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Innovative Applications of O.R.

Operational asymptotic stochastic dominance[☆]

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ABSTRACT

Levy (2016) proposes asymptotic first-degree stochastic dominance as a distribution ranking criterion for all non-satiable decision makers with infinite investment horizons. Given Levy's setting, this paper defines and offers the equivalent distributional conditions for asymptotic second-degree stochastic dominance, as well as operational asymptotic first- and second-degree stochastic dominance. Interestingly, the operational asymptotic stochastic dominance provides a full rank over assets with lognormal returns and different means. Empirical applications show that our conditions can be readily implemented in practice.

1. Introduction

The dramatic increase in life expectancy in many countries has become a major challenge not only in management, but also in finance and economics. Searching for the optimal investment over the long run has become an important issue in the life-cycle planning for an aging society. In a recent important study, Levy (2016) proposed incorporating a long investment horizon with the concept of stochastic dominance to find the preferred investment strategies. He defined "asymptotic first-degree stochastic dominance" (asymptotic-FSD, henceforth) as the distribution ranking criterion for all non-satiable investors when the investment horizon goes to infinity. His approach helps us understand how preferences affect risky choices in the very long run.

To find the moment condition for asymptotic-FSD, Levy (2016) employed the distribution assumption: the log returns of portfolios follow normal distributions. This assumption is commonly adopted since, by the central limit theorem, the terminal wealth distribution is lognormal in the very long run, assuming that the returns per period are independently and identically distributed.[1] The necessary and sufficient distribution conditions for asymptotic-FSD under lognormal distribution assumptions are found by Huang, Tzeng, Wang, and Zhao (2019). They show that the condition is the same as that for first-degree stochastic dominance (FSD) when the investment horizon is finite, i.e., higher geometric means together with the same volatilities. While the concept of asymptotic-FSD is helpful for solving the investment problem in an aging society, the corresponding distribution condition is too rigid to be applied.

The purpose of this paper is to extend asymptotic-FSD by placing common constraints on the preferences to search for consensus rules under the lognormal distribution assumption. Three sets of investors are considered. The first set only contains risk-averse investors since risk aversion is commonly assumed in theoretical research and frequently observed in empirical studies. We propose a new notion of "asymptotic second-degree stochastic dominance" (asymptotic-SSD, henceforth), which is the distribution ranking criterion for all non-satiable and risk-averse decision makers when

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[1] Fama and French (2018) examine this prophecy of the central limit theorem by conducting bootstrap simulation experiments based on actual long-horizon U.S. stock market returns. Their numerical results verify that the distributions of gross returns of the U.S. market portfolio converge toward lognormal as the investment horizon increases. Moreover, Levy (2016) conducts a goodness-of-fit test of the empirical distribution of the stock returns for 20 possible theoretical distributions. In Section 3.2 therein, he finds that "in this horse race, for a horizon of 20 years, the lognormal distribution provides the best fit. Moreover, for the 20-year horizon or longer, the deviations between the theoretical lognormal distribution and the empirical distribution are negligible."

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the investment period goes to infinity. Following the lognormal assumption as in Levy (2016), we find that the necessary and sufficient distribution condition in the indefinite long run is the same as that for a finite investment horizon. That is, the strategy which generates a higher geometric mean and a lower volatility is the dominant strategy for all risk-averse investors.

The second set of investors has strictly positive and finite marginal utilities. The constraint of bounded marginal utility on preferences is similar to those in almost stochastic dominance pioneered by Leshno and Levy (2002). Leshno and Levy (2002) argue that most decision makers have non-extreme preferences, i.e., the rates of marginal substitution for any wealth level are bounded.[2] We relax this preference assumption by assuming that marginal utilities are positive and bounded. We define "operational asymptotic first-degree stochastic dominance" (operational asymptotic-FSD, henceforth) as the distribution ranking rule for all investors with positive and bounded marginal utility functions. We find that the dominant portfolio is the one generating the highest mean of wealth.

The third set of investors considered in this paper is not only characterized by strictly positive and finite marginal utilities but is also risk-averse. "Operational asymptotic second-degree stochastic dominance" (operational asymptotic-SSD, henceforth) is defined as the distribution ranking rule for all investors in this set. The dominant portfolio under a lognormal assumption is the one which either generates the highest mean of wealth or the highest geometric mean if the means of the wealth are the same. With the lognormal assumption, operational asymptotic-SSD is a full ranking criterion: it can always rank any two portfolios.

We explore the applicability of our newly proposed stochastic dominance rules in practice. For investments in the long run, the usefulness of these conditions in pairwise comparisons of assets is illustrated with numerical examples. We further show that operational asymptotic-FSD and -SSD can be conveniently employed in comparing multiple assets. By adopting the data for 5 industry portfolios covering the period from July 1927 to December 2015, we find that the Healthcare, Medical Equipment, and Drugs Industry dominates other industry portfolios in terms of operational asymptotic-FSD, assuming that the underlying dynamics of the assets persists in the future. For investments in the short run, we provide two sufficient conditions ensuring that one asset generates a larger expected utility than another. The usefulness of these conditions in pairwise comparisons of assets for finite investment horizons is also illustrated with numerical examples.

Our study contributes to the understanding of the dominant strategy that maximizes the investor's terminal wealth in the very long run. To the best of our knowledge, Levy (2016) was the first to propose the notion of asymptotic stochastic dominance, making stochastic dominance applicable to the comparison of the limiting performances with infinite horizons. Later on, Huang et al. (2019) offered the distribution condition for asymptotic-FSD, whereas Levy (2019) examines the case where marginal utilities are bounded from both below and above. Complementary to this emerging literature, our study offers a more complete picture of how different utility conditions shape the criterion of asymptotic stochastic dominance. We concede that, as in Levy (2016), Levy (2019) and Huang et al. (2019), our study also relies on the assumption of a lognormal distribution. However, given that the lognormal distribution has been the workhorse in traditional investment studies, our study can serve as a standard benchmark for future research on this topic.

The structure of the paper is as follows. Section describes the model setting, reviews asymptotic-FSD, and defines and shows the moment condition of asymptotic-SSD. Section 3 defines and provides the moment conditions of operational asymptotic-FSD and -SSD. Section 4 provides empirical applications of different notions of asymptotic stochastic dominance in ranking assets. Section 5 discusses the practicability of operational asymptotic stochastic dominance in finite investment horizons. Section 6 concludes the paper. All proofs are relegated to the Appendix.

[2] They established almost stochastic dominance as the distribution ranking criterion for all decision makers with non-extreme preferences.

2. Asymptotic-FSD and -SSD

Assume that an investor adopts a buy-and-hold strategy to maximize her utility of wealth at time T. Let x_t denote the rate of gross portfolio return at time t. Thus, the terminal wealth of a uni-dollar of her investment at T, denoted as W_T, should be given by

$$\log W_T = \sum_{t=1}^{T} \log x_t.$$

Let F_T and G_T be two cumulative distribution functions of W_T. Let U denote the von Neumann–Morgenstern utility function, and U' and U'' respectively denote the first and second derivatives of U. Furthermore, let $E_F U(W_T)$ and $E_G U(W_T)$ be the expected utility of W_T under F_T and G_T, respectively.

2.1. Asymptotic-FSD

Levy (2016) defined asymptotic-FSD as follows:

Definition 1. F_T dominates G_T by asymptotic-FSD if and only if

$$\lim_{T\to\infty} [E_F U(W_T) - E_G U(W_T)] \geq 0 \ \textit{for all } U \textit{ with } U' \geq 0,$$

and for some non-decreasing U there is a strict inequality.

To further understand the property of the dominant strategy in the long run, Levy (2016) placed assumptions on the portfolio return distributions. Assume that the x_t's are independently and identically distributed (i.i.d.) and that $\log x_t$ follows a normal distribution $N(\mu, \sigma^2)$. Thus, $\log W_T$ follows a normal distribution $N(T\mu, T\sigma^2)$.

Let F_T and G_T be two lognormal distributions of W_T with mean $e^{T(\mu_F+\sigma_F^2/2)}$ and $e^{T(\mu_G+\sigma_G^2/2)}$, respectively. Huang et al. (2019) provided the necessary and sufficient conditions to rank F_T and G_T for asymptotic-FSD:

Theorem 1 (Huang et al., 2019)**.** *Assume that F_T and G_T are lognormal distributions. For $T\to\infty$, F_T dominates G_T by asymptotic-FSD if and only if*

$$\mu_F > \mu_G \text{and} \sigma_F = \sigma_G.$$

Theorem 1 confirms that asymptotic-FSD yields exactly the same condition as FSD with a finite horizon as found by Levy (1973, Theorem 4).

2.2. Asymptotic-SSD

Here, we relax the strict requirement $\sigma_F = \sigma_G$ associated with asymptotic-FSD by concentrating on utility functions exhibiting risk aversion, which leads us to the notion of asymptotic-SSD.

Definition 2. F_T dominates G_T by asymptotic-SSD if and only if

$$\lim_{T\to\infty} [E_F U(W_T) - E_G U(W_T)] \geq 0 \ \textit{for all } U \textit{ with } U' \geq 0 \textit{ and } U'' \leq 0, \quad (1)$$

and for some non-decreasing and concave U there is a strict inequality.

Definition 2 is parallel to the standard definition of SSD for prospects with finite horizons. The utility class used to define asymptotic-SSD excludes the risk-loving attitudes but allows the marginal utility to be unbounded when wealth is close to zero. The equivalent conditions on distributions for asymptotic-SSD are offered in the following theorem.

Theorem 2. *Assume that F_T and G_T are lognormal distributions. For $T\to\infty$, F_T dominates G_T by asymptotic-SSD, if and only if*

$$\mu_F+\frac{\sigma_F^2}{2}\geq\mu_G+\frac{\sigma_G^2}{2},\quad \sigma_F\leq\sigma_G, \tag{2}$$

and at least one of the above inequalities is strict.

Theorem 2 confirms that relative to the distribution condition for asymptotic-FSD, the distribution condition for asymptotic-SSD is less demanding. To ensure that F is preferable to G for all non-decreasing and concave utility functions, the theorem allows for $\sigma_F<\sigma_G$ because convex utility functions are removed from the underlying utility class but still excludes $\sigma_F>\sigma_G$ because otherwise one can always pick up a CRRA utility function (i.e., a utility function with constant relative risk aversion) to obtain a contradiction. It is worthwhile noting here that the asymptotic-SSD condition (2) is exactly the same as the condition for SSD with a finite horizon found by Levy (1973, Theorem 5).

3. Operational asymptotic-FSD and -SSD

To gain insights and tractability, we relax the strict requirement $\sigma_F=\sigma_G$ associated with asymptotic-FSD by placing an additional commonly adopted constraint on marginal utility: the marginal utility is positive and bounded, i.e.,

$$0<\inf_w U'(w)\leq\sup_w U'(w)<\infty. \tag{3}$$

Condition (3) is closely related to the growing literature on almost stochastic dominance which was pioneered by Leshno and Levy (2002), who find that some extreme utility functions may exhibit pathological preferences that violate most decision makers' choices.[3] They suggest that these pathological and extreme preferences can be excluded by requiring that the ratio of the sup to the inf of the marginal utility be bounded.[4]

Almost first-degree stochastic dominance is a decision criterion for decision makers with preferences satisfying

$$0<\sup_w U'(w)\leq\inf_w U'(w)\left(\frac{1}{\varepsilon}-1\right), \tag{4}$$

where $\varepsilon\in(0,1/2)$.[5] The larger ε represents a smaller set of decision makers. In particular, in the limit $\varepsilon\to0$, the set of decision makers includes all preferences with $U'(w)>0$, while in the limit $\varepsilon\to1/2$, only preferences with constant marginal utility (i.e., risk-neutral preferences) are included in the set. Levy (2016) also employs condition (4) and the corresponding almost stochastic dominance rule in his empirical investigation. Our condition (3) is weaker than (4), in the sense that (3) always holds true if (4) is satisfied, irrespective of ε.[6]

3.1. Operational asymptotic-FSD

A variant of asymptotic-FSD, referred to as operational asymptotic-FSD is defined as follows:

Definition 3. F_T dominates G_T by operational asymptotic-FSD if and only if

$$\lim_{T\to\infty}[E_FU(W_T)-E_GU(W_T)]\geq0 \tag{5}$$

for all increasing U satisfying condition (3), and for some increasing U subject to (3) there is a strict inequality.

The following theorem provides the equivalent conditions on distributions for operational asymptotic-FSD.

Theorem 3. *Assume that F_T and G_T are lognormal distributions. For $T\to\infty$, F_T dominates G_T by operational asymptotic-FSD, if and only if*

$$\mu_F+\frac{\sigma_F^2}{2}>\mu_G+\frac{\sigma_G^2}{2}. \tag{6}$$

Condition (6) amounts to requiring that the mean of W_T under F is greater than that under G for all T. Thus, a novel point revealed by Theorem 3 is that for assets subject to lognormal distributions and decision makers subject to (3), the dominance in utility in the limit $T\to\infty$ is actually equivalent to the dominance in the mean. In other words, the ranking implied by risk-neutral preference will become dominant in the very long run, even if the utility function itself is not neutral (linear) in wealth.

Without allowing the marginal utility to go to infinity, the utility loss caused by the violation area always becomes dominated by the utility gain derived from the normal area when $T\to\infty$ under condition (6).[7] Since condition (6) imposes no restriction on the size of σ_F relative to σ_G, operational asymptotic-FSD is strictly weaker than asymptotic-SSD.

In Theorem 3, we have imposed a uniform lower bound on $U'(w)$. Levy (2019) imposed the lower bound on $U'(w)$ from a different angle. He finds that for F_T to dominate G_T as $T\to\infty$ under $\sigma_F>\sigma_G$, $\mu_F>\mu_G$ and $\mu_F/\sigma_F<\mu_G/\sigma_G$, the lower bound on U' can decay according to the power law

$$U'(w)>w^{-\beta}\ \text{as}\ w\to\infty,\ \text{where}$$

$$\beta=1+\frac{1}{2\mu_F}\left(\frac{\mu_F-z_0}{\sigma_F}\right)^2-\frac{z_0}{\mu_F}and z_0=\frac{\mu_F/\sigma_F-\mu_G/\sigma_G}{1/\sigma_F-1/\sigma_G}. \tag{7}$$

Our ranking rule (6) is robust to the power-law decay of $U'(w)$ with a small exponent.

Proposition 1. *Assume that F_T and G_T are lognormal distributions. Under condition (6), F_T dominates G_T for $T\to\infty$ for all U with*

$$U'>0,\ \sup_w U'(w)<\infty,\ \text{and}\ \ U'(w)>w^{-\beta}\ \text{for}\ w\to\infty,$$

as long as $\beta\in[0,1)$ is small enough such that

$$(1-\beta)\left(\mu_F+(1-\beta)\frac{\sigma_F^2}{2}\right)>\mu_G+\frac{\sigma_G^2}{2}. \tag{8}$$

When the exponent β that governs the power-law decay of $U'(w)$ is big, condition (6) alone would fail to ensure the dominance of F_T over G_T in the long run. In particular, when $\beta\geq1$,

[3] Almost stochastic dominance has been widely applied in economics and finance. For example, Bali, Demirtas, Levy, and Wolf (2009) and Levy (2009) employ almost stochastic dominance to evaluate the performance of stocks and bonds in the long run. Bali, Brown, and Demirtas (2013) use almost stochastic dominance to show that some types of hedge funds outperform stocks and bonds. Post and Kopa (2013) derive general linear formulations for almost stochastic dominance.

[4] Following Leshno and Levy (2002), various types of almost stochastic dominance have been developed by employing different constraints on preferences. Please see Lizyayev and Ruszczynsk (2012), Tzeng, Huang, and Shih (2013), and Tsetlin, Winkler, Huang, and Tzeng (2013).

[5] Leshno and Levy (2002) show that $E_FU(W)\geq E_GU(W)$ for all U satisfying (4) if and only if $\int_{F(w)>G(w)}[F(w)-G(w)]dw\leq\varepsilon\int|F(w)-G(w)|dw$.

[6] Let $\mathcal{U}=\{U(w)|U(w)\ \text{satisfies (3)}\}$ and $\mathcal{U}_\varepsilon=\{U(w)|U(w)\ \text{satisfies (4)}\}$. There holds $\mathcal{U}=\cup_{\varepsilon\in(0,1/2]}\mathcal{U}_\varepsilon$.

[7] When $\sigma_F\neq\sigma_G$, F_T and G_T always have an intersection point, which divides the region bounded by F_T and G_T into two parts. There always exists one part such that the FSD condition is violated. The area of the violated part is termed the "violation area".

Table 1
Moment conditions for F dominate G.

Stochastic dominance	Moment conditions
Asymptotic-FSD	$\mu_F > \mu_G$ & $\sigma_F = \sigma_G$
Asymptotic-SSD	$\mu_F + \sigma_F^2/2 \geq \mu_G + \sigma_G^2/2$ & $\sigma_F \leq \sigma_G$, with at least one inequality being strict
Operational asymptotic-FSD	$\mu_F + \sigma_F^2/2 > \mu_G + \sigma_G^2/2$
Operational asymptotic-SSD	$\mu_F + \sigma_F^2/2 > \mu_G + \sigma_G^2/2$ or $\mu_F + \sigma_F^2/2 = \mu_G + \sigma_G^2/2$ & $\mu_F > \mu_G$

Note: $\log x_t$ follows $N(\mu_F, \sigma_F^2)$ and $N(\mu_G, \sigma_G^2)$ under F and G, respectively. The conditions for asymptotic-FSD and -SSD are exactly the same as the conditions for FSD and SSD with a finite horizon found by Levy (1973, Theorems 4 and 5), respectively.

Table 2
Applicability of operational asymptotic stochastic dominance.

Stochastic dominance	Case I ($\sigma_F < \sigma_G$)	Case II ($\sigma_F > \sigma_G$)
Asymptotic-FSD	–	–
Asymptotic-SSD	$F \succ G$	–
Operational asymptotic-FSD	$F \succ G$	$F \succ G$
Operational asymptotic-SSD	$F \succ G$	$F \succ G$

Note: In Case I, $\mu_F = 0.0895$, $\sigma_F = 0.1619$, $\mu_G = 0.0715$ and $\sigma_G = 0.1813$. In Case II, $\mu_F = 0.0895$, $\sigma_F = 0.1619$, $\mu_G = 0.0461$ and $\sigma_G = 0.0721$. $F \succ G$ means that the asset F dominates the asset G in terms of the applied asymptotic stochastic dominance in the first column. A dash indicates that the ranking between the assets F and G is not available based on the applied asymptotic stochastic dominance.

$\mu_F > \mu_G$ is a necessary condition for F_T to dominate G_T (see Lemma A2 in the Appendix). Under $\mu_F > \mu_G$ and $\sigma_F > \sigma_G$, sufficient conditions for F_T to dominate G_T in the long run can be either $\mu_F/\sigma_F \geq \mu_G/\sigma_G$, as shown by Levy (2016), or $\mu_F/\sigma_F < \mu_G/\sigma_G$ together with condition (7), as shown by Levy (2019).

3.2. Operational asymptotic-SSD

Operational asymptotic-SSD is formally defined as follows:

Definition 4. F_T dominates G_T by operational asymptotic-SSD if and only if

$$\lim_{T\to\infty} [E_F U(W_T) - E_G U(W_T)] \geq 0 \tag{9}$$

for all increasing and concave U satisfying condition (3), and for some increasing and concave U subject to (3) there is a strict inequality.

The following theorem provides the equivalent conditions on distributions for operational asymptotic-SSD.

Theorem 4. *Assume that F_T and G_T are lognormal distributions. For $T \to \infty$, F_T dominates G_T by operational asymptotic-SSD, if and only if either (6) holds true or $\mu_F + \sigma_F^2/2 = \mu_G + \sigma_G^2/2$ with $\mu_F > \mu_G$.*

Relative to operational asymptotic-FSD, operational asymptotic-SSD only adds in the comparison for the special case where $\mu_F + \sigma_F^2/2 = \mu_G + \sigma_G^2/2$.

4. Empirical illustrations

A summary of the distribution conditions for various notions of asymptotic stochastic dominance is provided in Table 1. This section is devoted to illustrate the usefulness of these newly proposed (operational) asymptotic stochastic dominance in ranking individual assets. Three cases are examined. In the first two cases, we discuss the comparisons between two assets. In Case I, $\mu_F > \mu_G$ and $\sigma_F < \sigma_G$, whereas in Case II, $\mu_F > \mu_G$ but $\sigma_F > \sigma_G$. For the third case, multiple assets are compared. We employ the returns of 5 industry portfolios to illustrate the application of Theorems 2–4.

In Case I, assets F_T and G_T are chosen as the MSCI World index and S&P 500 index based on the data in Bodie, Kane, and Marcus (2013), respectively. Assuming that the gross returns of F_T and G_T follow log normal distributions, we have[8]

$$\mu_F = 0.0895, \quad \sigma_F = 0.1619;$$

$$\mu_G = 0.0715, \quad \sigma_G = 0.1813.$$

In this example, $\mu_F + \sigma_F^2/2 = 0.1026 > 0.0879 = \mu_G + \sigma_G^2/2$. According to Levy (1973, Theorem 5) and our Theorem 2, all risk-averse agents would prefer the MSCI World index to the S&P 500 index not only for any finite T but also when T approaches infinity, assuming that the underlying dynamics of the two assets persists in the future. By Theorems 3 and 4, decision makers defined by operational asymptotic-FSD and -SSD also prefer the MSCI World index to the S&P 500 index when T approaches infinity.

Consider Case II where the asset with the maximum geometric mean also has a higher volatility. We keep μ_F and σ_F as before and assume instead

$$\mu_G = 0.0461, \quad \sigma_G = 0.0721$$

to capture the mean and volatility of the returns on the long-term U.S. Treasury Bonds as shown in Bodie et al. (2013).[9] In this case, asymptotic-FSD and -SSD as respectively shown in Theorems 1 and 2 cannot serve to provide any investment advice for the long run. By contrast, operational asymptotic-FSD and -SSD shed light on the choice of assets in this case. Since $\mu_F + \sigma_F^2/2 = 0.1026 > 0.0487 = \mu_G + \sigma_G^2/2$, all decision makers defined by operational asymptotic-FSD and -SSD would prefer the MSCI World index to long-term U.S. Treasury Bonds for long-run investments.

The ranking of assets based on the four notions of asymptotic stochastic dominance in the above two cases is summarized in Table 2. As shown in this table, the asymptotic-FSD proposed by Levy (2016) or the asymptotic-SSD extended by us cannot rank the assets in both cases. By contrast, operational asymptotic-FSD and -SSD are able to fully rank these assets for investors with preferences subject to condition (3). This contrast justifies the importance of the notion of operational asymptotic stochastic dominance.

Case III simulates the situation where investors make decisions on choosing among multiple assets. We employ the returns of 5 industry portfolios obtained from French's website. The data period extends from July 1927 to December 2015. Table 3 shows that among these 5 industry portfolios, their σ are all distinct. In other words, investors cannot use the conditions $\mu_F > \mu_G$ and $\sigma_F = \sigma_G$ of asymptotic-FSD in Theorem 1 to select among industry portfolios. Moreover, none of the industry portfolios has the highest $\mu + \sigma^2/2$ and lowest σ. Thus, investors cannot employ the conditions of asymptotic-SSD in Theorem 2 to select the portfolios.

Operational asymptotic-FSD and -SSD can provide investment advice for retirement. Under the assumption that the log gross returns of these industry portfolios follow normal distributions, investors should invest in the Healthcare, Medical Equipment, and Drugs Industry according to Theorems 3 and 4, since this industry portfolio has the highest $\mu + \sigma^2/2$.

[8] As reported in Bodie et al. (2013, Fig. 5.3), the annualized means and volatilities of the rates of return of these two assets for the period from 1926 to 2010 are 10.81% and 18.06% for F and 9.19% and 19.96% for G, respectively.

[9] The annualized means and volatilities of the rates of return of the long-term U.S. Treasury Bonds for the period from 1926 to 2010 are 4.99% and 7.58%, respectively.

Table 3
Descriptive statistics of 5 industry portfolios.

	Cnsmr	Manuf	HiTec	Hlth	Other
μ	0.0086	0.0080	0.0078	0.0093	0.0070
σ	0.0531	0.0552	0.0563	0.0560	0.0638
Correlation	1.0000				
	0.8731	1.0000			
	0.8104	0.8086	1.0000		
	0.7787	0.7460	0.7094	1.0000	
	0.8774	0.8917	0.7988	0.7398	1.0000
$\mu+\sigma^2/2$	0.0100	0.0096	0.0094	0.0109	0.0090

Note: Cnsmr includes Consumer Durables, NonDurables, Wholesale, Retail, and Some Services. Manuf includes Manufacturing, Energy, and Utilities. HiTec includes Business Equipment, Telephone and Television Transmission. Hlth includes Healthcare, Medical Equipment, and Drugs. Other includes all other industries. In addition, μ and σ represent the mean and volatility of the log gross return, respectively.

5. Implications for short-run investment

Our study inherits the framework as proposed by Levy (2016), Levy (2019) and Huang et al. (2019) which assumes that investors only care about their terminal wealth in the very long run. In practice, investors' utility may also depend on their wealth in the short run.[10] In this section, we extend Theorem 3 to give hints on the investment over a finite horizon. Assuming $\mu_F+\sigma_F^2/2>\mu_G+\sigma_G^2/2$, we address two mutually dual questions: (i) given $T<\infty$, what is the preference condition making F_T more preferable than G_T? (ii) Given a preference condition, what is the horizon T required for investors to prefer F_T to G_T?

5.1. Preference conditions under a finite horizon

Proposition 2. *Assume that F_T and G_T are lognormal distributions and that condition (6) holds true. Given $T<\infty$, $E_F U(W_T)\geq E_G U(W_T)$, if*

$$\frac{\sup_w U'(w)}{\inf_w U'(w)} \leq \Lambda(\mu_F,\mu_G,\sigma_F,\sigma_G,T), \tag{10}$$

where $\Lambda(\mu_F,\ \mu_G,\ \sigma_F,\ \sigma_G,\ T)$ increases to infinity faster than $e^{(\mu_F+\sigma_F^2/2-\mu_G-\sigma_G^2/2)T}$.

Condition (10) is obtained from the proof of Theorem 3. The explicit expression of $\Lambda(\mu_F,\ \mu_G,\ \sigma_F,\ \sigma_G,\ T)$ is given by Eqs. (B5) and (B6) for the respective cases $\sigma_F>\sigma_G$ and $\sigma_F<\sigma_G$ in the Appendix. The existing literature has some suggestions on the upper bound of the ratio of the sup to the inf of marginal utilities. Levy, Leshno, and Leibovitch (2010) used experimental data based on a sample of 200 respondents to estimate ε as shown in (4). Their estimated ε is about 5.9%, suggesting that an upper bound of $\sup_w U'(w)/\inf_w U'(w)$ for all decision makers whose preferences are not pathological is $1/5.9\%-1=15.9492$.

In Table 4, we numerically calculate $\Lambda(\mu_F,\ \mu_G,\ \sigma_F,\ \sigma_G,\ T)$ for different T in Case I and Case II introduced in Section 4. In Case I, if the considered investment horizon is 3 years, all non-satiable investors with $\sup_w U'(w)/\inf_w U'(w)\leq 30.9009$ would prefer F_T (MSCI World index) to G_T (S&P 500 index). This set of investors includes all non-pathological and non-satiable investors suggested by Levy et al. (2010). In Case II, if the considered investment horizon is 4 years, all non-satiable investors with $\sup_w U'(w)/\inf_w U'(w)\leq 22.5819$ would prefer F_T (MSCI World index) to G_T (long-term U.S. Treasury Bonds). This set of investors also includes all non-pathological investors suggested by Levy et al. (2010). The upper bound of $\sup_w U'(w)/\inf_w U'(w)$ increases quickly in T. For example, when $T=10$, the corresponding upper bounds in Cases I and II are 891.6645 and 176.1834, respectively. These two sets of investors are much larger than the sets of non-pathological investors suggested by Levy et al. (2010).

[10] We thank an anonymous referee for drawing our attention to this issue.

Table 4
The upper bound of $\sup_w U'(w)/\inf_w U'(w)$.

Year	Case I $\mu_F>\mu_G$ and $\sigma_F<\sigma_G$	Case II $\mu_F>\mu_G$ and $\sigma_F>\sigma_G$
0.01	1.2098	1.1625
0.1	1.8276	1.6106
1	6.8903	4.5683
2	15.8891	8.7328
3	30.9009	14.5398
4	55.2142	22.5819
5	93.6464	33.5867
6	153.2561	48.4811
7	244.3354	68.4509
8	381.8151	95.0123
9	587.2560	130.1009
10	891.6645	176.1834
20	3.90×10^{04}	2590.1131
30	1.21×10^{06}	2.85×10^{04}
40	3.27×10^{07}	2.78×10^{05}
50	8.08×10^{08}	2.56×10^{06}
60	1.89×10^{10}	2.27×10^{07}
70	4.28×10^{11}	1.98×10^{08}
80	9.43×10^{12}	1.69×10^{09}
90	2.03×10^{14}	1.44×10^{10}
100	3.84×10^{15}	1.21×10^{11}

Note: In Case I, $\mu_F=0.0895$, $\sigma_F=0.1619$, $\mu_G=0.0715$ and $\sigma_G=0.1813$. In Case II, $\mu_F=0.0895$, $\sigma_F=0.1619$, $\mu_G=0.0461$ and $\sigma_G=0.0721$.

Benartzi and Thaler (1995) proposed a prospect theory value function

$$U(W_T)=\begin{cases} W_T-W_0, & \text{if } W_T>W_0,\\ \lambda(W_T-W_0), & \text{if } W_T<W_0,\end{cases} \tag{11}$$

where W_0 denotes the initial wealth and λ denotes the loss aversion parameter. For this utility function, $\sup_w U'(w)/\inf_w U'(w)=\lambda$. The commonly adopted value of λ in the literature ranges from 2.25 to 5 (Abdellaoui, Bleichrodt, & Paraschiv, 2007). If the considered investment horizon is 1 year, Table 4 shows that investors with prospect preferences described as in Eq. (11) would prefer F_T to G_T in Case I since $\sup_w U'(w)/\inf_w U'(w)=6.8903>5$ when $T=1$. However, we do not have a conclusion for Case II when the adopted value of λ is greater than 4.6. If the considered investment horizon is 2 years, Table 4 indicates that investors with prospect preferences described as in Equation (11) and $\lambda\leq 5$ would prefer F_T to G_T in both cases.

5.2. Finite horizon under preference conditions

Proposition 3. *Assume that F_T and G_T are lognormal distributions and condition (6) holds true. Given $\varepsilon\in(0,1/2)$, $E_F U(W_T)\geq E_G U(W_T)$ for all U satisfying condition (4), if $T\geq T^*$, where T^* uniquely solves*

$$\Lambda(\mu_F,\mu_G,\sigma_F,\sigma_G,T^*)=\frac{1}{\varepsilon}-1.$$

Proposition 3 is a dual version of Proposition 2. To get a sense of the empirical magnitude of T^*, we numerically calculate T^* for two representative sets of investors. The first set includes the utility functions subject to condition (4) with $\varepsilon=5.9\%$ as suggested by Levy et al. (2010), and the second assumes $\varepsilon=1/6$ or equivalently $\sup_w U'(w)/\inf_w U'(w)=5$ to account for the loss aversion as described in Eq. (11). We fix $\mu_F=0.0895$ and $\sigma_F=0.1619$ as above and let μ_G and σ_G take values of 0.02, 0.04, ..., 0.18 and 0.04, 0.08, ..., 0.36, respectively. To compare T^* with a benchmark portfolio in practice, we estimate the average investment period

Table 5
Minimal investment horizons to achieve a dominance.

Panel A. $\frac{\sup_w U'(w)}{\inf_w U'(w)} = 15.9492$ (percentage for the cases of $T^* < 3.38$ is 50.62%)									
$\sigma_G \backslash \mu_G$	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18
0.04	*2.56†*	*4.47*	*9.71*	*34.55*	*1464.64*	49.17	11.65	5.06	2.81†
0.08	*1.23†*	*2.19†*	*4.98*	*20.30*	4982.92	18.20	4.71	2.12†	1.20†
0.12	*0.36†*	*0.66†*	*1.62†*	*8.51*	96.71	3.37†	1.02†	0.49†	0.33†
0.16	*0.00†*	*0.00†*	*0.00†*	*0.04†*	0.04†	0.00†	0.00†	0.00†	0.00†
0.20	*0.43†*	*0.92†*	*3.27†*	*233.03*	5.57	1.20†	0.51†	0.28†	0.18†
0.24	*2.43†*	*6.14*	*36.14*	171.24	10.26	3.31†	1.61†	0.95†	0.63†
0.28	*8.47*	*28.41*	*698.23*	55.30	11.91	5.02	2.75†	1.73†	1.19†
0.32	*27.64*	*162.25*	250.70	33.16	11.97	6.08	3.66	2.44†	1.74†
0.36	*104.47*	1285.58	73.86	23.69	11.33	6.59	4.29	3.01†	2.23†
Panel B. $\frac{\sup_w U'(w)}{\inf_w U'(w)} = 5$ (percentage for the cases of $T^* < 3.38$ is 66.67%)									
$\sigma_G \backslash \mu_G$	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18
0.04	*0.90†*	*1.57†*	*3.42*	*12.41*	*793.37*	17.85	4.11	1.77†	0.98†
0.08	*0.43†*	*0.77†*	*1.74†*	*7.15*	2818.18	6.40	1.65†	0.74†	0.42†
0.12	*0.12†*	*0.23†*	*0.57†*	*2.98†*	34.11	1.18†	0.36†	0.17†	0.10†
0.16	*0.00†*	*0.00†*	*0.00†*	*0.02†*	0.01†	0.00†	0.00†	0.00†	0.00†
0.20	*0.15†*	*0.32†*	*1.14†*	*82.92*	1.95†	0.42†	0.18†	0.10†	0.06†
0.24	*0.85†*	*2.15†*	*12.78*	63.11	3.60	1.16†	0.56†	0.33†	0.22†
0.28	*2.98†*	*10.14*	*333.02*	20.11	4.20	1.76†	0.96†	0.60†	0.42†
0.32	*10.01*	*67.36*	110.99	12.09	4.25	2.14†	1.29†	0.86†	0.61†
0.36	*43.44*	741.88	29.44	8.68	4.05	2.33†	1.51†	1.06†	0.78†

Note: For each pair of (μ_G, σ_G), the horizon T^* (in years) required to guarantee $E_F U(W_T) \geq E_G U(W_T)$ or $E_G U(W_T) \geq E_F U(W_T)$ is reported given that μ_F and σ_F are fixed to be 0.0895 and 0.1619, respectively. The figures in *italic* typeface indicate that $E_F U(W_T) \geq E_G U(W_T)$, and the other figures indicate that $E_G U(W_T) \geq E_F U(W_T)$. The superscript dagger † indicates that T^* is shorter than 3.38 years, which is the typical investment holding period for large-value-stock funds calculated as the inverse of the size-weighted turnover ratios of the U.S. large-value-stock funds.

of large-value-stock funds in the U.S. based on the data provided by the investment research company Morningstar.[11] Since the investment targets of large-value-stock funds are usually large and long-life firms, it is plausible that their managers adopt strategies to maximize the long-term returns, the goal of which is closely related to that of operational asymptotic-FSD.

To implement the estimation, we calculate a size-weighted turnover ratio for the collected 304 U.S. large-value-stock funds as 29.6% and derive its inverse to obtain the average asset holding period of 3.38 years. This number suggests that for large-value-stock funds, once an investment decision is made, the chosen assets are held for the following 3.38 years on average. Or, it could be viewed as the case where managers of large-value-stock funds make investment decisions by maximizing the terminal payoffs of assets for an average investment horizon of 3.38 years.[12]

The results of T^* corresponding to the two values of $\sup_w U'(w)/\inf_w U'(w)$ are shown in Panels A and B of Table 5, where italic and upright typefaces indicate $E_F U(W_T) \geq E_G U(W_T)$ and $E_G U(W_T) \geq E_F U(W_T)$, respectively. In Table 5, a superscript dagger beside the figure of T^* indicates the situation where $T^* < 3.38$. More than half the values of T^* are shorter than 3.38 years. The percentages are 50.62% and 66.67% for $\sup_w U'(w)/\inf_w U'(w)$ to be 15.9492 and 5 in Panels A and B, respectively. This evidence implies that in the majority of scenarios, investors can apply the moment condition of operational asymptotic-FSD to make investment decisions based on a relatively short horizon.

There are two interesting observations worth discussing in Table 5. First, although most values of T^* are within reasonable levels, the values of T^* on the diagonal from the left bottom to the right top are extremely large. This is because for those entries, the differences between $\mu_F + \sigma_F^2/2$ and $\mu_G + \sigma_G^2/2$ are so small that a long enough period is needed to achieve a dominance in expected utility. If the difference between $\mu_F + \sigma_F^2/2$ and $\mu_G + \sigma_G^2/2$ is not so small, then these T^* will be at reasonable levels as shown in the off-diagonal entries.[13] Second, the values of T^* are particularly short for the scenarios with $\sigma_G = 0.16$, which is approximately equal to $\sigma_F = 0.1619$. This is because the moment conditions in these scenarios are quite close to those for the asymptotic-FSD and FSD provided that the difference between σ_F and σ_G can be ignored. Since the FSD should hold for all investment horizons T, it can be expected that extremely short periods T^* are sufficient to guarantee operational asymptotic-FSD for the scenarios with $\sigma_G = 0.16$.

To sum up, while the operational asymptotic-FSD proposed is originally defined for an infinite horizon, we discover that the distribution condition of operational asymptotic-FSD can also be employed to achieve a dominance for a finite horizon, if a rational value of $\sup_w U'(w)/\inf_w U'(w)$ for investors is considered. For example, suppose that $\mu_F = 0.0895$, $\sigma_F = 0.1619$, $\mu_G = 0.02$ and $\sigma_G = 0.08$. Table 5, Panel A, shows that for 1.23 years, all non-pathological investors as suggested by Levy et al. (2010) would prefer F_T to G_T. Thus, all non-pathological investors could follow our rule to hold portfolio F_T instead of G_T for at least 1.23 years. Assume that after 2 years of investment, portfolio G_T starts to pay dividends and results in a change in the estimated μ_G and σ_G,

[11] The classification of different types of funds and the data for the asset size and turnover ratio of each fund can be found on the Morningstar website (http://news.morningstar.com/fund-category-returns). All data are updated to and collected at the end of August in 2018.

[12] If we alternatively focus on funds that are larger in size, we will derive a lower turnover ratio. For example, the turnover ratios of the three largest-size value-stock funds of which the sum of the asset values represents 25.26% of the total asset value of the whole sample are 25%, 13%, and 9%. The corresponding investment holding periods are 4.00 years, 7.69 years, and 11.11 years, respectively.

[13] Let us illustrate this with two examples. For $(\mu_G, \sigma_G) = (0.1, 0.04)$ in Panel A, the values of $\mu_F + \sigma_F^2/2$ and $\mu_G + \sigma_G^2/2$ are 0.1026 and 0.1008, respectively. Due to the small difference, more than 1000 years are needed to obtain $E_F U(W_T) \geq E_G U(W_T)$ for all U subject to condition (4). By contrast, for $(\mu_G, \sigma_G) = (0.02, 0.04)$ in Panel A, $\mu_G + \sigma_G^2/2 = 0.0208$ differs from $\mu_F + \sigma_F^2/2 = 0.1026$ to a large extent. As a result, a relatively short horizon of 1.23 years is sufficient to achieve the dominance.

e.g., $\mu_G = \sigma_G = 0.04$. From Table 5, our rule suggests that all non-pathological investors could keep holding portfolio F_T for another 4.47 years.

6. Conclusion

In this article, by assuming that the return is independent and identically lognormally distributed, we have extended the concept of asymptotic-FSD in three ways. For non-satiable and risk-averse investors, we have defined asymptotic-SSD and provided the corresponding distribution conditions. For most decision makers with positive and bounded marginal utility, we have defined operational asymptotic-FSD. We have demonstrated that the necessary and sufficient condition for operational asymptotic-FSD is equivalent to choosing the highest mean of final wealth among investment strategies. Furthermore, for most risk-averse decision makers with positive and bounded marginal utility, we have defined operational asymptotic-SSD and provided the corresponding necessary and sufficient condition. We have investigated the applicability of the proposed operational asymptotic-FSD and -SSD for fully ranking assets, and analyzed the possibility to apply the operational asymptotic stochastic dominance over a reasonably short horizon. The numerical findings support the view that our conditions could be practically crucial for the investment decision with either infinite or finite horizons, particularly for investors with rational upper and lower bounds of their marginal utilities.

Overall, our contribution is to clarify what kinds of moment conditions are relevant to long-run performance in a setting without precise knowledge on preferences. Although our paper are helpful in theoretically understanding the distribution ranking criteria in the long run, our analysis inherits two limitations in this line of research as Levy (2016), Levy (2019) and Huang et al. (2019): the log-normal distribution and the buy-and-hold strategy. These assumptions are useful in theoretical analyses but may be over-simplistic for applications.

A potential extension of the notions of (operational) asymptotic-FSD and -SSD is to assume more general distributions beyond lognormals. Denote the mean and variance of the i.i.d. random variables x_t by μ and σ^2, and the cumulative distribution function of $\frac{\log W_T - \mu T}{\sigma\sqrt{T}}$ by $P_T(y)$. The central limit theorem has justified the fact that $\lim_{T\to\infty} P_T(y) = \Phi(y)$ for every y, where the latter is the cumulative distribution function of the standard normal distribution $N(0, 1)$. Unfortunately, as clarified by Merton and Samuelson (1974, p. 79), this does not necessarily imply that

$$\lim_{T\to\infty}\int_{-\infty}^{\infty} U\left(e^{T\mu+\sqrt{T}\sigma y}\right) dP_T(y) = \lim_{T\to\infty}\int_{-\infty}^{\infty} U\left(e^{T\mu+\sqrt{T}\sigma y}\right) d\Phi(y).$$

In other words, it is not condition-free to say that the expected utility generated by general distributions, $EU(W_T)$, should converge to the expected utility generated by the limiting lognormal distribution in the long run. However, if for both assets F and G, $P_T(y)$ converges to $\Phi(y)$ quickly and uniformly across all utility functions within the defined utility class, in the sense that for any $\varepsilon > 0$, there exists a constant T_ε such that

$$\left|\int_{-\infty}^{\infty} U\left(e^{T\mu+\sqrt{T}\sigma y}\right) d(P_T(y) - \Phi(y))\right| < \varepsilon \tag{12}$$

holds true for all $T > T_\varepsilon$ and all underlying utility functions, then one can adapt the proof slightly by plugging a triangle inequality to show that our Theorems 1–4 extend straightforwardly to such distributions (see Appendix C). Better extensions with weaker restrictions on distributions are promising avenues for future research.

Another potential extension of the notions of (operational) asymptotic-FSD and -SSD is to allow for dynamic strategies in asset comparisons. In the practice of asset-liability management (ALM), institutional investors such as pension insurance firms commonly require a dynamic investment plan to meet their liabilities while pursuing profit. Dynamic programming allows researchers to study the strategy of asset allocation by incorporating various transaction costs and risks. However, the existing approaches rely on a precise specification of preferences (Gondzio & Kouwenberg, 2001; Kouwenberg, 2001; Sodhi, 2005), and the results are usually sensitive to preference parameters. Our approach, which requires no specific parametrization of preferences, calls for more research in ALM to relax the preference assumptions. For example, Kouwenberg (2001) uses parameters to take into consideration risk aversion of the pension funds in developing stochastic programming models. A starting point to incorporate the stochastic dominance idea in ALM could be examining the efficient/optimal asset allocation (Longarela, 2015; Bruni, Cesarone, Scozzari, & Tardella, 2017; Kallio & Hardoroudi, 2019) for a set of investors whose degree of risk aversion is not precisely known.

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Appendix

A. Preliminary results

Let $\Phi(x)$ denote the cumulative distribution function of the standard normal distribution and $\Psi(x) = 1 - \Phi(x)$.

Lemma A1. *Let F_T and G_T be lognormal distributions.*

(i) *If $\mu_F + \frac{\sigma_F^2}{2} > \mu_G + \frac{\sigma_G^2}{2}$, then $\lim_{T\to\infty}(E_F W_T - E_G W_T) = \infty$.*
(ii) *If $\mu_F + \frac{\sigma_F^2}{2} < \mu_G + \frac{\sigma_G^2}{2}$, then $\lim_{T\to\infty}(E_F W_T - E_G W_T) = -\infty$.*

Proof. This lemma follows straightforwardly from the fact that $E_F W_T - E_G W_T = e^{(\mu_F+\sigma_F^2/2)T} - e^{(\mu_G+\sigma_G^2/2)T}$. □

Lemma A2. *Let F_T be a lognormal distribution, $M > 0$, and*

$$U_M(w) = \begin{cases} w, & \text{if } w \le M, \\ M\log\left(\frac{w}{M}\right) + M, & \text{if } w > M. \end{cases}$$

Then, $\lim_{T\to\infty}\frac{1}{T}E_F U_M(W_T) = M\mu_F$.

Proof. Simple manipulation yields

$$E_F U_M(W_T) = \int_{-\infty}^{\infty} U_M\left(e^{T\mu_F+\sqrt{T}\sigma_F y}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy = I + II,$$

where $0 < I = \int_{-\infty}^{\frac{\log M - T\mu_F}{\sqrt{T}\sigma_F}} e^{T\mu_F+\sqrt{T}\sigma_F y}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy < M\Phi\left(\frac{\log M - T\mu_F}{\sqrt{T}\sigma_F}\right)$, and

$$\begin{aligned} II &= \int_{\frac{\log M - T\mu_F}{\sqrt{T}\sigma_F}}^{\infty}\left[M\left(T\mu_F + \sqrt{T}\sigma_F y\right) + M(1 - \log M)\right]\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy \\ &= M(T\mu_F + 1 - \log M)\Phi\left(-\left(\frac{\log M - T\mu_F}{\sqrt{T}\sigma_F}\right)\right) \\ &\quad + M\sigma_F\sqrt{\frac{T}{2\pi}}e^{-\frac{1}{2}\left(\frac{\log M - T\mu_F}{\sqrt{T}\sigma_F}\right)^2}. \end{aligned}$$

Since $\lim_{T\to\infty}\frac{I}{T} = 0$ and $\lim_{T\to\infty}\frac{II}{T} = M\mu_F$, we obtain the result. □

Lemma A3. *Let F_T and G_T be lognormal distributions satisfying $\mu_F + \frac{1}{2}\sigma_F^2 > \mu_G + \frac{1}{2}\sigma_G^2$. Let w_0 be the intersection point of F_T and G_T such* *that*

$$F_T(w_0) = G_T(w_0) \Leftrightarrow w_0 = e^{T\left(\frac{\mu_F}{\sigma_F}-\frac{\mu_G}{\sigma_G}\right)/\left(\frac{1}{\sigma_F}-\frac{1}{\sigma_G}\right)}.$$

Denote $I = \int_0^{w_0}[G_T(w) - F_T(w)]dw$ *and* $II = \int_{w_0}^{\infty}[G_T(w) - F_T(w)]dw$.

(i) If $\sigma_F < \sigma_G$*, then* $I > 0$ *and* $II < 0$*. Let* $m > 0$ *satisfy* $\mu_F + \frac{1}{2}\sigma_F^2 = \mu_G + \frac{1}{2}\sigma_G^2 + m(\sigma_G - \sigma_F)$*. We have*

$$I = e^{(\mu_G+\frac{1}{2}\sigma_G^2)T} \times\left[e^{m(\sigma_G-\sigma_F)T}\Phi\left(\sqrt{T}\left(m+\frac{\sigma_G-\sigma_F}{2}\right)\right) - \Phi\left(\sqrt{T}\left(m-\frac{\sigma_G-\sigma_F}{2}\right)\right)\right],$$

$$II = e^{(\mu_G+\frac{1}{2}\sigma_G^2)T} \times\left[e^{m(\sigma_G-\sigma_F)T}\Psi\left(\sqrt{T}\left(m+\frac{\sigma_G-\sigma_F}{2}\right)\right) - \Psi\left(\sqrt{T}\left(m-\frac{\sigma_G-\sigma_F}{2}\right)\right)\right].$$

(ii) If $\sigma_F > \sigma_G$*, then* $I < 0$ *and* $II > 0$*. Let* $m > 0$ *satisfy* $\mu_F + \frac{1}{2}\sigma_F^2 = \mu_G + \frac{1}{2}\sigma_G^2 + m(\sigma_F - \sigma_G)$*. We have*

$$I = e^{(\mu_G+\frac{1}{2}\sigma_G^2)T} \times\left[e^{m(\sigma_F-\sigma_G)T}\Psi\left(\sqrt{T}\left(m+\frac{\sigma_F-\sigma_G}{2}\right)\right) - \Psi\left(\sqrt{T}\left(m-\frac{\sigma_F-\sigma_G}{2}\right)\right)\right],$$

$$II = e^{(\mu_G+\frac{1}{2}\sigma_G^2)T} \times\left[e^{m(\sigma_F-\sigma_G)T}\Phi\left(\sqrt{T}\left(m+\frac{\sigma_F-\sigma_G}{2}\right)\right) - \Phi\left(\sqrt{T}\left(m-\frac{\sigma_F-\sigma_G}{2}\right)\right)\right].$$

Proof. Simple manipulations yield

$$I = \int_0^{w_0}[G_T(w) - F_T(w)]dw = \int_0^{w_0} wd[F_T(w) - G_T(w)]$$
$$= \int_{-\infty}^{\sqrt{T}\left(\frac{\mu_F-\mu_G}{\sigma_G-\sigma_F}\right)}\left(e^{T\mu_F+\sqrt{T}\sigma_F y} - e^{T\mu_G+\sqrt{T}\sigma_G y}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy$$
$$= e^{\left(\mu_F+\frac{\sigma_F^2}{2}\right)T}\Phi\left(\sqrt{T}\left(\frac{\mu_F-\mu_G}{\sigma_G-\sigma_F}-\sigma_F\right)\right) - e^{\left(\mu_G+\frac{\sigma_G^2}{2}\right)T}\Phi\left(\sqrt{T}\left(\frac{\mu_F-\mu_G}{\sigma_G-\sigma_F}-\sigma_G\right)\right)$$

and

$$II = \int_{w_0}^{\infty}[G_T(w) - F_T(w)]dw = \int_{w_0}^{\infty} wd[F_T(w) - G_T(w)]$$
$$= \int_{\sqrt{T}\left(\frac{\mu_F-\mu_G}{\sigma_G-\sigma_F}\right)}^{\infty}\left(e^{T\mu_F+\sqrt{T}\sigma_F y} - e^{T\mu_G+\sqrt{T}\sigma_G y}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy$$
$$= e^{\left(\mu_F+\frac{\sigma_F^2}{2}\right)T}\Psi\left(\sqrt{T}\left(\frac{\mu_F-\mu_G}{\sigma_G-\sigma_F}-\sigma_F\right)\right) - e^{\left(\mu_G+\frac{\sigma_G^2}{2}\right)T}\Psi\left(\sqrt{T}\left(\frac{\mu_F-\mu_G}{\sigma_G-\sigma_F}-\sigma_G\right)\right).$$

The results follow straightforwardly. □

B. Proofs of theorems and rropositions

Proof of Theorem 2. To prove the necessity, we look for a contradiction to Definition 2 if condition (2) is violated. If, by contradiction, $\sigma_F > \sigma_G$, we can choose $\gamma < 2\min\{\frac{\mu_G-\mu_F}{\sigma_F^2-\sigma_G^2}, 0, -\frac{\mu_G}{\sigma_G^2}\}$ such that $\mu_F + \frac{\gamma}{2}\sigma_F^2 < \mu_G + \frac{\gamma}{2}\sigma_G^2 < 0$. Since $E(\frac{1}{\gamma})W_T^{\gamma} = \frac{1}{\gamma}e^{\gamma(\mu+\frac{\gamma}{2}\sigma^2)T}$, we further have $E_F(\frac{1}{\gamma})W_T^{\gamma} < E_G(\frac{1}{\gamma})W_T^{\gamma}$ for any T and, moreover, $\lim_{T\to\infty}[E_F(\frac{1}{\gamma})W_T^{\gamma} - E_G(\frac{1}{\gamma})W_T^{\gamma}] = -\infty$, which is a contradiction of Definition 2. Therefore, it must be the case that $\sigma_F \le \sigma_G$. If this is an equality, then $\mu_F > \mu_G$ must hold. If $\sigma_F < \sigma_G$, then $\mu_F + \frac{1}{2}\sigma_F^2 \ge \mu_G + \frac{1}{2}\sigma_G^2$ is necessary, because otherwise we take $U(w) = w$ and will obtain a contradiction.

To prove the sufficiency, we differentiate between two cases.

Case 1. $\sigma_F = \sigma_G$. In this case, $\mu_F > \mu_G$ and F_T dominates G_T by FSD. Since FSD is stronger than asymptotic-SSD, F_T naturally dominates G_T.

Case 2. $\sigma_F < \sigma_G$. In this case, F_T intersects G_T from below. Due to the concavity of U, we have for all $y \in (-\infty, \infty)$ that

$$U\left(e^{T\mu_F+\sqrt{T}\sigma_F y}\right) - U\left(e^{T\mu_G+\sqrt{T}\sigma_G y}\right)$$
$$\ge U'\left(e^{T\mu_F+\sqrt{T}\sigma_F y}\right)\left[e^{T\mu_F+\sqrt{T}\sigma_F y} - e^{T\mu_G+\sqrt{T}\sigma_G y}\right]$$
$$\ge U'\left(e^{T\left(\frac{\mu_F\sigma_G-\mu_G\sigma_F}{\sigma_G-\sigma_F}\right)}\right)\left[e^{T\mu_F+\sqrt{T}\sigma_F y} - e^{T\mu_G+\sqrt{T}\sigma_G y}\right],$$

which leads us to

$$E_F U(W_T) - E_G U(W_T) = \int_{-\infty}^{\infty}\left(U\left(e^{T\mu_F+\sqrt{T}\sigma_F y}\right) - U\left(e^{T\mu_G+\sqrt{T}\sigma_G y}\right)\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy \ge U'\left(e^{T\left(\frac{\mu_F\sigma_G-\mu_G\sigma_F}{\sigma_G-\sigma_F}\right)}\right)(E_F W_T - E_G W_T) \ge 0, \tag{B1}$$

provided that $\mu_F + \frac{\sigma_F^2}{2} \ge \mu_G + \frac{\sigma_G^2}{2}$. This proves that $\lim_{T\to\infty}[E_F U(W_T) - E_G U(W_T)] \ge 0$ for all U satisfying $U' \ge 0$ and $U'' \le 0$. Moreover, since $\sigma_F < \sigma_G$, then $0 > \mu_F + \frac{\gamma}{2}\sigma_F^2 > \mu_G + \frac{\gamma}{2}\sigma_G^2$ as long as $\gamma < 2\min\{\frac{\mu_F-\mu_G}{\sigma_G^2-\sigma_F^2}, 0, -\frac{\mu_F}{\sigma_F^2}\}$. Since $E\left(\frac{1}{\gamma}\right)W_T^{\gamma} = \frac{1}{\gamma}e^{\gamma(\mu+\frac{\gamma}{2}\sigma^2)T}$, we further have $E_F\left(\frac{1}{\gamma}\right)W_T^{\gamma} > E_G\left(\frac{1}{\gamma}\right)W_T^{\gamma}$ for any T and, moreover, $\lim_{T\to\infty}[E_F\left(\frac{1}{\gamma}\right)W_T^{\gamma} - E_G\left(\frac{1}{\gamma}\right)W_T^{\gamma}] = +\infty$. This fact shows that we can choose a utility function such that (1) is strict. In both cases, we have verified the sufficiency. □

Proof of Theorem 3. To prove the necessity, we look for a contradiction of Definition 3 if $\mu_F + \frac{\sigma_F^2}{2} \le \mu_G + \frac{\sigma_G^2}{2}$. Indeed, if $\mu_F + \frac{\sigma_F^2}{2} < \mu_G + \frac{\sigma_G^2}{2}$, then agents with $U(w) = w$ would strictly prefer G_T to F_T when $T \to \infty$ according to Lemma A1. If $\mu_F + \frac{\sigma_F^2}{2} = \mu_G + \frac{\sigma_G^2}{2}$, we differentiate between three cases.

Case 1. $\mu_F = \mu_G$. In this case, $\sigma_F = \sigma_G$ and F_T becomes identical to G_T. There exists no utility function such that (5) could become a strict inequality.

Case 2. $\mu_F > \mu_G$. In this case, we take $U(w) = 2w - U_M(w)$, where $U_M(w)$ is specified in Lemma A2. It is easy to see that $U(w)$ satisfies $1 \le U'(w) < 2$ for all $w \ge 0$. Moreover,

$$E_F U(W_T) - E_G U(W_T) = -T\left(\frac{1}{T}E_F U_M(W_T) - \frac{1}{T}E_G U_M(W_T)\right),$$

in which $\frac{1}{T}E_F U_M(W_T) - \frac{1}{T}E_G U_M(W_T) \to M(\mu_F - \mu_G) > 0$ as $T \to \infty$ by Lemma A2, which in turn implies that $\lim_{T\to\infty} E_F U(W_T) - EU_G(W_T) = -\infty$.

Case 3. $\mu_F < \mu_G$. In this case, we take $U(w) = w + U_M(w)$, where $U_M(w)$ is specified in Lemma A2. It is easy to see that $U(w)$ satisfies $1 < U'(w) \le 2$ for all $w \ge 0$. Moreover,

$$E_F U(W_T) - EU_G(W_T) = T\left(\frac{1}{T}E_F U_M(W_T) - \frac{1}{T}E_G U_M(W_T)\right), \tag{B2}$$

in which $\frac{1}{T}E_F U_M(W_T) - \frac{1}{T}E_G U_M(W_T) \to M(\mu_F - \mu_G) < 0$ as $T \to \infty$ by Lemma A2, which in turn implies that $\lim_{T\to\infty} E_F U(W_T) - E_G U(W_T) = -\infty$.

To prove the sufficiency, we also differentiate between three cases.

Case 1. $\sigma_F=\sigma_G$. In this case, we have $\mu_F>\mu_G$ and F_T dominates G_T by FSD for all T, which naturally implies that F_T dominates G_T by asymptotic-FSD.

Case 2. $\sigma_F>\sigma_G$. We have

$$\begin{aligned}
&E_FU(W_T)-E_GU(W_T)\\
&=\int_0^{w_0}[G_T(w)-F_T(w)]U'(w)dw\\
&\quad+\int_{w_0}^{\infty}[G_T(w)-F_T(w)]U'(w)dw\\
&\geq \sup_w U'(w)\int_0^{w_0}[G_T(w)-F_T(w)]dw\\
&\quad+\inf_w U'(w)\int_{w_0}^{\infty}[G_T(w)-F_T(w)]dw.
\end{aligned}$$

By Lemma A3, we further have

$$\begin{aligned}
&E_FU(W_T)-E_GU(W_T)\\
&\geq \sup_w U'(w)e^{(\mu_G+\frac{1}{2}\sigma_G^2)T}\\
&\quad\times\left[e^{m(\sigma_F-\sigma_G)T}\Psi\left(\sqrt{T}\left(m+\frac{\sigma_F-\sigma_G}{2}\right)\right)\right.\\
&\quad\left.-\Psi\left(\sqrt{T}\left(m-\frac{\sigma_F-\sigma_G}{2}\right)\right)\right]\\
&\quad+\inf_w U'(w)e^{(\mu_G+\frac{1}{2}\sigma_G^2)T}\\
&\quad\times\left[e^{m(\sigma_F-\sigma_G)T}\Phi\left(\sqrt{T}\left(m+\frac{\sigma_F-\sigma_G}{2}\right)\right)\right.\\
&\quad\left.-\Phi\left(\sqrt{T}\left(m-\frac{\sigma_F-\sigma_G}{2}\right)\right)\right]\\
&=e^{[(\mu_G+\frac{1}{2}\sigma_G^2)+m(\sigma_F-\sigma_G)]T}\inf_w U'(w)X(T),
\end{aligned}$$

where

$$\begin{aligned}
X(T)=&\frac{\sup_w U'(w)}{\inf_w U'(w)}\left[\Psi\left(\sqrt{T}\left(m+\frac{\sigma_F-\sigma_G}{2}\right)\right)\right.\\
&\left.-\frac{\Psi\left(\sqrt{T}\left(m-\frac{\sigma_F-\sigma_G}{2}\right)\right)}{e^{m(\sigma_F-\sigma_G)T}}\right]\\
&+\left[\Phi\left(\sqrt{T}\left(m+\frac{\sigma_F-\sigma_G}{2}\right)\right)-\frac{\Phi\left(\sqrt{T}\left(m-\frac{\sigma_F-\sigma_G}{2}\right)\right)}{e^{m(\sigma_F-\sigma_G)T}}\right]\\
&\to 1>0,\quad as T\to\infty. \qquad \text{(B3)}
\end{aligned}$$

Thus, $\lim_{T\to\infty}[E_FU(W_T)-E_GU(W_T)]=\infty$ as long as $U(w)$ satisfies (3).

Case 3. $\sigma_F<\sigma_G$. Similar to the above, we have

$$\begin{aligned}
&E_FU(W_T)-E_GU(W_T)\\
&\geq \inf_w U'(w)\int_0^{w_0}[G_T(w)-F_T(w)]dw\\
&\quad+\sup_w U'(w)\int_{w_0}^{\infty}[G_T(w)-F_T(w)]dw.
\end{aligned}$$

By Lemma A3, we further have

$$E_FU(W_T)-E_GU(W_T)\geq e^{[(\mu_G+\frac{1}{2}\sigma_G^2)+m(\sigma_G-\sigma_F)]T}\inf_w U'(w)X(T),$$

where

$$\begin{aligned}
X(T)=&\left[\Phi\left(\sqrt{T}\left(m+\frac{\sigma_G-\sigma_F}{2}\right)\right)-\frac{\Phi\left(\sqrt{T}\left(m-\frac{\sigma_G-\sigma_F}{2}\right)\right)}{e^{m(\sigma_G-\sigma_F)T}}\right]\\
&+\frac{\sup_w U'(w)}{\inf_w U'(w)}\left[\Psi\left(\sqrt{T}\left(m+\frac{\sigma_G-\sigma_F}{2}\right)\right)\right.\\
&\left.-\frac{\Psi\left(\sqrt{T}\left(m-\frac{\sigma_G-\sigma_F}{2}\right)\right)}{e^{m(\sigma_G-\sigma_F)T}}\right]\\
&\to 1>0,\quad as T\to\infty. \qquad \text{(B4)}
\end{aligned}$$

Thus, $\lim_{T\to\infty}[E_FU(W_T)-E_GU(W_T)]=\infty$ as long as $U(w)$ satisfies (3). □

Proof of Proposition 1. The case where $\sigma_F=\sigma_G$ is trivial, since condition (6) implies that $\mu_F>\mu_G$ and F_T dominates G_T by FSD for all T. When $\sigma_F\neq\sigma_G$, we adopt the notation in Lemma A3 to write $E_FU(W_T)-E_GU(W_T)=\int_0^{w_0}[G_T(w)-F_T(w)]U'(w)dw+\int_{w_0}^{\infty}[G_T(w)-F_T(w)]U'(w)dw$. We differentiate between two cases.

Case 1. $\sigma_F>\sigma_G$. $\int_0^{w_0}[G_T(w)-F_T(w)]U'(w)dw$ is negative and satisfies

$$\begin{aligned}
&\int_0^{w_0}[G_T(w)-F_T(w)]U'(w)dw\\
&\geq \sup_w U'(w)\int_0^{w_0}[G_T(w)-F_T(w)]dw\\
&=\sup_w U'^{\left(\mu_F+\frac{\sigma_F^2}{2}\right)T}\Psi\left(\sqrt{T}\left(m+\frac{\sigma_F-\sigma_G}{2}\right)\right)\\
&\quad-\sup_w U'^{\left(\mu_G+\frac{\sigma_G^2}{2}\right)T}\Psi\left(\sqrt{T}\left(m-\frac{\sigma_F-\sigma_G}{2}\right)\right).
\end{aligned}$$

To evaluate $\int_{w_0}^{\infty}[G_T(w)-F_T(w)]U'(w)dw$, we differentiate between two cases. First, $\mu_F/\sigma_F\leq\mu_G/\sigma_G$ so that $w_0\geq 1$. We have for $\beta\in[0,1)$ that

$$\begin{aligned}
&\int_{w_0}^{\infty}[G_T(w)-F_T(w)]U'(w)dw\\
&\geq\int_{w_0}^{\infty}w^{-\beta}[G_T(w)-F_T(w)]dw\\
&=\frac{1}{1-\beta}e^{(1-\beta)\left(\mu_F+(1-\beta)\frac{\sigma_F^2}{2}\right)T}\Phi\left(\sqrt{T}\left(m+\frac{\sigma_F-\sigma_G}{2}-\beta\sigma_F\right)\right)\\
&\quad-\frac{1}{1-\beta}e^{(1-\beta)\left(\mu_G+(1-\beta)\frac{\sigma_G^2}{2}\right)T}\Phi\left(\sqrt{T}\left(m-\frac{\sigma_F-\sigma_G}{2}-\beta\sigma_G\right)\right).
\end{aligned}$$

Combining the two terms and noticing that $\mu_F+(1-\beta)\sigma_F^2/2>\mu_G+(1-\beta)\sigma_G^2/2$ implies $m+(\sigma_F-\sigma_G)/2-\beta\sigma_F\geq(1-\beta)(\sigma_F-\sigma_G)/2>0$, we see that condition (8) is sufficient for $\lim_{T\to\infty}E_FU(W_T)-E_GU(W_T)\geq 0$. Second, $\mu_F/\sigma_F>\mu_G/\sigma_G$ so that $w_0\to 0$ as $T\to\infty$. We have for $\beta\in[0,1)$ that

$$\begin{aligned}
&\int_1^{\infty}[G_T(w)-F_T(w)]U'(w)dw\\
&\geq\int_1^{\infty}w^{-\beta}[G_T(w)-F_T(w)]dw\\
&=\frac{1}{1-\beta}[F_T(1)-G_T(1)]\\
&\quad+\frac{1}{1-\beta}e^{(1-\beta)\left(\mu_F+(1-\beta)\frac{\sigma_F^2}{2}\right)T}\Phi\left(\sqrt{T}\left(\frac{\mu_F}{\sigma_F}+(1-\beta)\sigma_F\right)\right)\\
&\quad-\frac{1}{1-\beta}e^{(1-\beta)\left(\mu_G+(1-\beta)\frac{\sigma_G^2}{2}\right)T}\Phi\left(\sqrt{T}\left(\frac{\mu_G}{\sigma_G}+(1-\beta)\sigma_G\right)\right).
\end{aligned}$$

Therefore, condition (8) is sufficient for $\lim_{T\to\infty}E_FU(W_T)-E_GU(W_T)\geq 0$.

Case 2. $\sigma_F<\sigma_G$. $\int_{w_0}^{\infty}[G_T(w)-F_T(w)]U'(w)dw$ is negative, and satisfies

$$\begin{aligned}
&\int_{w_0}^{\infty}[G_T(w)-F_T(w)]U'(w)dw\geq\sup_w U'^{\infty}_{w_0}[G_T(w)-F_T(w)]dw\\
&=\sup_w U'^{(\mu_F+\frac{1}{2}\sigma_F^2)T}\Psi\left(\sqrt{T}\left(m+\frac{\sigma_G-\sigma_F}{2}\right)\right)
\end{aligned}$$

$$-\sup_w U'^{(\mu_G+\frac{1}{2}\sigma_G^2)T}\Psi\left(\sqrt{T}\left(m-\frac{\sigma_G-\sigma_F}{2}\right)\right).$$

Under condition (6), it must be the case that $\mu_F > \mu_G$, which in turn implies that $w_0 \to \infty$ as $T \to \infty$. Thus, we have for $\beta \in [0, 1)$ that

$$\int_0^{w_0}[G_T(w)-F_T(w)]U'(w)dw \geq \int_1^{w_0} w^{-\beta}[G_T(w)-F_T(w)]dw$$

$$= \frac{1}{1-\beta}[F_T(1)-G_T(1)]$$

$$+\frac{1}{1-\beta}e^{(1-\beta)\left(\mu_F+(1-\beta)\frac{\sigma_F^2}{2}\right)T}\Phi\left(\sqrt{T}\left(m+\frac{\sigma_G-\sigma_F}{2}+\beta\sigma_F\right)\right)$$

$$-\frac{1}{1-\beta}e^{(1-\beta)\left(\mu_G+(1-\beta)\frac{\sigma_G^2}{2}\right)T}\Phi\left(\sqrt{T}\left(m-\frac{\sigma_G-\sigma_F}{2}+\beta\sigma_G\right)\right)$$

$$-\frac{1}{1-\beta}e^{(1-\beta)\left(\mu_F+(1-\beta)\frac{\sigma_F^2}{2}\right)T}\Psi\left(\sqrt{T}\left(\frac{\mu_F}{\sigma_F}+(1-\beta)\sigma_F\right)\right)$$

$$+\frac{1}{1-\beta}e^{(1-\beta)\left(\mu_G+(1-\beta)\frac{\sigma_G^2}{2}\right)T}\Psi\left(\sqrt{T}\left(\frac{\mu_G}{\sigma_G}+(1-\beta)\sigma_G\right)\right).$$

Therefore, condition (8) is sufficient for $\lim_{T\to\infty} E_F U(W_T) - E_G U(W_T) \geq 0$. □

Proof of Theorem 4. We only need to deal with the special case where $\mu_F + \sigma_F^2/2 = \mu_G + \sigma_G^2/2$ that is excluded from Theorem 3.

To prove the necessity, we look for a contradiction of Definition 4 if $\mu_F \leq \mu_G$. If $\mu_F = \mu_G$, then $\sigma_F = \sigma_G$ and F_T becomes identical to G_T. There exists no utility function such that (9) could become a strict inequality. If $\mu_F < \mu_G$, then we take $U(w) = w + U_M(w)$, where $U_M(w)$ is specified in Lemma A2. It is easy to see that $U(w)$ is increasing, concave, and satisfies $1 < U'(w) \leq 2$ for all $w \geq 0$. By (B2), $\lim_{T\to\infty}[E_F U(W_T) - E_G U(W_T)] = -\infty$, a contradiction.

To prove the sufficiency, notice that $\mu_F > \mu_G$ implies that $\sigma_F < \sigma_G$. In this case, by (B1), we have $\lim_{T\to\infty}[E_F U(W_T) - E_G U(W_T)] \geq 0$ for all concave U. Moreover, we take $U(w) = w + U_M(w)$ and use Lemma A2 to get $\lim_{T\to\infty}[E_F U(W_T) - E_G U(W_T)] = \infty$, a strict inequality in (9). □

Proof of Proposition 2. From the proof of Theorem 3, we see that under condition (6), if $X(T) \geq 0$, then $E_F U(W_T) \geq E_G U(W_T)$. Simple manipulation yields $X(T) \geq 0$ if and only if $\frac{\sup_w U'(w)}{\inf_w U'(w)} \leq \Lambda(\mu_F, \mu_G, \sigma_F, \sigma_G, T)$, where

$$\Lambda(\mu_F,\mu_G,\sigma_F,\sigma_G,T) = \frac{\Phi\left(\sqrt{T}\left(m+\frac{\sigma_F-\sigma_G}{2}\right)\right) - \frac{\Phi\left(\sqrt{T}\left(m-\frac{\sigma_F-\sigma_G}{2}\right)\right)}{e^{m(\sigma_F-\sigma_G)T}}}{\frac{\Psi\left(\sqrt{T}\left(m-\frac{\sigma_F-\sigma_G}{2}\right)\right)}{e^{m(\sigma_F-\sigma_G)T}} - \Psi\left(\sqrt{T}\left(m+\frac{\sigma_F-\sigma_G}{2}\right)\right)} \tag{B5}$$

with m satisfying $\mu_F + \frac{1}{2}\sigma_F^2 = \mu_G + \frac{1}{2}\sigma_G^2 + m(\sigma_F - \sigma_G)$ when $\sigma_F > \sigma_G$, and

$$\Lambda(\mu_F,\mu_G,\sigma_F,\sigma_G,T) = \frac{\Phi\left(\sqrt{T}\left(m+\frac{\sigma_G-\sigma_F}{2}\right)\right) - \frac{\Phi\left(\sqrt{T}\left(m-\frac{\sigma_G-\sigma_F}{2}\right)\right)}{e^{m(\sigma_G-\sigma_F)T}}}{\frac{\Psi\left(\sqrt{T}\left(m-\frac{\sigma_G-\sigma_F}{2}\right)\right)}{e^{m(\sigma_G-\sigma_F)T}} - \Psi\left(\sqrt{T}\left(m+\frac{\sigma_G-\sigma_F}{2}\right)\right)} \tag{B6}$$

with m satisfying $\mu_F + \frac{1}{2}\sigma_F^2 = \mu_G + \frac{1}{2}\sigma_G^2 + m(\sigma_G - \sigma_F)$ when $\sigma_F < \sigma_G$. Standard calculus shows that $\Lambda(\mu_F, \mu_G, \sigma_F, \sigma_G, T)$ is strictly increasing in T, and satisfies $\lim_{T\to\infty}\frac{\Lambda(\mu_F,\mu_G,\sigma_F,\sigma_G,T)}{e^{m|\sigma_F-\sigma_G|T}} \geq 1$. □

C. The triangle inequality

For assets F and G, let P_{FT} and P_{GT} be the cumulative distribution functions of $\frac{\log W_T - \mu_F T}{\sigma_F\sqrt{T}}$ and $\frac{\log W_T - \mu_G T}{\sigma_G\sqrt{T}}$, respectively. By the triangle inequality, we have

$$\int_{-\infty}^{\infty}U\left(e^{T\mu_F+\sqrt{T}\sigma_F y}\right)dP_{FT}(y) - \int_{-\infty}^{\infty}U\left(e^{T\mu_G+\sqrt{T}\sigma_G y}\right)dP_{GT}(y)$$

$$\leq \left|\int_{-\infty}^{\infty}U\left(e^{T\mu_F+\sqrt{T}\sigma_F y}\right)d(P_{FT}(y)-\Phi(y))\right|$$

$$+\left|\int_{-\infty}^{\infty}U\left(e^{T\mu_G+\sqrt{T}\sigma_G y}\right)d(P_{GT}(y)-\Phi(y))\right|$$

$$+\int_{-\infty}^{\infty}U\left(e^{T\mu_F+\sqrt{T}\sigma_F y}\right)d\Phi(y) - \int_{-\infty}^{\infty}U\left(e^{T\mu_G+\sqrt{T}\sigma_G y}\right)d\Phi(y)$$

$$\leq 2\varepsilon + \int_{-\infty}^{\infty}U\left(e^{T\mu_F+\sqrt{T}\sigma_F y}\right)d\Phi(y) - \int_{-\infty}^{\infty}U\left(e^{T\mu_G+\sqrt{T}\sigma_G y}\right)d\Phi(y)$$

for all $T > T_\varepsilon$, and similarly

$$\int_{-\infty}^{\infty}U\left(e^{T\mu_F+\sqrt{T}\sigma_F y}\right)dP_{FT}(y) - \int_{-\infty}^{\infty}U\left(e^{T\mu_G+\sqrt{T}\sigma_G y}\right)dP_{GT}(y)$$

$$\geq -2\varepsilon + \int_{-\infty}^{\infty}U\left(e^{T\mu_F+\sqrt{T}\sigma_F y}\right)d\Phi(y) - \int_{-\infty}^{\infty}U\left(e^{T\mu_G+\sqrt{T}\sigma_G y}\right)d\Phi(y)$$

for all $T > T_\varepsilon$. Letting $T \to \infty$ first and $\varepsilon \to 0$ next, we have that

$$\lim_{T\to\infty}\left[\int_{-\infty}^{\infty}U\left(e^{T\mu_F+\sqrt{T}\sigma_F y}\right)dP_{FT}(y) - \int_{-\infty}^{\infty}U\left(e^{T\mu_G+\sqrt{T}\sigma_G y}\right)dP_{GT}(y)\right] \geq 0 \tag{C7}$$

if and only if

$$\lim_{T\to\infty}\left[\int_{-\infty}^{\infty}U\left(e^{T\mu_F+\sqrt{T}\sigma_F y}\right)d\Phi(y) - \int_{-\infty}^{\infty}U\left(e^{T\mu_G+\sqrt{T}\sigma_G y}\right)d\Phi(y)\right] \geq 0. \tag{C8}$$

Since our Theorems 1–4 provide the conditions for (C8), all these theorems can be applied equally well to (C7) under the assumption of (12).

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