, and \$20.8033 for the three following successor nodes at time 1. For each node at time 1, the corresponding FV and the equity value volatility can be derived by substituting the prevailing equity price and the FV volatility into Equations (5) and (6). Take node D, for example. The equity value, which is estimated as the equity price \$20.8033 multiplied by the number of outstanding shares, 10,000, can be treated as the down-and-out call option value on the FV. By substituting the equity value and the FV volatility 0.1220 into Equations (5) and (6), we obtain an FV of \$661.11 thousand dollars and an EPV of 0.3948. The DP for the subsequent time step can be solved by Equation (7) to obtain 0.95%. By adopting the method introduced in Section 3.1 and the EPV for each node at time 1, we construct the successor nodes at time 2. The above procedure is repeatedly applied to construct the entire tree.

The backward induction procedure in Equations (13)–(15) is then applied to evaluate the CB. The CB value for each node at maturity is obtained via Equation (13). For example, a CB holder converts her CB at node A to earn the conversion value \$123.8221, which equals the product of the after-conversion equity price \$61.9111 (calculated via Equation (18)) and the conversion ratio, 2. Otherwise, the bond is redeemed at par value if the prevailing equity price is

[12]Vassalou and Xing's (2004) iteration method is not used in this hypothetical example since for simplicity, we make no assumption concerning historical equity prices.

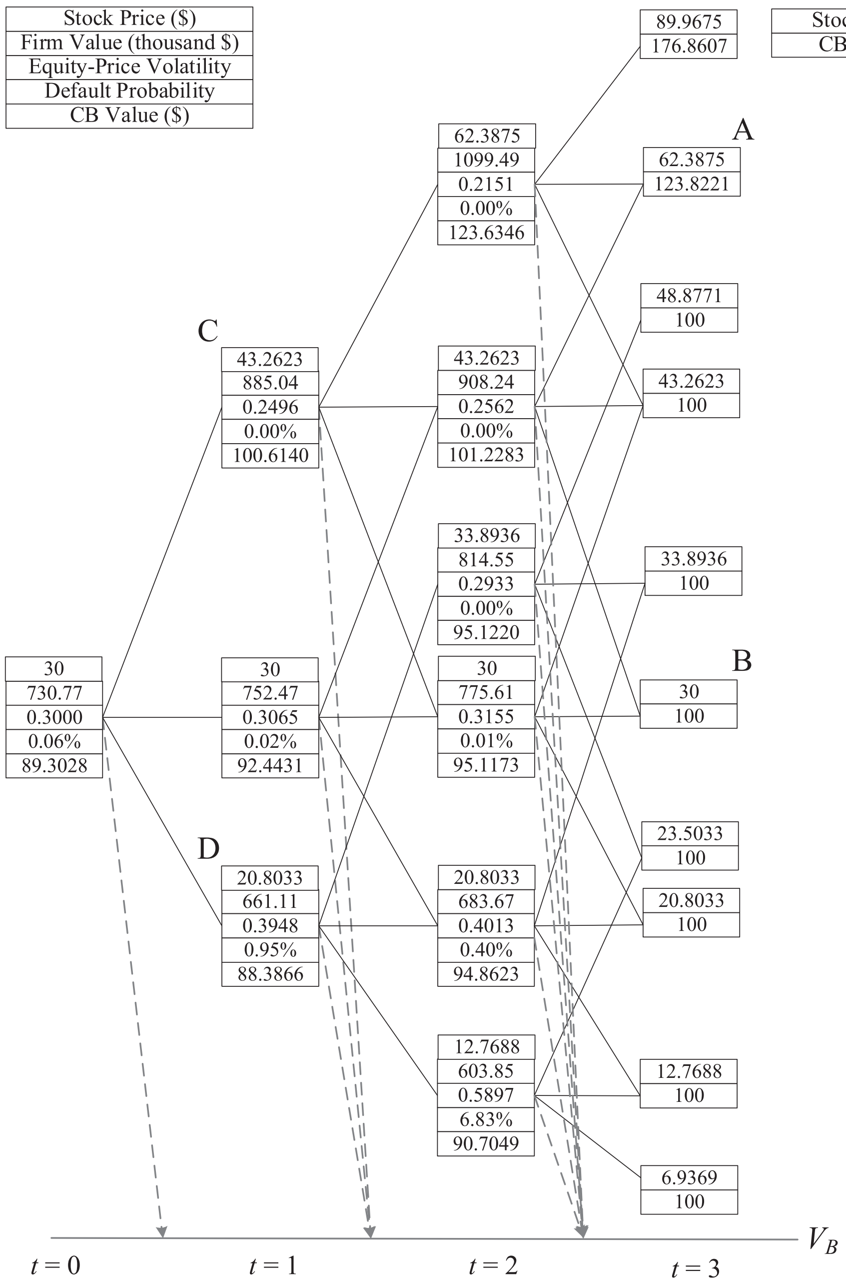

FIGURE 3 Three-time-step example of proposed trinomial equity-price tree. Each tree node at the last time step is represented by a two-row rectangle, whose meanings are listed in the upper-rightmost legend. All other tree nodes are represented by five-row rectangles, whose meanings are listed in the upper-leftmost legend. The parameter values are $T = 3$ (years), $n = 3$, $F = 100$, $\theta = 2$, $CP_t = 113$, $PP_t = 0$, $S_0 = 30$, $\sigma_{S,0} = 0.3$, $r = 5\%$, $N_O = 10{,}000$, $N_B = 4800$, $N_C = 200$, $\omega = 0.32$, $q = c_B = c_S = \phi = 0$, $D = F\,(N_B + N_C) = 500{,}000$, and $x = 1$. The time-0 firm value V_0 ($730.77 thousand dollars) and σ_V (0.1220) are determined by substituting S_0 and $\sigma_{S,0}$ into Equations (5) and (6). The solid or gray dashed lines emitting from each node indicate branches to other nodes or reaching to the default boundary, respectively

low, for example, node B, in which case the equity price is $30. The conversion, redemption strategy, and hence the CB value for the node before maturity is determined by Equation (14). Take node D for example: its holding value is evaluated via Equation (15) as $88.3866. By substituting its holding value, conversion value (far below than its holding value due to its low equity price), and the call price into Equation (14), the CB value at node D is solved as $88.3866. By repeatedly applying the aforementioned backward induction procedure back to the root node of the tree, our tree yields the CB pricing result $88.9191.

Since our model formulates reasonable relationships among stochastically evolving equity prices, FVs, equity volatilities, and default probabilities, favorable features that are generally observed in financial markets, such as a

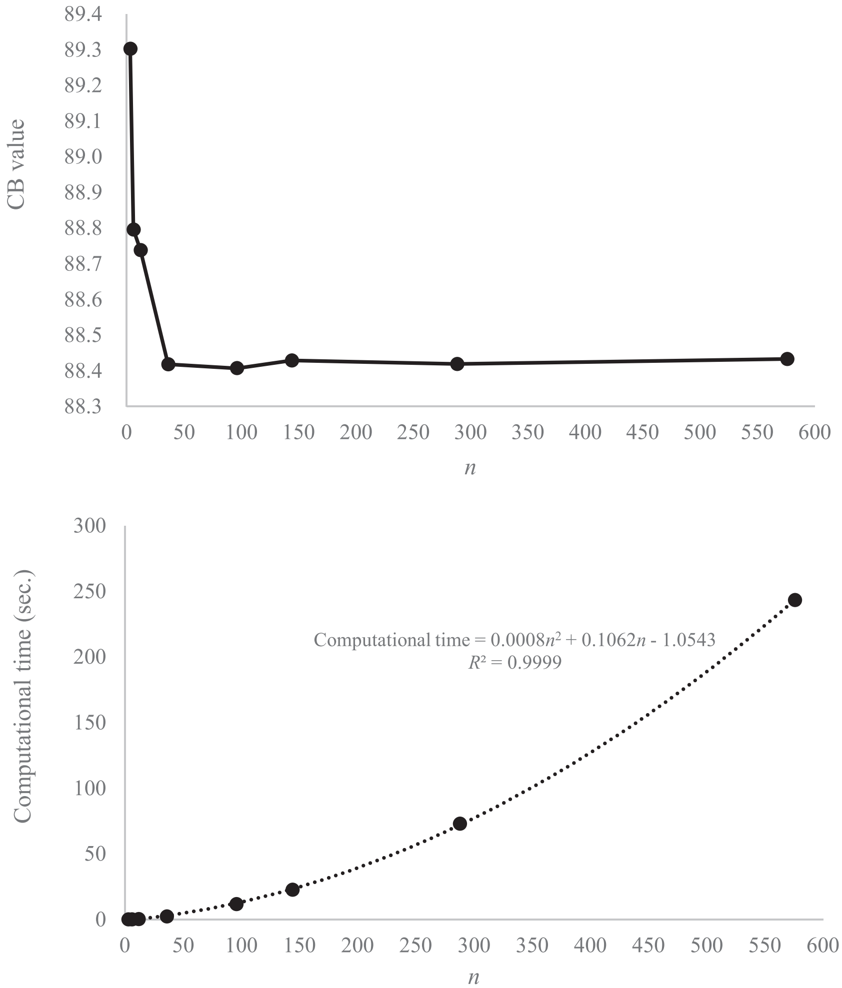

FIGURE 4 Convergence and time complexity analyses of the proposed CB pricing model. The pricing error convergence behavior and the running time complexity of our CB pricing model with respect to the number of time steps n are analyzed based on the hypothetical example examined in Figure 3. The upper panel shows that the generated CB value approaches its convergent result quickly and the lower panel implies that the time complexity of our CB pricing model is $O(n^2)$. CB, convertible bond.

higher equity price implying a higher FV, a lower equity volatility,[13] and a lower DP can be captured by our model as in Figure 3. However, these features are not captured well in other CB pricing models in the literature. Taking nodes C and D at $t = 1$ for comparison, a high equity price of $43.2623 implies a high FV of $885.04 thousand dollars, a low equity volatility of 0.2496, and an almost zero low DP, whereas a low equity price of $20.8033 implies a low FV of $661.11 thousand dollars, a high equity volatility of 0.3948, and a high DP of 0.95%. Note that the proposed model's ability to capture the negative (positive) relationships between the DP and the FV or the equity price (the EPV) cannot be fully achieved by existing structural or reduced-form CB pricing models in the literature.

3.6 | Convergence and time complexity analysis of our CB pricing model

We examine the convergence property and the running time of our tree model in Figure 4 with the hypothetical example defined in Figure 3. The upper panel suggests that our pricing results converge to around 88.4 quickly as the number of steps

[13]The negative relationship between the equity price and its volatility is commonly attributed to the leverage effect. Christie (1982) first termed this observation the *leverage effect*.

increases. Since Equation (14) is checked for each node at every time step to determine whether the call-back or conversion decisions are executed, both the numbers of call-back and conversion opportunities increase with the increment of the number of time steps, n. Theoretically speaking, the increment of n (or the numbers of call-back and conversion opportunities) makes our pricing model's optimal conversion and call-back boundaries finer and more accurate along the time dimension and thus generates convergent pricing results, as shown in Figure 4. The running time is proportional to the quadratic of the number of time steps, as in the CRR binomial tree. Without the recombination property constructed by following Dai (2009) and Ritchken and Trevor (1999), the number of tree nodes and hence the running time of our equity-price-tree-based CB pricing model would grow explosively, as argued in Dai and Lyuu (2010).

3.7 | Comparison with traditional structural and reduced-form CB pricing model

This section explains the theoretical advantages of the proposed equity-price-tree-based CB pricing model by comparing the lattice structures and schemes for modeling default events among structural, reduced-form, and our CB pricing models.[14] In the upper-left of Figure 5, a structural CB pricing model treats a CB as a contingent claim on the FV V_t, that is, the FV is the most exogenous driver of the value of the firm securities. However, this is improper since the FV is nontradable, as argued in Chen et al. (2013); therefore, the Greek letters generated by firm-value-based CB pricing models are impractical to use. Besides, since CB contracts are stipulated based on the equity price (treated as a call on the FV in structural default models) rather than the FV, CBs can be theoretically regarded as a call on a call on the FV, which complicates the pricing procedure. In the upper-right panel, in contrast, the conversion option can be easily modeled as a call option on the equity price in a reduced-form CB pricing model. However, dilution effects and default risks related to FVs are hard to model due to the lack of information on FVs and capital structures in a reduced-form CB pricing model. This stream of CB pricing models typically employs time-varying (independent of firm-value and equity-value) default probabilities calibrated from the prevailing risky and riskless term structures of interest rates.

However, our equity-price-tree-based CB pricing model in the bottom panel accommodates advantageous features from both structural and reduced-form default models. Our tree structure employs techniques from Dai (2009) and Ritchken and Trevor (1999) to account for stochastic default intensity and EPV. Thus, the first-passage default model can be integrated into our equity-price tree to infer each node's implied FV and DP. The bottom panel briefly illustrates a cookbook recipe for the tree construction of our model. Our model first uses the first-passage default model and historical equity prices (method [M1]) to calibrate the FV and its constant volatility for the root node. At other nodes of the equity-price tree, we calibrate the corresponding FV and EPV with the [M2] method (the first-passage default model and Equation (6)). Then Equations (7) and (8) are employed to derive the DP ε and the corresponding jump intensity ϑ given the implied FV of each node.

In Table 1, we compare the pricing results generated by a typical reduced-form CB pricing model (illustrated in Figure 6) and our model with or without dilutions. To achieve a fair comparison, the time-varying DP for each time step of the reduced-form CB pricing model is the weighted average of the default probabilities of the tree nodes at that time step in our model. A comparison of the second and the third columns of Table 1 shows that ignoring the dilution effect overprices the CB, especially when the conversion is likely to happen due to the high equity price.[15] In addition, unlike our CB pricing model, the reduced-form CB pricing model adopts time-varying default rates that are independent of the stochastically involving equity price or FV. The price differences between these two models are significant, especially for scenarios with low prevailing equity prices and thus high default risks. Compared with the results of our CB pricing model without the dilution effect, we believe that the reduced-form CB pricing model underprices CBs in this hypothetic example since it ignores the relationship between the equity-price and default probabilities and thus overweights (underweights) the default probabilities for nodes with high (low) equity prices. As mentioned above, for a CB pricing model that fails to capture the negative (positive) relationships between the

[14]Since decomposition CB pricing models do not actually model default events, we do not discuss this category of models here. In addition, we argue that the risky discount rates employed by this category of models are independent of the stochastic evolving firm or equity value, or are even simply constant; thus decomposition CB pricing models clearly overestimate (underestimate) default risks when firm or equity values are high (low).

[15]It is important to model dilution effects when pricing CBs. Fields and Mais (1991) find that shareholder wealth is related to the size of the private CB issue; the mean (median) ratio of the issue size divided by the preissue equity value is 23.6% (16.9%). More recently, Kazmierczak (2017) also suggests that the median ratio of the private/public CB issue size divided by the asset of profitable firms is 22.2%/19.3%. The high ratios of dilution effects could influence CB holders' conversion policies.

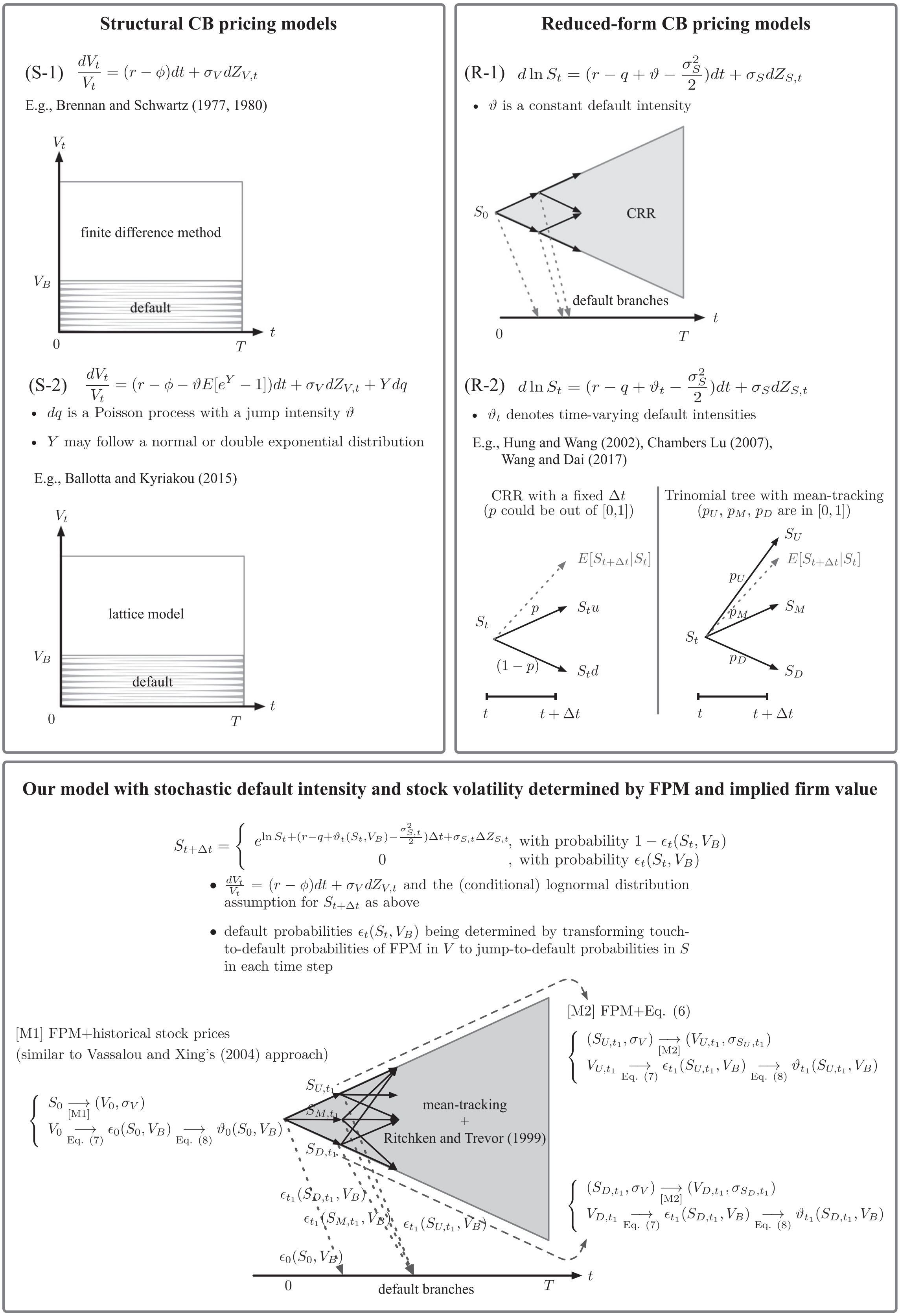

FIGURE 5 (See caption on next page)

TABLE 1 Pricing comparison between our CB pricing model and the reduced-form CB pricing model with time-varying default probabilities

CB value	Our CB pricing model	Our CB pricing model without considering dilution effect	Reduced-form CB pricing model given comparable time-varying default probabilities (does not consider dilution effect)
$S_0 = 10$	83.2593 (−0.0000%)	83.2593	78.7978 (−5.3585%)
$S_0 = 20$	84.3968 (−0.0097%)	84.4050	81.1416 (−3.8663%)
$S_0 = 30$	88.4294 (−0.0517%)	88.4752	87.1465 (−1.5018%)
$S_0 = 40$	96.3316 (−0.4497%)	96.7667	96.0070 (−0.7851%)
$S_0 = 50$	105.8459 (−0.6610%)	106.5502	106.1759 (−0.3513%)
$S_0 = 60$	119.0785 (−0.7679%)	120.0000	120.0000 (0.0000%)

Note: The default probability for each time step of the reduced-form CB pricing model illustrated in Figure 6 is the weighted average of the default probabilities at that time step in our CB pricing model illustrated in Figure 3. Since the design of reduced-form default models does not involve modeling a firm's asset value and capital structure, they are unable to estimate the impacts of dilution effects. The examined CB example here is the same as that investigated in Figure 3: $T = 3$ (years), $F = 100$, $\theta = 2$, $CP_t = 113$, $PP_t = 0$, $\sigma_{S,0} = 0.3$ (as for the comparative reduced-form CB pricing model, $\sigma_S = \sigma_{S,0} = 0.3$ being a constant over any time point), $r = 5\%$, $N_O = 10{,}000$, $N_B = 4800$, $N_C = 200$, $\omega = 0.32$, $q = c_B = c_S = \phi = 0$, $D = F(N_B + N_C) = 500{,}000$, $x = 1$, and $n = 144$. The figures in parentheses are the percentage differences relative to the CB values generated by our CB pricing model without considering the dilution effect.

Abbreviation: CB, convertible bond.

stochastically evolving default probabilities and issuing FVs or equity prices (EPV), it is likely that the CB pricing model generates unreasonable CB prices.

4 | EXTENSION TO A TWO-FACTOR CB PRICING TREE MODEL

This section describes how to incorporate Vasicek's (1977) interest rate model into the one-factor model introduced in Section 3 to construct a two-factor tree for pricing CBs subject to the interest rate risk without losing the ability to endogenously model the relationships among the default risk, FV, stochastic-volatility equity price, and dilution effect.

Under the risk-neutral measure, the interest rate stochastic process follows

$$dr_t = a(b - r_t)dt + \sigma_r dZ_{r,t}, \tag{19}$$

and its counterpart in the discrete time framework

$$\Delta r_t = a(b - r_t)\Delta t + \sigma_r \Delta Z_{r,t}, \tag{19'}$$

where r_t denotes the short rate, σ_r denotes the instantaneous standard deviation of the short rate, $dZ_{r,t}$ is a Brownian motion, and a and b represent the mean-reverting speed and long-term interest rate, respectively. Moreover, the

FIGURE 5 Illustration of proposed and traditional CB pricing models based on different default models. For simplicity, a constant riskless interest rate is considered here. For structural CB pricing models illustrated in the upper-left panel, CBs are derivatives on the firm value and defaults occur as long as the firm value falls to the default boundary, V_B. For reduced-form CB pricing models in the upper-right panel, time-varying default probabilities are first calibrated by the prevailing risky and riskless term structures of interest rates. Then the stochastic equity price (the asset class most related to the CB conversion) is employed instead to construct the pricing model since the firm value is generally nontradable. Our CB pricing model in the lower panel synthesizes the advantages of these two streams of models by integrating the first-passage default model (FPM) into an equity-price tree. First, the framework of stochastic equity prices is still utilized in our model such that we can deal with the CB conversion precisely and avoid the nontradable issue of the firm value. Second, equipped with the implied firm value at each node, we follow the structural default model to model dilution effects due to CB conversions and to formulate the theoretical relationship among default probabilities, firm value, and (stochastic-volatility) equity prices. CB, convertible bond.

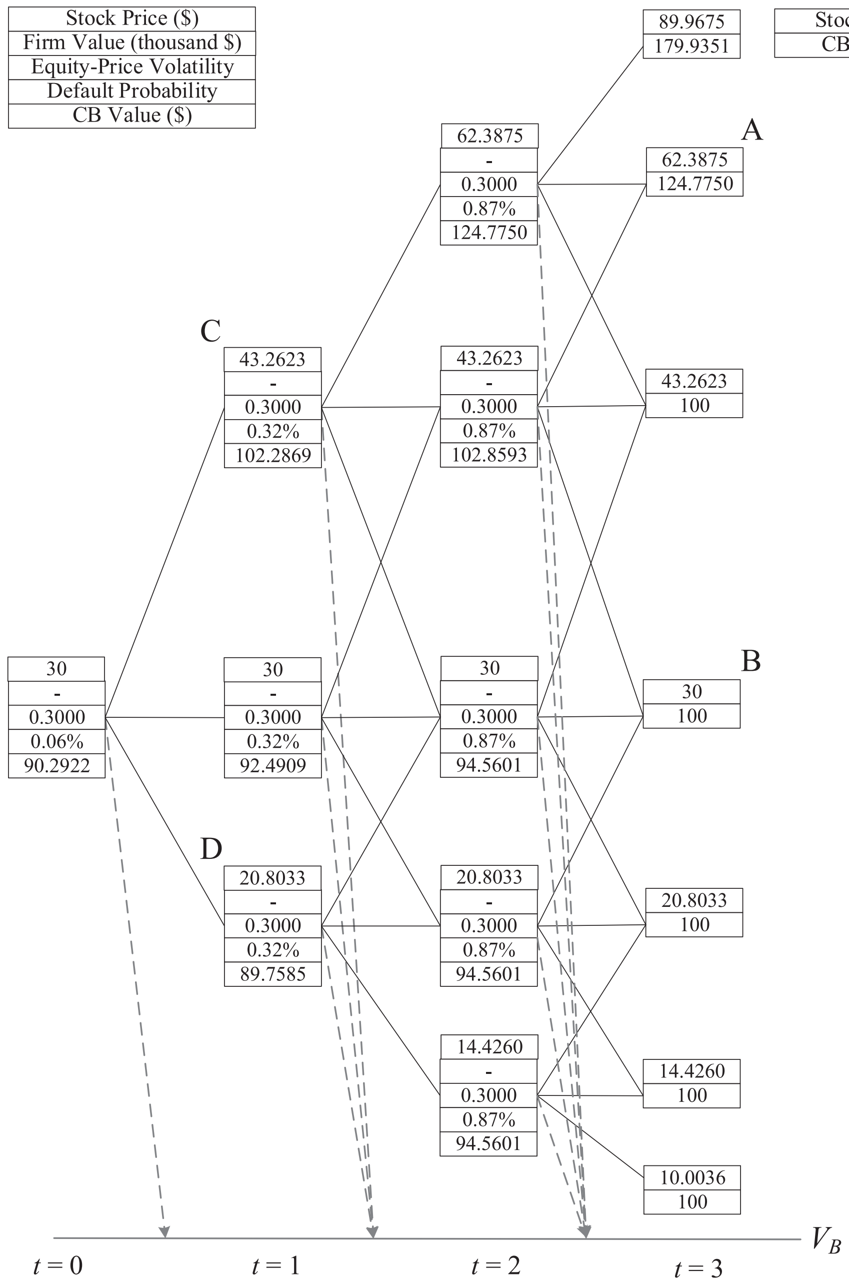

FIGURE 6 Pricing the same three-time-step example in Figure 3 based on the reduced-form CB pricing model given time-varying default probabilities. For a fair comparison with Figure 3, the default probability for each time step here is the weighted average of the default probabilities at the same time step in Figure 3. Specifically, we calculate the weighted average default probability for the time step of $(t, t + 1]$ by summing the product of the default probability of each node at the time t and the probability of reaching that node, provided no occurrence of default before t in our CB pricing model. The resulting default probabilities for the time steps of (0, 1], (1, 2], and (2, 3] are 0.06%, 0.32%, and 0.87%, respectively. Moreover, since the reduced-form CB pricing model does not take the firm value and capital structure into consideration, the dilution effect due to CB conversions cannot be modeled. CB, convertible bond.

correlation between $\Delta Z_{r,t}$ in Equation (19′) and $\Delta Z_{S,t}$ in Equation (2) is assumed to be ρ. Under the stochastic interest rate framework, the FV process becomes

$$\frac{dV_t}{V_t} = (r_t - \phi)dt + \sigma_V dZ_{V,t}. \tag{20}$$

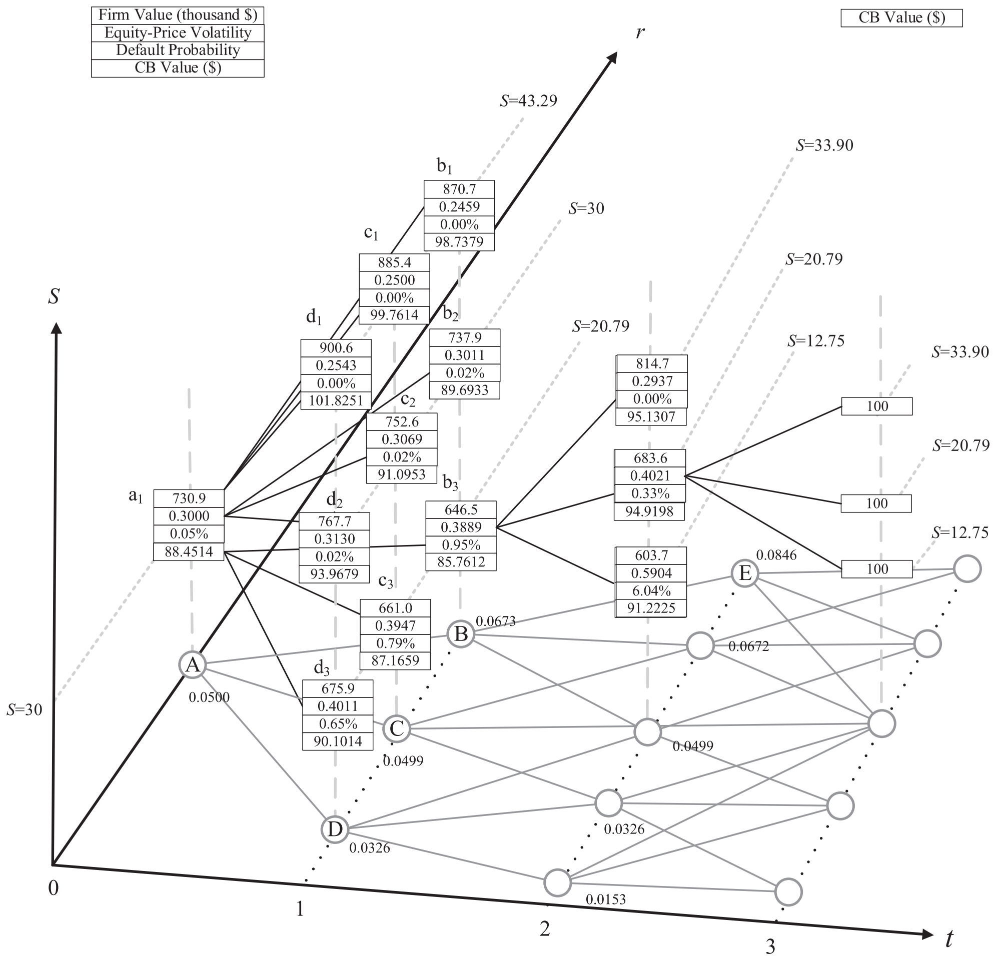

FIGURE 7 Three-time-step example of two-factor tree model. The three axes denote the equity price S, short rate r, and time t. Each tree node at maturity is represented by a rectangle that lists the corresponding CB value. All other tree nodes are represented by four-row rectangles, whose meanings are represented in the upper-leftmost legend. The parameter settings mirror those in Figure 3 in addition to the settings of the Vasicek short rate model and the correlation as follows: $a = 0.05$, $b = 0.05$, $\sigma_r = 0.01$, $r_0 = 5\%$, and $\rho = 0.1$. V_0 (\$730.9 thousand dollars) and σ_V (0.1225) are solved based on Equations (21) and (6) given the values of S_0 and $\sigma_{S,0}$. The interest rate tree is denoted by gray lines and circles. The two-factor tree is denoted by black lines and rectangles. The branches representing the scenarios of reaching the default boundary are omitted for simplicity. CB, convertible bond.

We adopt Hull and While (1994) interest rate tree construction method to construct an interest rate tree in the $r-t$ plane of Figure 7 (denoted by gray lines and circles) to implement the Vasicek model in Equation (19′). Then we incorporate the equity-price factor by building a three-dimensional tree based on this interest rate tree. Specifically, we add the trinomial equity-price tree introduced in Section 3 to model the equity-price evolution to form our two-factor tree, the branches and the nodes of which are plotted with black lines and rectangles. For example, node a_1 in the two-factor tree projects to node A in the interest rate tree; nodes b_1, b_2, and b_3 project to node B. Thus the short rates for nodes a_1 and A equal 0.0500; the short rates for b_1, b_2, b_3, and B are all equal to 0.0673. Nine descendant nodes emit from node a_1 in the two-factor tree. The branches to nodes b_1, b_2, and b_3 reflect the scenarios in which the short rate moves from 0.0500 (node A) to 0.0673 (node B) and the equity price moves from \$30.00 to \$43.29, \$30.00, and \$20.79, respectively. In addition, we must also modify the down-and-out call option formula in Equation (5) and the first-passage DP function in Equation (7) by incorporating the Vasicek interest rate model when constructing our $S-r$ two-factor CB pricing tree model.

To determine the relationship among the FV, the equity price, and its volatility under the stochastic interest rate environment instead of the constant one discussed in Section 3, we replace the down-and-out call option pricing formula under the constant interest rate in Equation (5) by the semiclosed-form pricing formulas in Bernard et al. (2008) as

$$E_t = P(t, T)\left(E^T\left[(V_T - D)^+ - (V_T - D)^+ 1_{V_{\min} \le V_B}\right]\right), \tag{21}$$

where $P(t, T)$ denotes the value at time t of a risk-free unit-face-value zero-coupon bond that matures at time T governed by the Vasicek interest rate process, and $E^T[\cdot]$ represents the expectation under the T-forward-neutral measure. This equation is numerically evaluated with

$$\begin{aligned} E^T[(V_T - D)^+] &= \mathcal{K}(M_T, U_T, D) - D \times N\left(\frac{M_T - \ln(D)}{\sqrt{U_T}}\right), \\ E^T\left[(V_T - D)^+ 1_{V_{\min} < V_B}\right] &= \sum_{j=0}^{n_T} \sum_{i=0}^{n_r} \mathcal{K}\left(\hat{\mu}_{t_j,T}, \hat{u}_{t_j,T}, D\right) q^d(i, j) - D \times \sum_{j=0}^{n_T} \sum_{i=0}^{n_r} N\left(\frac{M_T - \ln(D)}{\sqrt{\hat{u}_{t_j,T}}}\right) q^d(i, j), \end{aligned} \tag{22}$$

where the interval $[t, T]$ is equally subdivided into n_T subperiods, and the interest rate space is discretely expressed by $(n_r + 1)$ representative values located equally spaced between the prespecified short rate lower bound $r_{\min}$ and upper bound $r_{\max}$. The definition of $\mathcal{K}(\cdot,\cdot,\cdot)$ in Equation (22) is

$$\mathcal{K}(m, \sigma, d) = \exp\left(m + \frac{\sigma^2}{2}\right) N\left(\frac{m + \sigma^2 - \ln(d)}{\sigma}\right).$$

Here, $\hat{\mu}_{t_j,T}$ and $\hat{u}_{t_j,T}$ are, respectively, the mean and the variance of $\ln(V_t)$ conditional on the filtration F_{t_j}

$$\hat{\mu}_{t_j,T} = E^T[\ln V_t | F_{t_j}],$$

$$\hat{u}_{t_j,T} = \mathrm{var}^T(\ln V_t | F_{t_j}),$$

and M_T and U_T are, respectively, the unconditional mean and the variance of $\ln(V_T)$ under the T-forward-neutral measure. Finally, term $q^d(i, j)$ is computed iteratively according to Bernard et al. (2008). Detailed formulas for $\hat{\mu}_{t_j,T}$, $\hat{u}_{t_j,T}$, M_T, U_T, and $q^d(i, j)$ are presented in Appendix C.

Similar to the one-factor model introduced in Section 3, we employ the equity price and its volatility $\sigma_{S,0}$ observed or accessed from the market on the issuance date to derive the FV $V_0 = \$730.9$ thousand dollars and its volatility $\sigma_V = 0.1225$ via Equations (21) and (6).[16] Then for each possible short rate at time 1, we construct a trinomial branching structure by the method introduced in Section 3.1 that connects to b_1, b_2, and b_3 (reflecting the short rate at node B), c_1, c_2, and c_3 (node C), and d_1, d_2, and d_3 (node D). For each two-factor-tree node at time step 1, say b_1, the FV volatility σ_V and the simulated equity price \$43.29 are substituted into Equations (21) and (6) to calculate the implied FV \$870.7 thousand dollars and the EPV 0.2459. The above steps are repeatedly applied to form our two-factor tree. For each node of the tree, the DP (or the probability for the continuous process Equation 20 to reach the default boundary within a time step) under the Vasicek short rate model can be evaluated by the method proposed in Collin-Dufresne and Goldstein (2001). Specifically, the probability for the FV V_t to fall below the default boundary V_B at time τ before time s is

$$p_t\left(\tau \le s\right) = \sum_{i=1}^{n} \varpi_i, \tag{23}$$

where

$$\varpi_1 = N(a_1),$$

[16]Again, Vassalou and Xing's (2004) iteration method is not used in this hypothetical example since for simplicity, we make no assumption concerning historical equity prices.

$$\varpi_i = N(a_i) - \sum_{j=1}^{i-1} \omega_j N(b_{ij}), \quad i = 2, 3, \ldots, n,$$

$$a_i = \frac{\ln\left(\frac{V_B}{V_t}\right) - M\left(\frac{i(s-t)}{n}, (s-t)\right)}{\sqrt{S\left(\frac{i(s-t)}{n}\right)}},$$

$$b_{ij} = \frac{M\left(\frac{j(s-t)}{n}, (s-t)\right) - M\left(\frac{i(s-t)}{n}, (s-t)\right)}{\sqrt{S\left(\frac{i(s-t)}{n}\right) - S\left(\frac{j(s-t)}{n}\right)}},$$

with

$$\begin{aligned} M(u, v) = &\left(\frac{ab - \phi - \rho\sigma_V\sigma_r}{a} - \frac{\sigma_r^2}{a^2} - \frac{\sigma_V^2}{2}\right)u + \left(\frac{\rho\sigma_V\sigma_r}{a^2}\right)\exp(-av)(\exp(au) - 1) \\ &+\left(\frac{r_t}{a} - \frac{b}{a} + \frac{\sigma_r^2}{a^3}\right)(1 - \exp(-au)) - \left(\frac{\sigma_r^2}{2a^3}\right)\exp(-av)\exp(au) \\ &(1 - \exp(-au))^2, \\ S(u) \quad = &\left(2\frac{\rho\sigma_V\sigma_r}{a} + \frac{\sigma_r^2}{a^2} + \sigma_V^2\right)u - 2\left(\frac{\rho\sigma_V\sigma_r}{a^2} + \frac{\sigma_r^2}{a^3}\right)(1 - \exp(-au)) \\ &+\left(\frac{\sigma_r^2}{2a^3}\right)(1 - \exp(-2au)). \end{aligned}$$

The CB value can now be evaluated by the two-factor tree model in Figure 7 with 10-nomial backward induction similar to Equation (15). Taking node a_1, for example, its holding value of a CB is

$$HV^{\mathrm{a}_1} = e^{-r^{\mathrm{a}_1}\Delta t}\left(\epsilon^{\mathrm{a}_1}\omega xF + (1 - \epsilon^{\mathrm{a}_1})\begin{pmatrix} P_{\mathrm{Uu}}^{\mathrm{a}_1}CV^{\mathrm{b}_1} + P_{\mathrm{Um}}^{\mathrm{a}_1}CV^{\mathrm{c}_1} + P_{\mathrm{Ud}}^{\mathrm{a}_1}CV^{\mathrm{d}_1} \\ +P_{\mathrm{Mu}}^{\mathrm{a}_1}CV^{\mathrm{b}_2} + P_{\mathrm{Mm}}^{\mathrm{a}_1}CV^{\mathrm{c}_2} + P_{\mathrm{Md}}^{\mathrm{a}_1}CV^{\mathrm{d}_2} \\ +P_{\mathrm{Du}}^{\mathrm{a}_1}CV^{\mathrm{b}_3} + P_{\mathrm{Dm}}^{\mathrm{a}_1}CV^{\mathrm{c}_3} + P_{\mathrm{Dd}}^{\mathrm{a}_1}CV^{\mathrm{d}_3} \end{pmatrix}\right), \tag{24}$$

where r^{a_1} denotes the short rate 0.0500 at node a_1 (or node A at the interest rate tree) and ϵ^{a_1}, evaluated based on Equation (23) given $t = 0$ and $s = \Delta t$, denotes the DP of 0.05% within the subsequent Δt time period. In addition, $P_{\mathrm{Um}}^{\mathrm{X}}$ is the branching probability for node X in which the equity price goes up (denoted by the lower script U) and the interest rate goes to the middle (denoted by the lower script m) without involving defaults at the next time step; for example, $P_{\mathrm{Um}}^{\mathrm{a}_1}$ denotes the probability of moving from node a_1 to c_1. Other branching probabilities P_{Ij}^{X} for $I = \mathrm{U, M, D}$ and $j = \mathrm{u, m, d}$ can be interpreted in the same way. In addition, if the examined node (e.g., node a_1 in Equation 24) is on a coupon-payment date, the resulting holding value in Equation (24) is adjusted upward by $\frac{c_{\mathrm{B}}}{2}F$ to reflect the coupon income received by CB holders.

To determine the nine branching probabilities that simultaneously calibrate the correlation ρ between the logarithmic equity price and the short rate, we modify the method proposed in Wang and Dai (2017).[17] Specifically, each branching probability is first derived by multiplying the corresponding marginal probability of the equity price with that of the interest rate as if they were independent, after which the term ε is added to or subtracted from some branching probabilities to calibrate the correlation without changing the marginal probabilities of the equity price and the interest rate as illustrated in Table 2. The intuitions for the three possible ε probability adjustments illustrated in panels A, B, and C are as follows. Since P_{U} and P_{D} (P_{u} and P_{d}) are smaller than P_{M} (P_{m}) in the mean-tracking method in Figure 2 (Hull & While 1994 model),[18] the adjusted branching probabilities $P_{Ij} = P_I \times P_j - \varepsilon$ could be negative for

[17]Wang and Dai (2017) simulate Black et al. (1990) lognormal interest rate process with a binomial tree, whereas this paper employs Hull and While (1994) trinomial tree model to simulate the Vasicek interest rate process.
[18]Note that in the two-factor model, one must modify the approach to determine P_{U}, P_{M}, and P_{D} in Section 3.1.2 by replacing the constant r with the stochastic evolving r_t when calculating $E[\ln S_{t+\Delta t}|S_t, r_t]$.

TABLE 2 Branching probability adjustment to calibrate correlation between logarithmic equity price and interest rate

$S_{t+\Delta t} \setminus r_{t+\Delta t}$	r_u	r_m	r_d	Marginal probability for $S_{t+\Delta t}$
Panel A: Normal case				
$\ln S_U$	$P_{Uu} = P_U P_u - \varepsilon$	$P_{Um} = P_U P_m$	$P_{Ud} = P_U P_d + \varepsilon$	P_U
$\ln S_M$	$P_{Mu} = P_M P_u$	$P_{Mm} = P_M P_m$	$P_{Md} = P_M P_d$	P_M
$\ln S_D$	$P_{Du} = P_D P_u + \varepsilon$	$P_{Dm} = P_D P_m$	$P_{Dd} = P_D P_d - \varepsilon$	P_D
Marginal probability for $r_{t+\Delta t}$	P_u	P_m	P_d	1
Panel B: If P_{Uu} is negative in the normal case				
$\ln S_U$	$P_{Uu} = P_U P_u$	$P_{Um} = P_U P_m$	$P_{Ud} = P_U P_d$	P_U
$\ln S_M$	$P_{Mu} = P_M P_u - \varepsilon$	$P_{Mm} = P_M P_m$	$P_{Md} = P_M P_d + \varepsilon$	P_M
$\ln S_D$	$P_{Du} = P_D P_u + \varepsilon$	$P_{Dm} = P_D P_m$	$P_{Dd} = P_D P_d - \varepsilon$	P_D
Marginal probability for $r_{t+\Delta t}$	P_u	P_m	P_d	1
Panel C: If P_{Dd} is negative in the normal case				
$\ln S_U$	$P_{Uu} = P_U P_u - \varepsilon$	$P_{Um} = P_U P_m$	$P_{Ud} = P_U P_d + \varepsilon$	P_U
$\ln S_M$	$P_{Mu} = P_M P_u + \varepsilon$	$P_{Mm} = P_M P_m$	$P_{Md} = P_M P_d - \varepsilon$	P_M
$\ln S_D$	$P_{Du} = P_D P_u$	$P_{Dm} = P_D P_m$	$P_{Dd} = P_D P_d$	P_D
Marginal probability for $r_{t+\Delta t}$	P_u	P_m	P_d	1

Note: In the three panels, the term ε is added to or subtracted from different branching probabilities to calibrate the correlation without triggering the infeasible branching probability problem. Superscript X for the branching probability P_{Ij}^{X} for $I = \mathrm{U, M, D}$ and $j = \mathrm{u, m, d}$ is omitted for simplicity.

$I = \mathrm{U, D}$ and $j = \mathrm{u, d}$. To address this negative probability problem, we adjust the middle branching probabilities associated with $\ln S_M$ instead of the corner branching probabilities as in panel B or C. Our numerical experiments verify that this approach generates feasible branching probabilities under all of our parameter settings.

For an arbitrary node with equity price S_t and interest rate r_t,

$$\rho = \frac{E[\ln(S_{t+\Delta t})r_{t+\Delta t}|S_t, r_t] - E[\ln S_{t+\Delta t}|S_t, r_t]E[r_{t+\Delta t}|S_t, r_t]}{\sqrt{\mathrm{var}(\ln S_{t+\Delta t}|S_t, r_t)\mathrm{var}(r_{t+\Delta t}|S_t, r_t)}}$$

for the equity price and the interest rate at the next time step. In addition, based on the two-factor tree structure, one can express

$$\begin{aligned} E[\ln(S_{t+\Delta t})r_{t+\Delta t}|S_t, r_t] &= P_{Uu}\ln(S_U)r_u + P_{Um}\ln(S_U)r_m + P_{Ud}\ln(S_U)r_d \\ &\quad + P_{Mu}\ln(S_M)r_u + P_{Mm}\ln(S_M)r_m + P_{Md}\ln(S_M)r_d \\ &\quad + P_{Du}\ln(S_D)r_u + P_{Dm}\ln(S_D)r_m + P_{Dd}\ln(S_D)r_d, \\ E[\ln S_{t+\Delta t}|S_t, r_t] &= P_U \ln S_U + P_M \ln S_M + P_D \ln S_D, \\ E[r_{t+\Delta t}|S_t, r_t] &= P_u r_u + P_m r_m + P_d r_d, \\ \mathrm{var}(\ln S_{t+\Delta t}|S_t, r_t) &= P_U(\ln S_U)^2 + P_M(\ln S_M)^2 + P_D(\ln S_D)^2 - (E[\ln S_{t+\Delta t}|S_t, r_t])^2, \end{aligned}$$

$$\mathrm{var}(r_{t+\Delta t}|S_t, r_t) = P_u r_u^2 + P_m r_m^2 + P_d r_d^2 - (E[r_{t+\Delta t}|S_t, r_t])^2,$$

where the outgoing branches from the node with the equity price S_t (interest rate r_t) connect to three equity prices S_U, S_M, and S_D (interest rates r_u, r_m, and r_d) with probabilities P_U, P_M, and P_D (P_u, P_m, and P_d), respectively, in the individual equity-price (interest rate) tree. By substituting the above expressions of $E[\ln(S_{t+\Delta t})r_{t+\Delta t}|S_t, r_t]$, $E[\ln S_{t+\Delta t}|S_t, r_t]$, $E[r_{t+\Delta t}|S_t, r_t]$, $\mathrm{var}(\ln S_{t+\Delta t}|S_t, r_t)$, and $\mathrm{var}(r_{t+\Delta t}|S_t, r_t)$ into the definition of ρ, the adjustment term can be solved as

$$\varepsilon = \frac{\rho\sqrt{\mathrm{var}(\ln S_{t+\Delta t}|S_t, r_t)\mathrm{var}(r_{t+\Delta t}|S_t, r_t)}}{\gamma},$$

where

$$\gamma = \begin{cases} -\ln(S_\mathrm{U})r_\mathrm{u} + \ln(S_\mathrm{U})r_\mathrm{d} + \ln(S_\mathrm{D})r_\mathrm{u} - \ln(S_\mathrm{D})r_\mathrm{d} & \text{for Case A in Table 2,} \\ -\ln(S_\mathrm{M})r_\mathrm{u} + \ln(S_\mathrm{M})r_\mathrm{d} + \ln(S_\mathrm{D})r_\mathrm{u} - \ln(S_\mathrm{D})r_\mathrm{d} & \text{for Case B in Table 2,} \\ -\ln(S_\mathrm{U})r_\mathrm{u} + \ln(S_\mathrm{U})r_\mathrm{d} + \ln(S_\mathrm{M})r_\mathrm{u} - \ln(S_\mathrm{M})r_\mathrm{d} & \text{for Case C in Table 2.} \end{cases}$$

5 | NUMERICAL RESULTS

5.1 | Illustrative example for a two-factor tree model

To better understand how our two-factor tree model works, Table 3 lists the CB values (CV_t), equity-price volatilities ($\sigma_{S,t}$), default probabilities (ϵ_t), and FVs (V_t) for each node (except the last period) of the two-factor tree in Figure 7 for further analysis. Note that each tree node maps to a prevailing equity price S_t and short rate r_t. Similar to the results in Figure 3, the negative (positive) relationships between the DP and the FV or the equity price (the EPV) are retained under the stochastic interest rate environment. For example, at time 1, the increment of the equity price from \$20.7908 to \$43.2885 increases the FV from \$646,523 to \$870,735, decreases the DP from 0.95% to 0.00%, decreases the EPV from 0.3889 to 0.2459, and hence increases the CB value from \$85.7612 to \$98.7379 given a prevailing short rate of 0.0673.

In addition, the impact of the stochastic interest rate can also be examined. A decrement of the prevailing short rate from 0.0673 to 0.0326 increases the CB value from \$85.7612 to \$90.1014, increases the EPV from 0.3889 to 0.4011, decreases the DP from 0.95% to 0.65%, and increases the FV from \$646,523 to \$675,929 given a prevailing equity price of \$20.7908 at time 1. However, it can also be found in Table 3 that the positive relationship between the risk-free interest rate and the DP may not hold at time 2, the time step just before the maturity. This is because there are two opposite impacts on the DP caused by a decrement of the risk-free interest rate. One is the smaller-discount-rate effect: a smaller risk-free interest rate (thus a smaller discount rate) yields a higher FV (a phenomenon evident in Table 3), which reduces the likelihood of default since the distance between the FV and the default boundary increases. The other is the smaller-drift-term effect: a smaller risk-free interest rate means a smaller drift term for the FV in Equation (20), which increases the probability of the FV reaching the default boundary in the subsequent Δt period of time. The smaller-drift-term effect only influences the subsequent single time step, whereas the smaller-discount-rate effect can be accumulated in multiple time steps backward from the maturity and thus plays a dominating role to determine the DP with respect to the change of the risk-free interest for most nodes. Nonetheless, for nodes at the time step just before the maturity, both the smaller-discount-rate and smaller-drift-term effects take effect only for the remaining Δt period of time. These two opposite forces may even result in a nonmonotonic relationship between the risk-free interest rate and the DP especially when the equity price is relatively low. For example, for time 2 in Table 3, the default probabilities are 6.06%, 6.03%, and 6.04% for the interest rates being 0.0846, 0.0672, and 0.0499, respectively, given an equity price of \$12.7508.

5.2 | Sensitivity analyses

The respective impacts of changing initial equity prices S_0, equity-price volatilities $\sigma_{S,0}$, recovery rate ω, correlations between logarithmic equity prices and short rates ρ, and the parameters of Vasicek's interest rate models (a, b, σ_r, and r_0) on CB values are analyzed in Table 4.

Recall that a CB possesses attributes of both equity and debt, as it is a bond with a conversion option. Therefore, increments in the CB's underlying equity price S_0 significantly raise the values of both the embedded conversion option and hence the CB value. On the other hand, increments in the long-term interest rate level b and the initial short rate r_0 reduce the CB (debt) value. Increasing the underlying EPV $\sigma_{S,0}$ results in the following two effects: First, it increases the value of the conversion option and thus the CB value. Second, a higher $\sigma_{S,0}$ implies a higher probability of encountering a relatively low equity price, which yields a higher DP and thus diminishes the CB value. The net of these two different effects results in the hump-shaped CB values with respect to $\sigma_{S,0}$. To our knowledge, our CB pricing model is the first to appropriately capture these two different effects associated with the equity volatility. In contrast, the CB value monotonically decreases with the interest rate volatility σ_r. Increments in the mean-reversion parameter a implicitly

TABLE 3 Details for pricing 3-year CB with a two-factor tree in Figure 4

$t = 0$	
$S_t = 30.0000,\ r_t = 0.0500$	
CV_t	88.4514
$\sigma_{S,t}$	0.3000
ϵ_t	0.05%
V_t	730,868

$t = 1$							
$S_t \setminus r_t$	**0.0673**	**0.0499**	**0.0326**	**$S_t \setminus r_t$**	**0.0673**	**0.0499**	**0.0326**
	CV_t				$\sigma_{S,t}$		
43.2885	98.7379	99.7614	101.8251	43.2885	0.2459	0.2500	0.2543
30.0000	89.6933	91.0953	93.9679	30.0000	0.3011	0.3069	0.3130
20.7908	85.7612	87.1659	90.1014	20.7908	0.3889	0.3947	0.4011
ϵ_t				**V_t**			
43.2885	0.00%	0.00%	0.00%	43.2885	870,735	885,409	900,575
30.0000	0.02%	0.02%	0.02%	30.0000	737,893	752,558	767,711
20.7908	0.95%	0.79%	0.65%	20.7908	646,523	660,973	675,929

$t = 2$					
$S_t \setminus r_t$	**0.0846**	**0.0672**	**0.0499**	**0.0326**	**0.0153**
	CV_t				
62.4630	123.7320	123.7568	123.7814	123.8057	123.8297
43.2885	98.9096	100.054	101.2644	102.544	103.8959
33.9004	91.8919	93.4973	95.1307	96.7925	98.4834
30.0000	91.8896	93.4944	95.1271	96.7882	98.4781
20.7908	91.7215	93.3079	94.9198	96.5576	98.2214
12.7508	88.1077	89.6648	91.2225	92.7838	94.3513
$\sigma_{S,t}$					
$S_t \setminus r_t$	**0.0846**	**0.0672**	**0.0499**	**0.0326**	**0.0153**
62.4630	0.2122	0.2137	0.2153	0.2169	0.2185
43.2885	0.2521	0.2543	0.2565	0.2588	0.2612
33.9004	0.2880	0.2909	0.2937	0.2967	0.2996
30.0000	0.3096	0.3128	0.3160	0.3193	0.3227
20.7908	0.3930	0.3975	0.4021	0.4068	0.4115
12.7508	0.5883	0.5886	0.5904	0.5932	0.5969
ϵ_t					
$S_t \setminus r_t$	**0.0846**	**0.0672**	**0.0499**	**0.0326**	**0.0153**
62.4630	0.00%	0.00%	0.00%	0.00%	0.00%
43.2885	0.00%	0.00%	0.00%	0.00%	0.00%
33.9004	0.00%	0.00%	0.00%	0.00%	0.00%
30.0000	0.00%	0.01%	0.01%	0.01%	0.01%

TABLE 3 (Continued)

ϵ_t					
$S_t \setminus r_t$	**0.0846**	**0.0672**	**0.0499**	**0.0326**	**0.0153**
20.7908	0.27%	0.30%	0.33%	0.36%	0.39%
12.7508	6.06%	6.03%	6.04%	6.09%	6.17%
V_t					
$S_t \setminus r_t$	**0.0846**	**0.0672**	**0.0499**	**0.0326**	**0.0153**
62.4630	1,084,490	1,092,325	1,100,294	1,108,398	1,116,641
43.2885	892,744	900,579	908,548	916,653	924,895
33.9004	798,864	806,699	814,668	822,772	831,014
30.0000	759,860	767,695	775,664	783,768	792,010
20.7908	667,789	675,619	683,583	691,681	699,918
12.7508	588,338	595,910	603,653	611,560	619,624

Note: The convertible bond value CV_t, the equity-price volatility $\sigma_{S,t}$, the default probability ϵ_t, and the FV V_t for each node of the first three time steps of the two-factor tree are listed for analysis.

TABLE 4 Sensitivity analysis of CB prices with respect to different parameters

S_0	CB value	$\sigma_{S,0}$	CB value	ω	CB value	ρ	CB value
20	80.0660	0.2	83.5115	0.1	85.1407	−0.2	85.3553
25	81.9850	0.25	84.7077	0.2	85.2776	−0.1	85.3845
30	85.4420	0.3	85.4420	0.3	85.4146	0	85.4141
35	90.1307	0.35	85.3622	0.4	85.5516	0.1	85.4420
40	95.8399	0.4	84.0967	0.5	85.6886	0.2	85.4694
a	**CB value**	b	**CB value**	σ_r	**CB value**	r_0	**CB value**
0.025	85.4143	0.03	85.7346	0.0025	88.9563	0.03	89.3265
0.05	85.4279	0.04	85.5864	0.005	87.1775	0.04	87.3151
0.1	85.4420	0.05	85.4420	0.01	85.4420	0.05	85.4420
0.2	85.4566	0.06	85.2957	0.015	83.9643	0.06	83.6648
0.4	85.4712	0.07	85.1474	0.02	82.4884	0.07	81.9862

Note: Parameters not specified in the table follow the base case as follows: $S_0 = 30$, $\sigma_{S,0} = 0.3$, $n = 6$, $F = 100$, $\theta = 2$, $CP = 113$, $N_O = 10{,}000$, $N_B = 4800$, $N_C = 200$, $a = 0.05$, $b = 0.05$, $\sigma_r = 0.01$, $r_0 = 5\%$, $\rho = 0.1$, $\omega = 0.32$, $\phi = q = c_B = c_S = 0$, $D = F(N_B + N_C) = 500{,}000$, and $x = 1$.

Abbreviation: CB, convertible bond.

reduce the interest rate volatility and hence increase the CB value. Additionally, a higher recovery ω leads to a high CB value as expected. Finally, a CB is slightly more valuable with increments in the correlation between logarithmic equity prices and interest rates, which is consistent with the findings in Chambers and Lu (2007).

5.3 | Empirical cases

We next apply the proposed CB pricing model to evaluate a CB contract issued by DHR, which is examined in Wang and Dai (2017). We revisit this empirical case and calculate our CB pricing model for the DHR zero-coupon CB (thus $c_B = 0$) on January 22, 2009. See Wang and Dai (2017) for why the DHR CB contract and the examined date of January 22, 2009 are chosen. The parameter values employed to price the DHR CB on January 22, 2009 are summarized in Table 5.

TABLE 5 Parameters for pricing DHR zero-coupon CB on January 22, 2009

Name	Data source	Value and explanation					
(a) *CB parameters*							
Market price	Bloomberg	84.0000					
Years to maturity (T)		12					
Face value (F)		100			Normalized to 100		
Conversion ratio (θ)		1.45352 shares			For 100 dollars of face value		
Rating and seniority		A (senior and unsecured)					
Call schedule and call price (CP)		01/22/10	01/22/11	01/22/12	01/22/13	01/22/14	01/22/15
		77.128	78.970	80.857	82.789	84.767	86.792
		01/22/16	01/22/17	01/22/18	01/22/19	01/22/20	
		88.865	90.989	93.162	95.388	97.667	
Put schedule and put price (PP)		01/22/11					
		78.970					
(b) *Equity-price parameters*							
Prevailing equity price (S_0)	Yahoo! finance website	51.7400					
EPV ($\sigma_{S,0}$)		28.1562%			Standard deviation of log differences in daily equity prices ($\Delta \ln S_t$) in the previous 3 years		
Dividend yield (q)		0.0725%			Ratio of average annual cash dividends over average equity price in previous 3 years		
(c) *Interest rate parameters*							
Correlation between $d \ln S_t$ and dr_t (ρ)	Website of US Treasury Department	9.5214%			Correlation of daily $\Delta \ln S_t$ and Δr_t in previous 3 years, where Δr_t is calculated as daily difference of 1-month treasury yields		
Term structure of risk-free zero rates (%)	Bloomberg	3M	6M	1Y	2Y	3Y	4Y
		0.10	0.29	0.40	0.72	1.10	1.39
		5Y	7Y	10Y	12Y	15Y	
		1.62	2.06	2.69	3.08	3.60	
(d) *Estimated parameter values in the Vasicek model*							
a	Calibrated by this study	0.082914			Use best-fitting algorithm to minimize difference between prevailing term structure and theoretic term structure implied by Vasicek interest rate model		
b		0.088458					
σ_r		0.012590					
r_0		0.000659					
(e) *Capital structure information, payout ratio, and recovery rate*							
N_O (number of equity shares)	Estimated by this study	354,487,000 shares			DHR 2008 Annual Report (p. 68)		
N_C (number of examined CB contracts)		6,200,000 contracts (face value F assumed to be 100)			DHR 2008 Annual Report (p. 49)		
N_B (number of virtual coupon-bearing bond contracts, approximating for other liabilities)		70,615,660 contracts (face value F assumed to be 100)			DHR 2008 Annual Report (pp. 49 and 66)		
c_S (coupon rate for virtual coupon-bearing bond)		1.8434%			DHR 2008 Annual Report (pp. 65–66)		

TABLE 5 (Continued)

Name	Data source	Value and explanation	
ϕ (cash payment yield from firm value)		0.6070%	DHR 2008 Annual Report (pp. 65–67) and DHR equity price on December 31 of 2008
ω (recovery rate)	Wang and Dai (2017)	49.54%	

Abbreviations: CB, convertible bond; DHR, Danaher Corporation.

Parameters that pertain exclusively to our model are explained as follows. First, given the term structure of risk-free zero rates on that day, we can solve a, b, σ_r, and r_0 by minimizing the sum of the squared differences between the market risk-free zero rates and theoretical risk-free zero rates based on the Vasicek model. Next, the theoretical term structure based on the Vasicek model with the above solved a, b, σ_r, and r_0 is used as input to construct Hull and While (1994) interest rate tree. Second, according to DHR's annual reports (Danaher Corporation, 2007, 2008, 2009), we can estimate the parameters related to its capital structure, cash payment yield from FV, and average coupon rate for debts other than the examined CB. In Danaher Corporation (2009), by the end of 2008, there are 354,487,000 DHR shares outstanding. The total liability amount is $7,681,566,000, which consists of the examined zero-coupon CB with a principal of $620,000,000 and other liabilities of $7,061,566,000. Here we simplify other liabilities as a virtual coupon-bearing bond whose coupon rate is 1.8434% (paid semiannually), estimated as the annual interest expenses ($130,174,000) divided by the amount of other liabilities ($7,061,566,000). The cash payment yield from FV 0.6070% is estimated as the dividends payment ($38,259,000) plus the interest expense ($130,174,000) divided by the market value of the firm assets, which is approximated as the number of outstanding shares multiplied by the equity price on December 31 of 2008 ($56.61) plus the total liability amount. We follow Wang and Dai (2017) in assuming that the recovery rate of DHR is 49.54%, the average recovery rate for an A-rated firm. Furthermore, we follow Longstaff and Schwartz (1995) and Wang et al. (2014) to assume the default boundary $V_B = D$.

In addition, we employ Equation (21) to implement Vassalou and Xing's (2004) method to estimate a robust σ_V based on the FV over the past 3-year period, that is, from January 22, 2006 to January 21, 2009. To this end, on each trading day from January 22, 2006 to January 21, 2009, we collect the daily closing equity price for DHR from the Yahoo! finance website[19] and also the daily risk-free zero rates from Bloomberg and execute the calibration process to estimate a, b, σ_r, and r_0 of the Vasicek model. Moreover, the correlation of $\Delta \ln S_t$ and Δr_t,[20] ρ, is calculated based on daily data over the previous 3 years. Finally, we use the DHR annual report in the previous year (e.g., 2007) to estimate the corresponding parameters of the capital structure and cash payment yield from the FV used for each trading day in the following year (e.g., 2008). The estimation methodology is identical to that described in the preceding paragraph. These parameters are summarized in Table 6. We thus obtain the FV at time 0 (on January 22, 2009) and a constant σ_V of $25,395,950,363 and 24.0222%, respectively.

Finally, the theoretical CB value estimated by the proposed tree model is $84.1852 and $83.8555 given the number of time steps n of 24 and 48, respectively. Compared with the pricing result of $84.3198 in Wang and Dai (2017), our structural CB model yields pricing results closer to the actual market price of $84.0000 on January 22, 2009, perhaps due to its superior ability to formulate the stylized relationship among all stochastic processes and thus capture the consensus of DHR CB traders' default-event expectations.

6 | CONCLUSION AND FUTURE WORK

It is difficult but critical to price a CB and to simultaneously model the complex relationships among the DP (unobservable) issuing FV, equity price, EPV, and the dilution effect due to conversion. Our proposed two-factor (equity price and interest rate) CB pricing tree treats the issuer's equity value (derived as the product of the equity price and the number of outstanding shares) as a down-and-out call on the issuing FV to endogenously solve for the implied

[19]On the basis of the same time series of DHR equity prices, the EPV $\sigma_{S,0}$ on the CB pricing day of January 22, 2009 is estimated as the standard deviation of the log differences in daily equity prices ($\Delta \ln S_t$).

[20]Δr_t is calculated as the daily difference of 1-month Treasury yields.

TABLE 6 Capital structure and payout ratio parameters for determining σ_V based on Vassalou and Xing (2004)

Name	Data source	Value and explanation	
(a) *Capital structure information and payout ratio used in 2009/01/01–2009/1/21*			
N_O (number of equity shares)	Estimated by this study	354,487,000 shares	DHR 2008 Annual Report (p. 68)
N_C (number of examined CB contracts)		6,200,000 contracts (face value F assumed to be 100)	DHR 2008 Annual Report (p. 49)
N_B (number of virtual coupon-bearing bond contracts, approximating for other liabilities)		70,615,660 contracts (face value F assumed to be 100)	DHR 2008 Annual Report (pp. 49 and 66)
c_S (coupon rate for virtual coupon-bearing bond)		1.8434%	DHR 2008 Annual Report (pp. 65–66)
ϕ (cash payment yield from firm value)		0.6070%	DHR 2008 Annual Report (pp. 65–67) and DHR equity price on December 31 of 2008 (56.61)
(b) *Capital structure information and payout ratio used in 2008/01/01–2008/12/31*			
N_O (number of equity shares)	Estimated by this study	352,608,000 shares	DHR 2007 Annual Report (p. 63)
N_C (number of examined CB contracts)		6,060,000 contracts (face value F assumed to be 100)	DHR 2007 Annual Report (p. 45)
N_B (number of virtual coupon-bearing bond contracts, approximating for other liabilities)		77,802,470 contracts (face value F assumed to be 100)	DHR 2007 Annual Report (pp. 45 and 61)
c_S (coupon rate for virtual coupon-bearing bond)		1.4100%	DHR 2007 Annual Report (pp. 60–61)
ϕ (cash payment yield from firm value)		0.3661%	DHR 2007 Annual Report (pp. 60–62) and DHR equity price on December 31 of 2007 (87.74)
(c) *Capital structure information and payout ratio used in 2007/01/01–2007/12/31*			
N_O (number of equity shares)	Estimated by this study	341,223,000 shares	DHR 2006 Annual Report (p. 47)
N_C (number of examined CB contracts)		5,940,000 contracts (face value F is assumed to be 100)	DHR 2006 Annual Report (p. 33)
N_B (number of virtual coupon-bearing bond contracts, approximating for other liabilities)		56,254,910 contracts (face value F is assumed to be 100)	DHR 2006 Annual Report (pp. 33 and 45)
c_S (coupon rate for virtual coupon-bearing bond)		1.4191%	DHR 2006 Annual Report (pp. 44–45)
ϕ (cash payment yield from firm value)		0.3375%	DHR 2006 Annual Report (pp. 44–46) and DHR equity price on December 29 of 2006 (72.44)
(d) *Capital structure information and payout ratio used in 2006/01/22–2006/12/31*			
N_O (number of equity shares)	Estimated by this study	338,547,000 shares	DHR 2006 Annual Report (p. 47)
N_C (number of examined CB contracts)		5,820,000 contracts (face value F assumed to be 100)	DHR 2006 Annual Report (p. 33) (approximated)

TABLE 6 (Continued)

Name	Data source	Value and explanation	
N_B (number of virtual coupon-bearing bond contracts, approximating for other liabilities)		35,007,590 contracts (face value F assumed to be 100)	DHR 2006 Annual Report (pp. 33 and 45)
c_S (coupon rate for virtual coupon-bearing bond)		1.2835%	DHR 2006 Annual Report (pp. 44–45)
ϕ (cash payment yield from firm value)		0.2895%	DHR 2006 Annual Report (pp. 44–46) and DHR equity price on December 30 of 2005 (55.78)

Abbreviations: CB, convertible bond; DHR, Danaher Corporation.

FV and stochastic EPV given the simulated equity price in the tree model and the initially endogenously solved FV volatility. Combining the information of the implied FV and its volatility and the capital structure of the issuing firm allows us to determine the default rate and the dilution effect when pricing CBs. The proposed model captures the negative (positive) relationships between the stochastically evolving DP and the FV or the equity price (the EPV). Sensitivity analyses and empirical studies are presented that attest the robustness and the feasibility of the proposed CB pricing model.

Interesting yet challenging problems remain that are not well solved in past CB pricing literature or even in this paper. First, in addition to the standard call-back and conversion provisions triggered by hitting the optimal call and conversion boundaries, there are other more complex path-dependence provisions for pricing CBs. For example, some CB contracts may incorporate call-notice periods or soft calls (e.g., x days out of y days above a certain trigger level for being callable). Since these complex call provisions depend more strongly on the information of the realized price path than the standard call-back provision, it requires additional states in each tree node to represent different path-dependent call triggers. To our knowledge, modeling call/conversion policies with a call-notice period are studied in Grau et al. (2003). Lau and Kwok (2004) examine the impact of the soft call provisions on CB pricing, and Liu and Guo (2020) propose an approximation to estimate the probability of triggering soft calls. Second, it is also difficult to model a non-Markovian interest rate process like the LIBOR market model with the tree-based model that can easily deal with the interactive American-style call/conversion decisions. To our knowledge, no literature prices CBs under LIBOR models. Third, although we use a constant payout yield to model dividend payouts, sophisticated settings like stochastic dividends could be more proper for long-term CBs. However, pricing CBs with stochastic dividends has not yet been well addressed and is not easy to implement under the tree-based model. Finally, our CB pricing model does not consider sequential conversion. Although sequential conversion can be handled by introducing a forest of equity-price trees to represent the status of different percentages of CB contracts that have been converted, as proposed in Liu et al. (2021), it is highly complicated to combine their model with ours, and the high computational cost to combine these two models makes it almost infeasible.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are from three primary sources: (1) The details of DHR CB contract and the daily term structures of risk-free zero rates were inherited from EXHIBIT 13 in Wang and Dai (2017) at 10.3905/jod.2017.24.4.052 and collected from Bloomberg; (2) The historical time series of the DHR equity prices and the 1-month Treasury yields were, respectively, collected from the Yahoo! finance website (https://finance.yahoo.com/quote/DHR/history?p=DHR) and the website of the US Treasury Department (https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield); (3) The DHR financial statements were collected from its 2006–2008 annual reports, which are publicly available on its website at http://investors.danaher.com/annual-report-and-proxy/.

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APPENDIX A: SUMMARY OF NOTATIONS (TABLE A1)

TABLE A1 Summary of notations

<table>
<tr><th>Definition\notation</th><th>Our CB pricing model</th><th>Comparative reduced-form CB pricing models</th></tr>
<tr><td>Years to maturity for CB</td><td colspan="2">T</td></tr>
<tr><td>Face value for all bonds</td><td colspan="2">F</td></tr>
<tr><td>Conversion ratio for CB</td><td colspan="2">θ</td></tr>
<tr><td>Call price for CB (at time t)</td><td colspan="2">CP_t or CP</td></tr>
<tr><td>Put price for CB (at time t)</td><td colspan="2">PP_t or PP</td></tr>
<tr><td>Equity price</td><td colspan="2">S_t</td></tr>
<tr><td>Dividend yield</td><td colspan="2">q</td></tr>
<tr><td>Risk-free interest rate</td><td colspan="2">r_t</td></tr>
<tr><td>Correlation between stochastic equity prices and risk-free interest rates</td><td colspan="2">ρ</td></tr>
<tr><td>Coupon rate for CB</td><td colspan="2">c_{B}</td></tr>
<tr><td>Recovery rate for CB</td><td colspan="2">ω</td></tr>
<tr><td>Default probability at time t</td><td>$\epsilon_t(S_t, V_{\mathrm{B}})$</td><td>$\epsilon_t$ or ϵ</td></tr>
<tr><td>Equity-price volatility</td><td>$\sigma_{S,t}$</td><td>σ_S</td></tr>
<tr><td>Implied firm value</td><td>V_t</td><td></td></tr>
<tr><td>Implied firm value volatility</td><td>σ_V</td><td></td></tr>
<tr><td>Total debt amount</td><td>D</td><td></td></tr>
<tr><td>Default boundary</td><td>V_{B}</td><td></td></tr>
<tr><td>Number of equity shares</td><td>N_{O}</td><td></td></tr>
<tr><td>Number of examined CB contracts</td><td>N_{C}</td><td></td></tr>
<tr><td>Number of virtual coupon-bearing bond contracts, approximating for other liabilities</td><td>N_{B}</td><td></td></tr>
<tr><td>Coupon rate for virtual coupon-bearing bond</td><td>c_{S}</td><td></td></tr>
<tr><td>Cash payment yield from firm value</td><td>ϕ</td><td></td></tr>
</table>

Note: The definitions and notations for all parameters used in this paper are summarized in the following table.
Abbreviation: CB, convertible bond.

APPENDIX B: PROOF OF VALIDITY OF P_{U}, P_{M}, AND P_{D}

The branching probabilities can be solved from Equations (10) to (12) to obtain

$$P_{\mathrm{U}} = \frac{\beta^2 - \beta\eta\delta_S + \sigma_{S,t}^2\Delta t}{2\eta^2\delta_S^2},$$

$$P_{\mathrm{M}} = -\frac{\beta^2 - \eta^2\delta_S^2 + \sigma_{S,t}^2\Delta t}{\eta^2\delta_S^2},$$

$$P_{\mathrm{D}} = \frac{\beta^2 + \beta\eta\delta_S + \sigma_{S,t}^2\Delta t}{2\eta^2\delta_S^2}.$$

Since the sum of P_U, P_M, and P_D is 1, the validity of the trinomial branch construction method in Figure 2 can be proved by merely showing that P_U, P_M, and P_D are all larger than or equal to 0 given the condition $\frac{\eta}{2} \le \frac{\sigma_{S,t}\sqrt{\Delta t}}{\delta_S} \le \sqrt{\eta^2 - 1}$, where η is a positive integer.

Since the denominator of P_U is positive, we simply show that the numerator is also positive as follows:

$$\beta^2 - \beta\eta\delta_S + \sigma_{S,t}^2\Delta t = \left(\beta - \frac{1}{2}\eta\delta_S\right)^2 + \sigma_{S,t}^2\Delta t - \frac{1}{4}\eta^2\delta_S^2 = \left(\beta - \frac{1}{2}\eta\delta_S\right)^2 + \sigma_{S,t}^2\Delta t\left(1 - \frac{1}{4}\eta^2\frac{\delta_S^2}{\sigma_{S,t}^2\Delta t}\right) \ge 0,$$

where the last inequality is derived due to the inequality $\frac{\eta}{2} \le \frac{\sigma_{S,t}\sqrt{\Delta t}}{\delta_S}$.

For P_M, we intend to show that numerator $\beta^2 - \eta^2\delta_S^2 + \sigma_{S,t}^2\Delta t$ is nonpositive:

$$\beta^2 - \eta^2\delta_S^2 + \sigma_{S,t}^2\Delta t = \beta^2 - \delta_S^2\left(\eta^2 - \frac{\sigma_{S,t}^2\Delta t}{\delta_S^2}\right) < \delta_S^2\left(1 - \eta^2 + \frac{\sigma_{S,t}^2\Delta t}{\delta_S^2}\right),$$

where the last inequality is due to $\beta \in (-\delta_S, \delta_S)$, a consequence of applying the mean-tracking method to identify the middle descendant node. Moreover, since $\frac{\sigma_{S,t}\sqrt{\Delta t}}{\delta_S} \le \sqrt{\eta^2 - 1}$, one can derive

$$\delta_S^2\left(1 - \eta^2 + \frac{\sigma_{S,t}^2\Delta t}{\delta_S^2}\right) \le 0$$

and obtain the desired result.

The proof of P_D is similar to the proof of P_U as follows:

$$\beta^2 + \beta\eta\delta_S + \sigma_{S,t}^2\Delta t = \left(\beta + \frac{1}{2}\eta\delta_S\right)^2 + \sigma_{S,t}^2\Delta t - \frac{1}{4}\eta^2\delta_S^2 = \left(\beta + \frac{1}{2}\eta\delta_S\right)^2 + \sigma_{S,t}^2\Delta t\left(1 - \frac{1}{4}\eta^2\frac{\delta_S^2}{\sigma_{S,t}^2\Delta t}\right) \ge 0,$$

due to $\frac{\eta}{2} \le \frac{\sigma_{S,t}\sqrt{\Delta t}}{\delta_S}$.

The remaining task is to show that there always exists a positive integer η which satisfies $\frac{\eta}{2} \le \frac{\sigma_{S,t}\sqrt{\Delta t}}{\delta_S} = \frac{\sigma_{S,t}\sqrt{\Delta t}}{\sigma_V\sqrt{\Delta t}} \le \sqrt{\eta^2 - 1}$ ($\delta_S \equiv \sigma_V\sqrt{\Delta t}$ by definition) in Equation (9). As proved in Section V of Merton (1974), the debt of the issuer should be less risky than the issuer asset as a whole, which implies the equity of a levered issuer must be at least as risky as its asset (i.e., $\sigma_{S,t} \ge \sigma_V$), and the equality between $\sigma_{S,t}$ and σ_V holds only for an all-equity firm. As a result, one can infer that $\frac{\sigma_{S,t}\sqrt{\Delta t}}{\delta_S} \ge 1$ and then choose η to be at least 2 to satisfy $\frac{\eta}{2} \le \frac{\sigma_{S,t}\sqrt{\Delta t}}{\delta_S} \le \sqrt{\eta^2 - 1}$.

APPENDIX C: EXTENDED FORTET METHOD AND MOMENT CONDITIONS OF PROCESSES r_t AND l_t

Define the logarithmic FV process $\ln(V_t) \equiv l_t$; the CB issuer defaults once the FV falls below V_B (i.e., $l_t \le \ln(V_B) \equiv h$). Note that $l_0 > h$, as the issuer does not default at the CB issuance date. Then $q^d(i, j)$ can be computed by a recursive formula as

$$q^d(i, j) = \Phi(r_i, t_j) - \sum_{v=0}^{j-1}\sum_{u=0}^{n_r} q^d(u, v)\Psi(r_i, t_j, r_u, t_v),$$

where

$$q^d(i, 0) = \Phi(r_i, t_0),$$

$$\Phi(r,t)=f_{r_t}(r|r_0)N\left(\frac{h-\mu(r|l_0,r_0)}{\sqrt{\Sigma^2(r|l_0,r_0)}}\right),$$

$$\Psi(r,t,r',s)=f_{r_t}(r|r_s=r')N\left(\frac{h-\mu(r|l_s=h,r')}{\sqrt{\Sigma^2(r|l_s=h,r')}}\right),$$

given that time points t and s satisfy $s\le t\le T$, and f_{r_t} is the transition density of r defined as $f_{r_t}(r|r_s)=\frac{1}{\sqrt{2\pi\sigma_r}}e^{-(r-m)^2/2\sigma_r}$ with $m=E^T[r_t|r_s]$ and $\sigma_r=\mathrm{var}^T(r_t|r_s)$. As for $\mu(r_t|l_s,r_s)$, $\Sigma^2(r_t|l_s,r_s)$, m, and σ_r, the details to derive them are presented as follows.

First, the conditional moments $\mu(r_t|l_s,r_s)$ and $\Sigma^2(r_t|l_s,r_s)$ can be calculated as

$$\begin{cases}\mu(r_t|l_s,r_s)=E^T[l_t|\mathcal{F}_s]+\dfrac{\mathrm{cov}^T(l_t,r_t|\mathcal{F}_s)}{\mathrm{var}^T(r_t|\mathcal{F}_s)}(r_t-E^T[r_t|\mathcal{F}_s]),\\ \Sigma^2(r_t|l_s,r_s)=\mathrm{var}^T(l_t|\mathcal{F}_s)-\dfrac{\mathrm{cov}^T(l_t,r_t|\mathcal{F}_s)^2}{\mathrm{var}^T(r_t|\mathcal{F}_s)}.\end{cases}$$

In the above formulas, the conditional moments for process l_t are

$$\begin{cases}\hat{\mu}_{s,T}=E^T[l_t|\mathcal{F}_s]=l_s-\left(\phi+\dfrac{{\sigma_V}^2}{2}+\dfrac{\sigma_V\rho\sigma_r}{a}-b+\dfrac{{\sigma_r}^2}{a^2}\right)(t-s)-\dfrac{{\sigma_r}^2}{a^2}e^{-a(T-t)}B_{2a}(t-s)\\ \quad+\left(r_s-b+\dfrac{{\sigma_r}^2}{a^2}+\dfrac{{\sigma_r}^2}{a^2}e^{-a(T-t)}+\dfrac{\sigma_V\rho\sigma_r}{a}e^{-a(T-t)}\right)B_a(t-s),\\ \hat{u}_{s,T}=\mathrm{var}^T(l_t|\mathcal{F}_s)=\left({\sigma_V}^2+\dfrac{{\sigma_r}^2}{a^2}+2\dfrac{\sigma_V\rho\sigma_r}{a}\right)(t-s)-2\left(\dfrac{{\sigma_r}^2}{a^2}+\dfrac{\sigma_V\rho\sigma_r}{a}\right)B_a(t-s)\\ +\dfrac{{\sigma_r}^2}{a^2}B_{2a}(t-s),\\ \mathrm{cov}^T(l_t,r_t|\mathcal{F}_s)=\left(\dfrac{{\sigma_r}^2}{a}+\sigma_V\rho\sigma_r\right)B_a(t-s)-\dfrac{{\sigma_r}^2}{a}B_{2a}(t-s),\end{cases}$$

where $B_a(u)=\frac{1}{a}(1-e^{-au})$.

Second, by replacing s with 0 in the above expressions, we obtain the first two unconditional moments for the process l_t as

$$\begin{aligned}M_t&=\ln\left(\frac{V_0}{P(0,t)}\right)+\frac{{\sigma_r}^2}{4a^3}-\left(\frac{{\sigma_r}^2}{2a^2}+\frac{\rho\sigma_V\sigma_r}{a}+\frac{{\sigma_V}^2}{2}+\phi\right)t-\frac{{\sigma_r}^2}{4a^3}e^{-2at}\\&\quad+\left(\frac{{\sigma_r}^2}{2a^3}+\frac{\rho\sigma_V\sigma_r}{a^2}\right)e^{-a(T-t)}-\left(\frac{{\sigma_r}^2}{a^3}+\frac{\rho\sigma_V\sigma_r}{a^2}\right)e^{-aT}+\frac{{\sigma_r}^2}{2a^3}e^{-a(T+t)},\\U_t&=\left({\sigma_V}^2+\frac{{\sigma_r}^2}{a^2}+\frac{2\rho\sigma_V\sigma_r}{a}\right)t-\frac{3{\sigma_r}^2}{2a^3}-\frac{2\rho\sigma_V\sigma_r}{a^2}+\frac{2\sigma_r(\sigma_r+a\rho\sigma_V)}{a^3}e^{-at}-\frac{{\sigma_r}^2}{2a^3}e^{-2at}.\end{aligned}$$

Last, Bernard et al. (2008) also show the formulas for $m=E^T[r_t|r_s]$ and $\sigma_r=\mathrm{var}^T(r_t|r_s)$ as

$$E^T[r_t|r_s]=e^{-a(T-t)}r_s+\left(ba-\frac{{\sigma_r}^2}{a}\right)B_a(t-s)+\frac{{\sigma_r}^2}{a}e^{-a(T-t)}B_{2a}(t-s),$$

$$\mathrm{var}^T(r_t|r_s)={\sigma_r}^2B_{2a}(t-s).$$