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Semistatic hedging and pricing American floating strike lookback options

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Abstract
We price an American floating strike lookback option under the Black–Scholes model with a hypothetic static hedging portfolio (HSHP) composed of nontradable European options. Our approach is more efficient than the tree methods because recalculating the option prices is much quicker. Applying put–call duality to an HSHP yields a tradable semistatic hedging portfolio (SSHP). Numerical results indicate that an SSHP has better hedging performance than a delta-hedged portfolio. Finally, we investigate the model risk for SSHP under a stochastic volatility assumption and find that the model risk is related to the correlation between asset price and volatility.

KEYWORDS
American floating strike lookback option, dynamic hedging, model risk, put–call duality, semistatic hedging, stochastic volatility model

JEL CLASSIFICATION
G13

1 | INTRODUCTION

The lookback option is a useful exotic option in financial markets which can be used for risk management or trading. For example, with floating strike puts the investor can sell the underlying stock at the maximum price during the life of the option.[1] Even though many methods have been developed to price European-style lookback options,[2] it remains difficult to accurately price various types of lookback options; indeed, this has been the focus of recent literature. For example, Kim, Park, and Qian (2011) propose a binomial tree method for lookback options under the jump-diffusion process, Kimura (2011) deals with valuation and premium decomposition of American fractional lookback options, Leung (2013) derives an analytical pricing formula for the European floating strike lookback option under Heston's (1993) stochastic volatility (SV) model, and Chang and Li (2018) evaluate amnesiac lookback options with Monte Carlo simulation.

Although Lai and Lim (2004) provide an analytical solution of American fixed strike lookback options, their results cannot be exploited to price American floating strike lookback options because the equivalence property does not hold for American-style floating strike and fixed strike lookback options.[3] Moreover, lattice-based methods such as Babbs

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[1]Kimura (2011) indicates that the lookback feature has been incorporated into equity-indexed annuities for years. Moreover, Chang and Li (2018) mention that lookback options can also be found in the gold market, and suggest that the lookback feature in extremely volatile markets lends itself to use in cryptocurrency markets.

[2]For example, see Goldman, Sosin, and Gatto (1979), Conze and Viswanathan (1991), Heynen and Kat (1995), and Wong and Kwok (2003), among others.

[3]Eberlein and Papapantoleon (2005) provide a symmetric relationship between European-style floating strike and fixed strike lookback options for assets driven by general Lévy processes. However, the equivalence property does not hold for American-style lookback options.

(2000) can be applied to price American floating strike lookback options. We fill the gap in the literature by proposing two static hedging portfolios for the valuation and the hedge of American floating strike lookback options, respectively.

The basic idea underlying the static replication approach of Derman, Ergener, and Kani (1995) is to form a portfolio of standard European options to match the boundary conditions before maturity and the terminal condition at maturity of the exotic option that will be hedged. This approach has been extended to many types of options: American options (Chung & Shih, 2009; Chung, Shih, & Tsai, 2013), European-style Asian options (Albrecher, Dhaene, Goovaerts, & Schoutens, 2005), European-style installment options (Davis, Schachermayer, & Tompkins, 2001), and models other than the Black–Scholes (BS) model (e.g., see Andersen, Andreasen, & Eliezer, 2002; Fink, 2003; Nalholm & Poulsen, 2006; Takahashi & Yamazaki, 2009).

For our research question, the difficulty arises from the fact that the static hedging portfolio of an American floating strike lookback option generally depends on the maximum (or minimum) price observed so far. In other words, whenever a new maximum price is observed, one must liquidate the original static hedging portfolio and then immediately create a new hedging portfolio, which is not consistent with the meaning of "static." To solve this problem, as suggested by Schroder (1999) and Babbs (2000), we employ the underlying asset price as the numeraire and derive a new partial differential equation (PDE) where the relative price of an American floating strike lookback option with respect to the underlying asset price must follow.

By observing this new PDE and its boundary conditions, we discover that the pricing problem of the relative price of an American floating strike lookback put option (AFSLPO) can correspond to that of an American call option in a hypothetic world where the risk-free rate equals q and the dividend yield of the underlying asset equals r. Note that q and r denote the dividend yield and risk-free rate, respectively, in the real world. Therefore, we directly follow Chung and Shih (2009) to form a static hedging portfolio for the corresponding American call option.

It should be emphasized that the component European options utilized in the above static hedging portfolio do not exist in the real world. Instead, they are European options under the hypothetic world mentioned above. Thus we term our portfolio a *hypothetic static hedging portfolio* (HSHP hereafter). Once the HSHP is obtained, the current relative price of an AFSLPO (with respect to the current asset price) equals the value of HSHP and one can then multiply this portfolio value by the current asset price to get the cash price of the AFSLPO.

Although our HSHP is effective for pricing, it is not realistic for hedging. Fortunately, one can follow Fajardo and Mordecki (2006) to apply the put–call duality under the BS model to transform an HSHP into a portfolio composed of European options existing in the real world.[4] However, it should be noted that the portfolio transformed from an HSHP is not a static portfolio because it must be rebalanced whenever a new maximum stock price is observed. For this reason, we term this portfolio a *semistatic hedging portfolio* (SSHP henceforth).[5] One special feature of an SSHP is that the calls and puts of the newly rebalanced portfolio have different strike prices, depending on the new realized maximum of the asset price, but with the same portfolio weights.

To sum up, we offer contributions for market participants of AFSLPOs in three aspects. First of all, for pricing AFSLPOs, our HSHP approach is efficient and generates monotonically convergent option values with the increase of the number of time points with matched boundaries. Moreover, the proposed approach preserves an attractive feature of the static hedging approach: The recalculation of option prices is very quick with the passage of time and/or with changes in the asset prices. Our numerical analysis suggests that the recalculation time is generally less than 5% of the initial computational time due to the fact that there is no need to solve the static hedging portfolio weights again. In contrast, the recalculation time of most (if not all) existing numerical methods is the same as the initial computational time. Secondly, for hedging purposes, one can construct an SSHP for an AFSLPO. We also examine the hedging performance of the SSHP in terms of the distribution of hedging errors. The numerical analyses indicate that the hedging performance of our SSHP is far less risky than that of a delta-hedged portfolio (DHP hereafter). Finally, we investigate the model risk for our SSHP approach under Heston's (1993) SV model. We find that there is model risk and that it becomes serious especially when the asset price and its volatility are positively correlated. This is because a higher correlation leads to higher costs for rebalancing the SSHP and thus further deterioration in the hedging performance of our SSHP approach developed under the BS model.

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[4]The put–call duality is a general formula to explain the relationship between calls and puts with different strikes; see Equations 14 and 15 of Fajardo and Mordecki (2006). Under the BS model, the put–call duality is reduced to a simple symmetric relationship in which the price of a call option with underlying asset price S, strike price X, interest rate r, and dividend yield q is equal to the price of an otherwise identical put option with asset price X, strike price S, interest rate q, and dividend yield r. Refer to Carr, Ellis, and Gupta (1998) and McDonald and Schroder (1998) for the put–call symmetry of European and American options, respectively.

[5]Our numerical experiments suggest that the average number of portfolio rebalances is generally small for typical parameter values. Therefore, our SSHP is still practical for hedging an AFSLPO.

The rest of the paper is organized as follows: In Section 2, we explain how to formulate the HSHP for AFSLPOs and discuss the numerical efficiency of the proposed method; we then compare the hedging performance of the SSHP and DHP approaches for AFSLPOs in Section 3; next, we investigate the model risk for SSHP under Heston's SV model in Section 4; Section 5 concludes the paper.

2 | PRICING AMERICAN FLOATING STRIKE LOOKBACK OPTIONS USING HSHP

In this section, as a demonstration, we first show how to construct an HSHP for an AFSLPO. We then conduct detailed numerical analyses to investigate the pricing efficiency of the proposed approach in comparison with the tree method of Babbs (2000).

2.1 | Construction of HSHP for AFSLPO

The payoff of a lookback option depends on the maximum or minimum of the underlying asset reached over a specified time, for example, during the entire life of the option. Without loss of generality, we consider the valuation and hedging of an AFSLPO which has the expiry payoff of the form: $\max(S_{\max}(T) - S(T), 0)$, where $S_{\max}(\tau) = \max(S(y); T_0 \leq y \leq \tau)$, $S(y)$ is the stock price at time y, T_0 is the issue time point, and T is the maturity date.

Denote the price of an AFSLPO price at time t as $P_{\text{Lookback}}^A(S(t), S_{\max}(t), \sigma, r, q, t, T)$, where $S(t)$, $S_{\max}(t)$, σ, r, q, and T are the stock price, exercise price, return volatility, risk-free rate, dividend yield, and maturity date, respectively. Under the BS model, when the current stock price is in the holding region, the AFSLPO price satisfies the following PDE:

$$\frac{\sigma^2 S^2}{2}\frac{\partial^2 P_{\text{Lookback}}^A}{\partial S^2} + (r-q)S\frac{\partial P_{\text{Lookback}}^A}{\partial S} + \frac{\partial P_{\text{Lookback}}^A}{\partial t} = rP_{\text{Lookback}}^A. \tag{1}$$

As discussed by Babbs (2000), it is difficult to solve the above PDE numerically or analytically due to the presence of the path-dependent term $S_{\max}(t)$ as an argument of the price function or in the boundary conditions.

To solve this problem, as suggested by Schroder (1999) and Babbs (2000), we change the numeraire and derive a new PDE for which the relative price of an American floating strike lookback option must follow. Specifically, we adopt the underlying asset price as the numeraire and denote the relative price of an AFSLPO with $V_{\text{Lookback}}^A(1, u(t), \sigma, r, q, t, T)$:

$$V_{\text{Lookback}}^A(1, u(t), \sigma, r, q, t, T) = \frac{P_{\text{Lookback}}^A(S(t), S_{\max}(t), \sigma, r, q, t, T)}{S(t)}, \tag{2}$$

where $u(t) = S_{\max}(t)/S(t)$. Babbs (2000) has shown that, in the holding region (i.e., $1 \leq u(t) < u^*(t)$, where $u^*(t) = S_{\max}(t)/S^*(t)$ and $S^*(t)$ is the critical stock price at time t), $V_{\text{Lookback}}^A(.)$ satisfies the following PDE[6]

$$\frac{\sigma^2 u^2}{2}\frac{\partial^2 V_{\text{Lookback}}^A}{\partial u^2} + (q-r)u\frac{\partial V_{\text{Lookback}}^A}{\partial u} + \frac{\partial V_{\text{Lookback}}^A}{\partial t} = qV_{\text{Lookback}}^A, \tag{3}$$

with the terminal payoff of $V_{\text{Lookback}}^A(1, u(T), \sigma, r, q, T, T) = \max(u(T)-1, 0)$.

It should be emphasized that Equation (3) is the PDE satisfied by options in the corresponding hypothetic world where the risk-free rate equals q and the dividend yield of the underlying asset equals r. Moreover, the terminal payoff function implies that we can regard the relative price (denominated in units of the underlying asset) of the AFSLPO as the price of an American call with the strike price equaling one in the above hypothetic world. Denote the American call price in the hypothetic world as $C^A(u(t), 1, \sigma, q, r, t, T)$, where $u(t)$, 1, σ, q, r, t, and T are the asset price, strike price, return volatility, risk-free rate, dividend yield, current time point, and maturity date, respectively. Then the boundary conditions of $C^A(.)$ are given by

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[6]Note that our Equation (3) is slightly different from the Equation (30) of Babbs (2000) because the underlying variable in his paper is $\ln u(t)$ rather than $u(t)$,

$$C^A(u(T), 1, \sigma, q, r, T, T) = \max(u(T) - 1, 0), \tag{4}$$

$$C^A(u^*(t), 1, \sigma, q, r, t, T) = u^*(t) - 1, \tag{5}$$

$$\frac{\partial C^A(u^*(t), 1, \sigma, q, r, t, T)}{\partial u(t)}\Big|_{u(t) = u^*(t)} = 1, \tag{6}$$

and

$$\frac{\partial C^A(u(t), 1, \sigma, q, r, t, T)}{\partial u(t)}\Big|_{u(t) = 1^+} = 0. \tag{7}$$

Note that Equations 5 and 6 are the *value-matching* condition and the *smooth-pasting* condition at the early exercise boundary, respectively, and Equation (7) is the boundary condition when the asset price equals $S_{\max}(t)$.[7]

Since both European call and put options in the hypothetic world also satisfy the same PDE as Equation (3), we use them to construct a HSHP[8] to replicate $C^A(u(t), 1, \sigma, q, r, t, T)$. Consider the construction of an n-point HSHP which matches the boundary conditions (Equations 4–7) of the American call at $t_0, t_1, ..., t_n = T$, which are equal space spans from the current time point t_0 until the maturity date T. The first component in our HSHP is the corresponding European call option with a strike price being 1 and a maturity date T that matches terminal condition (4).

Next, to match the early exercise boundary condition at time t_i ($i = n - 1, n - 2, ..., 0$) on the knock-in boundary, we add w_i units of standard European call options maturing at time t_{i+1} with a strike price equaling $u^*(t_i)$ into the HSHP. Moreover, to match the boundary condition when the stock price equals $S_{\max}(t)$ (i.e., $u(t) = 1$), we further add $\hat{w}_i$ units of standard European put options maturing at time t_{i+1} with a strike price equaling one into the HSHP. We then solve three unknowns (w_i, $u^*(t_i)$, and $\hat{w}_i$) by matching boundary conditions (5)–(7) at time t_i.

Similar to the binomial option pricing models, we work backward to determine the number of the standard European options for the above n-point HSHP. For example, to match boundary conditions at time t_{n-1}, our HSHP must satisfy the following equations:

$$u^*_{n-1} - 1 = C^E(u^*_{n-1}, 1, \sigma, q, r, t_{n-1}, T) + w_{n-1}C^E(u^*_{n-1}, u^*_{n-1}, \sigma, q, r, t_{n-1}, T) + \hat{w}_{n-1}P^E(u^*_{n-1}, 1, \sigma, q, r, t_{n-1}, T), \tag{8}$$

$$1 = \Delta_C^E(u^*_{n-1}, 1, \sigma, q, r, t_{n-1}, T) + w_{n-1}\Delta_C^E(u^*_{n-1}, u^*_{n-1}, \sigma, q, r, t_{n-1}, T) + \hat{w}_{n-1}\Delta_P^E(u^*_{n-1}, 1, \sigma, q, r, t_{n-1}, T), \tag{9}$$

$$0 = \Delta_C^E(1, 1, \sigma, q, r, t_{n-1}, T) + w_{n-1}\Delta_C^E(1, u^*_{n-1}, \sigma, q, r, t_{n-1}, T) + \hat{w}_{n-1}\Delta_P^E(1, 1, \sigma, q, r, t_{n-1}, T), \tag{10}$$

where $u^*(t_{n-1})$, denoted as u^*_{n-1} for convenience, w_{n-1}, and $\hat{w}_{n-1}$ are three unknowns to be solved, and $C^E(\bullet)$, $P^E(\bullet)$, $\Delta_C^E(\bullet)$, and $\Delta_P^E(\bullet)$ are the European call price, put price, call delta, and put delta, respectively. Note that Equations 8–10 correspond to Equations 5–7, respectively. The formulas for European option prices and deltas in the hypothetic world are

$$C^E(u, X, \sigma, q, r, t, T) = ue^{-r(T-t)}N(d_1) - Xe^{-q(T-t)}N(d_2), \tag{11}$$

$$P^E(u, X, \sigma, q, r, t, T) = Xe^{-q(T-t)}N(-d_2) - ue^{-r(T-t)}N(-d_1), \tag{12}$$

$$\Delta_C^E(u, X, \sigma, q, r, t, T) = e^{-r(T-t)}N(d_1), \tag{13}$$

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[7]Refer to Theorem 3 in Goldman et al. (1979). This theorem implies that when the stock price approaches $S_{\max}(t)$, the value of $P_{\text{Lookback}}^A(S_{\max}(t), S(t), t)$ is unaffected by marginal changes in the maximum asset price.

[8]We call this portfolio a HSHP because the European call and put options utilized here do not exist in the real world. Since our main purpose is to price AFSLPO using the value of the static hedging portfolio, it does not matter even if these hypothetic European options are not available. In the next section, we will turn to form an SSHP for AFSLPO using actual European options traded in the market.
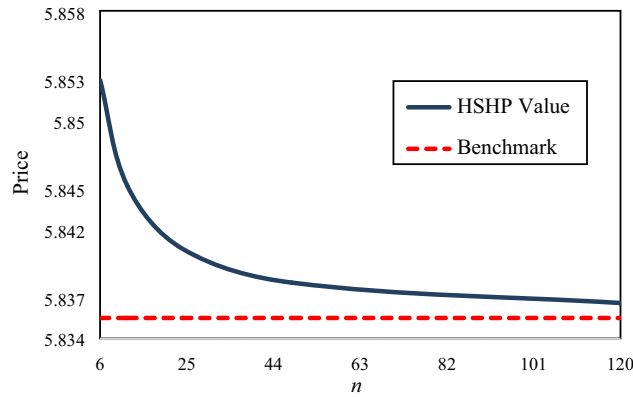
FIGURE 1 Convergence pattern of HSHP values to AFSLPO price. It shows a monotonic convergence of the HSHP values to the benchmark price of an AFSLPO. Here, we use the parameter settings of G. Chang, Kang, Kim, and Kim (2007): $S(t_0) = 50$, $u(t_0) = S_{\max}(t_0)/S(t_0) = 1.02$, $\sigma = 0.2$, $T-t_0 = 0.5$, $r = 0.05$, and $q = 0.05$ (corresponding to the 15th contract of Table 1). The benchmark AFSLPO price obtained from Equation (17) is 5.8355. AFSLPO: American floating strike lookback put option; HSHP: hypothetic static hedging portfolio [Color figure can be viewed at wileyonlinelibrary.com]

$$\Delta_P^E(u, X, \sigma, q, r, t, T) = -e^{-r(T-t)}N(-d_1), \tag{14}$$

where $N(\bullet)$ is the cumulative distribution function of the standard normal, $d_1 = (\ln(u/X) + (q - r + \sigma^2/2)(T - t))/\sigma\sqrt{T - t}$, and $d_2 = d_1 - \sigma\sqrt{T - t}$.

We follow similar procedures as those above and work backward to determine the portfolio weights and critical stock prices at each time point with matched boundary conditions. After solving all w_i, $\hat{w}_i$, and u_i^* ($i = n - 1, n - 2,...,0$), we approximate $C^A(u(t_0), 1, \sigma, q, r, t_0, T)$ using our HSHP as

$$\begin{aligned}
C_{\text{HSHP}}^A(u(t_0), 1, \sigma, q, r, t_0, T) = \ &C^E(u(t_0), 1, \sigma, q, r, t_0, T) + w_{n-1}C^E(u(t_0), u_{n-1}^*, \sigma, q, r, t_0, T) \\
&+ \hat{w}_{n-1}P^E(u(t_0), 1, \sigma, q, r, t_0, T) + w_{n-2}C^E(u(t_0), u_{n-2}^*, \sigma, q, r, t_0, t_{n-1}) \\
&+ \hat{w}_{n-2}P^E(u(t_0), 1, \sigma, q, r, t_0, t_{n-1}) + \cdots + w_0 C^E(u(t_0), u_0^*, \sigma, q, r, t_0, t_1) \\
&+ \hat{w}_0 P^E(u(t_0), 1, \sigma, q, r, t_0, t_1).
\end{aligned} \tag{15}$$

Finally, we multiply this value by the current asset price to get the cash price for the AFSLPO, that is,

$$\begin{aligned}
P_{\text{Lookback}}^A(S(t_0), S_{\max}(t_0), \sigma, r, q, t_0, T) &= V_{\text{Lookback}}^A(1, u(t_0), \sigma, r, q, t_0, T) \times S(t_0) \\
&\approx C_{\text{HSHP}}^A(u(t_0), 1, \sigma, q, r, t_0, T) \times S(t_0).
\end{aligned} \tag{16}$$

2.2 | Numerical efficiency of proposed method

In this subsection, we evaluate the numerical efficiency (in terms of speed and accuracy) of the HSHP approach for pricing an AFSLPO. We first investigate the accuracy and the convergence pattern of the proposed approach as n increases. Figure 1 shows the convergence pattern of the HSHP values (Equation (16)) to the benchmark price of an AFSLPO. The examined parameters are as follows: $S(t_0) = 50$, $u(t_0) = S_{\max}(t_0)/S(t_0) = 1.02$, $\sigma = 0.2$, $T - t_0 = 0.5$, $r = 0.05$, and $q = 0.05$. We compute the benchmark AFSLPO price (5.8355) based on the tree method of Babbs (2000) with an error adjustment, obtained by using the European pricing formula of Conze and Viswanathan (1991) as a control variate, as follows[9]

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[9]Hull and White (1988) propose reducing the pricing errors of tree method using the control variate technique. Moreover, Chung and Shackleton (2005) discuss how to choose good controls to reduce pricing errors effectively.

$$\text{Benchmark} = \text{Babbs's (2000) American option value}$$

$$+ \text{(Conze and Viswanathan (1991) European Closed-form option value} \tag{17}$$

$$- \text{Babbs's (2000) European option value),}$$

where Babbs's (2000) tree model is implemented with 1,000 time steps per day.

From Figure 1, we observe that the HSHP approach prices the AFSLPO accurately even at small values of n. For example, when $n = 6$, the value of the HSHP is 5.8530: The absolute (relative) pricing error is only 0.0175 (0.3%). Moreover, it is apparent that the HSHP values converge to the benchmark price of the AFSLPO when n increases; the convergence pattern seems monotonic.

In the following, we conduct a detailed efficiency analysis by comparing the accuracy and the computational time of the proposed method with those of the tree model of Babbs (2000). In Table 1, we price 36 AFSLPOs with the following parameter settings: $S(t_0) = 50$, $u(t_0) = S_{max}(t_0)/S(t_0) = 1.02$, $\sigma \in \{0.1, 0.2, 0.4\}$, $T - t_0 \in \{0.1, 0.3, 0.5\}$, and $(r, q) \in \{(0.025, 0.05), (0.05, 0.05), (0.05, 0.025), (0.05, 0)\}$. We measure the accuracy by the root-mean-squared error (RMSE) or the root-mean-squared relative error (RMSRE), where $RMSE = \sqrt{(1/36) \sum_{i=1}^{36} e_i^2}$, $RMSRE = \sqrt{(1/36) \sum_{i=1}^{36} \hat{e}_i^2}$, $e_i = |p_i^* - p_i|$ is the absolute error, $\hat{e}_i = (p_i^* - p_i)/p_i$ is the relative error, p_i is the benchmark value of the ith AFSLPO obtained based on Equation (17), and p_i^* is the estimated option price using the tree method or the static hedging approach. We measure the computational time by the CPU time required to calculate 36 AFSLPO prices using MATLAB with an Intel i7-6700K CPU and 16 GB RAM.

The numerical results include several points worth noting. First of all, consistent with Figure 1, Table 1 also indicates that the HSHP values converge to the benchmark value monotonically, where the convergence order is approximately of order $O(n)$. For example, the RMSEs (RMSREs) of the HSHP method are 0.0306, 0.0165, and 0.0087 (0.44%, 0.23%, and 0.12%) for $n = 6$, 12, and 24, respectively. The monotonic convergence pattern allows us to make the best use of the Richardson extrapolation method to further reduce the errors of the HSHP price estimates.[10] For instance, the RMSEs (RMSREs) of the extrapolated HSHP price estimates for $(n, 2n) = (6, 12)$ and $(12, 24)$ are 0.0028 and 0.0010 (0.03% and 0.01%), respectively, which are substantially less than those of the binomial tree prices and the unextrapolated HSHP values. In contrast, the RMSEs and RMSREs of Babbs (2000) in Table 1 imply that the price estimates obtained from the tree method are oscillatory around the benchmark value when Δt decreases.

Secondly, Table 1 shows that the computational time of the HSHP approach grows approximately at order $O(n)$, whereas the computational time of Babbs's (2000) tree method is approximately of order $O(n^2)$. For our HSHP approach, the computational times are 0.0317, 0.0583, and 0.1028 s for $n = 6$, 12, and 24, respectively. For Babbs's (2000) tree method, the computational times are 0.0123, 0.0432, and 0.1637 s for $\Delta t = 1/960$, $1/1,920$, and $1/3,840$, respectively. The quadratic growth in computational time of the tree method is consistent with findings in the literature (e.g., see Figlewski & Gao, 1999): As the total number of nodes of a binomial (or trinomial) tree is of order $O(n^2)$, the computational time of a tree model is also of order $O(n^2)$. The above analysis demonstrates that the proposed HSHP approach is more efficient than Babbs's (2000) tree method for large values of n.

Last, applying the HSHP method to price AFSLPOs has one unique advantage: Unlike the tree method, recalculation of AFSLPO prices is easy and quick when the asset price changes and/or time passes, because there is no need to solve the HSHP again. For example, if $S(t_0)$ in Table 1 increases by 1% instantaneously, the computational time of the HSHP method with $n = 24$ requires less than 0.01 s (0.01 s is about one-tenths of the original computational time) to recompute 36 AFSLPO prices because there is no need to solve the portfolio weights of the HSHPs. In contrast, when the stock price changes instantaneously, it still requires about 0.1637 s to recalculate the option prices for the tree method with $\Delta t = 1/3,840$.

3 | HEDGING PERFORMANCE OF SSHP AND DHP

It is emphasized above that as European options in the HSHP do not exist in the real world, they cannot be used as components of a hedging portfolio. To solve this problem, one can follow Fajardo and Mordecki (2006) to apply the put–call duality under the BS model to transform an HSHP into a tradable portfolio composed of European options

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[10]When the convergence order is $O(n)$, the extrapolated price estimate is $2 \times HSHP(2n) - HSHP(n)$, where $HSHP(2n)$ and $HSHP(n)$ are the HSHP price estimates for hedging frequencies equal to n and $2n$, respectively.

TABLE 1 Valuation of American floating strike lookback puts based on tree and HSHP approach

CN	σ	$T - t_0$	Tree $\Delta t = \frac{1}{960}$	$\Delta t = \frac{1}{1,920}$	$\Delta t = \frac{1}{3,840}$	HSHP $n = 6$	$n = 12$	$n = 24$	Benchmark
Panel A: $r = 0.025$; $q = 0.05$									
1	0.1	0.1	1.5938	1.5963	1.5954	1.5983	1.5969	1.5962	1.5955
2	0.1	0.3	2.5562	2.5576	2.5570	2.5687	2.5631	2.5602	2.5571
3	0.1	0.5	3.2873	3.2884	3.2880	3.3091	3.2990	3.2937	3.2880
4	0.2	0.1	2.8403	2.7606	2.7595	2.7667	2.7637	2.7620	2.7603
5	0.2	0.3	4.7923	4.7428	4.7421	4.7647	4.7543	4.7487	4.7426
6	0.2	0.5	6.2072	6.1672	6.1667	6.2055	6.1875	6.1776	6.1670
7	0.4	0.1	5.3628	5.3585	5.4027	5.3791	5.3706	5.3660	5.3611
8	0.4	0.3	9.4836	9.4812	9.5086	9.5422	9.5145	9.4993	9.4827
9	0.4	0.5	12.4380	12.4363	12.4584	12.5415	12.4931	12.4665	12.4374
Panel B: $r = 0.05$; $q = 0.05$									
10	0.1	0.1	1.5125	1.5150	1.5140	1.5146	1.5145	1.5145	1.5141
11	0.1	0.3	2.3403	2.3418	2.3411	2.3431	2.3422	2.3418	2.3411
12	0.1	0.5	2.9387	2.9399	2.9393	2.9430	2.9414	2.9405	2.9393
13	0.2	0.1	2.7677	2.6909	2.6898	2.6934	2.6922	2.6915	2.6906
14	0.2	0.3	4.5882	4.5420	4.5413	4.5515	4.5471	4.5446	4.5417
15	0.2	0.5	5.8724	5.8358	5.8352	5.8530	5.8452	5.8406	5.8355
16	0.4	0.1	5.2963	5.2916	5.3347	5.3077	5.3014	5.2980	5.2939
17	0.4	0.3	9.2824	9.2797	9.3095	9.3262	9.3055	9.2938	9.2808
18	0.4	0.5	12.1018	12.0997	12.1207	12.1804	12.1439	12.1234	12.1004
Panel C: $r = 0.05$; $q = 0.025$									
19	0.1	0.1	1.4597	1.4617	1.4605	1.4581	1.4594	1.4599	1.4606
20	0.1	0.3	2.2108	2.2118	2.2110	2.2023	2.2067	2.2088	2.2110
21	0.1	0.5	2.7433	2.7440	2.7433	2.7289	2.7363	2.7398	2.7434
22	0.2	0.1	2.7184	2.6429	2.6415	2.6408	2.6417	2.6419	2.6420
23	0.2	0.3	4.4637	4.4190	4.4181	4.4149	4.4171	4.4179	4.4183
24	0.2	0.5	5.6841	5.6492	5.6485	5.6439	5.6470	5.6481	5.6485
25	0.4	0.1	5.2531	5.2478	5.2899	5.2585	5.2544	5.2521	5.2494
26	0.4	0.3	9.1735	9.1702	9.1954	9.2007	9.1877	9.1799	9.1707
27	0.4	0.5	11.9441	11.9415	11.9615	11.9958	11.9721	11.9581	11.9417
Panel D: $r = 0.05$; $q = 0$									
28	0.1	0.1	1.4154	1.4171	1.4159	1.4109	1.4131	1.4146	1.4160
29	0.1	0.3	2.1065	2.1073	2.1064	2.0890	2.0977	2.1021	2.1066
30	0.1	0.5	2.5906	2.5911	2.5904	2.5615	2.5758	2.5830	2.5906
31	0.2	0.1	2.6738	2.5995	2.5980	2.5931	2.5959	2.5972	2.5982
32	0.2	0.3	4.3545	4.3113	4.3104	4.2913	4.3030	4.3069	4.3103
33	0.2	0.5	5.5224	5.4890	5.4882	5.4634	5.4763	5.4824	5.4881
34	0.4	0.1	5.2123	5.2066	5.2479	5.2117	5.2104	5.2093	5.2076
35	0.4	0.3	9.0731	9.0695	9.0938	9.0848	9.0789	9.0750	9.0695
36	0.4	0.5	11.8009	11.7980	11.8171	11.8270	11.8153	11.8076	11.7977
RMSE			0.0325	0.0010	0.0173	0.0306	0.0165	0.0087	
RMSRE			0.0104	0.0003	0.0028	0.0044	0.0023	0.0012	
Time			0.0123	0.0432	0.1637	0.0317	0.0583	0.1028	

Note. The accuracy and computational time for our HSHP approach and the binomial tree approach in Babbs (2000) for pricing AFSLPOs are compared. There are 36 contracts; CN denotes the contract number. The parameter values common in the four panels are $S(t_0) = 50$ and $u(t_0) = S_{max}(t_0)/S(t_0) = 1.02$. Time steps $\Delta t = 1/960$, $1/1,920$, and $1/3,840$ correspond to 4, 8, and 16 time steps per day in the binomial tree model of Babbs (2000). The benchmark option values are evaluated according to Equation (17). The root-mean-squared error (RMSE) is defined by $\mathrm{RMSE} = \sqrt{(1/36) \sum_{i=1}^{36} e_i^2}$, where $e_i = |p_i^* - p_i|$ is the absolute error, p_i is the benchmark value, and p_i^* is the estimated option price using the tree method or HSHP approach. The root-mean-squared relative error (RMSRE) is defined by $\mathrm{RMSRE} = \sqrt{(1/36) \sum_{i=1}^{36} \hat{e}_i^2}$, where $\hat{e}_i = (p_i^* - p_i)/p_i$ is the relative error. The CPU time reports the computational time for all 36 contracts using MATLAB with an Intel i7-6700K CPU and 16 GB RAM.

AFSLPO: American floating strike lookback put option; HSHP: hypothetic static hedging portfolio.

existing in the real world. For example, the European call $C^E(u(t_0), 1, \sigma, q, r, t_0, T)$ in Equation (15) has the same value (denominated in units of the underlying asset) as the European put $P^E(1, u(t_0), \sigma, r, q, t_0, T)$. Multiplying this value by the current stock price yields $P^E(1, u(t_0), \sigma, r, q, t_0, T)S(t_0) = P^E(S(t_0), S_{\max}(t_0), \sigma, r, q, t_0, T)$. Note that this European put with a strike price equal to $S_{\max}(t)$ is now tradable in the real world. Repeating this procedure for all component European options of the HSHP in Equation (15) yields an n-point semistatic hedge portfolio (SSHP(n)), composed of tradable European call and put options:

$$
\begin{aligned}
\text{SSHP}(n) = {}& P^E(S(t_0), S_{\max}(t_0), \sigma, r, q, t_0, T) \\
& + u^*_{n-1}w_{n-1}P^E(S(t_0), S^*_{n-1}(t_0), \sigma, r, q, t_0, T) + \hat{w}_{n-1}C^E(S(t_0), S_{\max}(t_0), \sigma, r, q, t_0, T) \\
& + u^*_{n-2}w_{n-2}P^E(S(t_0), S^*_{n-2}(t_0), \sigma, r, q, t_0, t_{n-1}) + \hat{w}_{n-2}C^E(S(t_0), S_{\max}(t_0), \sigma, r, q, t_0, t_{n-1}) + \cdots \\
& + u^*_0 w_0 P^E(S(t_0), S^*_0(t_0), \sigma, r, q, t_0, t_1) + \hat{w}_0 C^E(S(t_0), S_{\max}(t_0), \sigma, r, q, t_0, t_1),
\end{aligned}
\tag{18}
$$

where $S^*_i(t) = S_{\max}(t)/u^*_i$. Note that there are $2n + 1$ component options in an SSHP(n), including one European put option with strike price $S_{\max}(t)$ and maturity date T, n European put options with strike prices $S^*_i(t)$ and maturity date t_{i+1}, and n European call options with strike prices $S_{\max}(t)$ and maturity date t_{i+1}, respectively, for $i = 0, 1, 2, ..., n - 1$.

Note that for an issuer of an AFSLPO, he can hedge his liability by holding the corresponding SSHP(n) specified in Equation (18). However, in contrast to traditional static hedging approaches, purchasing the corresponding SSHP(n) at the issuance date is not a once-and-for-all hedging strategy since the maximum of the underlying stock price could change with the passage of time. Thus the issuer must rebalance the SSHP(n) when a new maximum of the stock price is observed.[11] For this reason we term the proposed method a "semistatic" hedging portfolio approach.

We conduct an analysis to compare the hedging performance for the SSHP and the DHP approaches. In our analysis, we simulate 2,000 asset price paths with 1,000 time steps per day. For the DHP, we calculate the delta values of the AFSLPO at the rebalancing time point, based on Babbs's (2000) tree model with 1,000 time steps per day. To demonstrate the superior hedging performance of the SSHP approach, we examine it with $n = 6$ only. Moreover, to evaluate the hedging errors on the early exercise boundary, we solve the accurate early exercise boundary $u^*(t)$ in $[t_0, T]$ by equalizing the benchmark option value in Equation (17) with the early exercise value $(S_{\max}(t) - S(t))^+ = S(t)(u(t)-1)^+$ at each rebalancing time point. Finally, both SSHP and DHP are rebalanced according to information on the prevailing asset price and realized maximum asset price every m days. In our analysis, $m = 0.1, 0.25, 0.5, 1, 2,$ and 4 are examined.

The hedging error on a simulated path for the SSHP approach is calculated via the following three-step method.[12]

Step 1. Suppose the issuer sells the AFSLPO at the benchmark option value P_{t_0} and uses the sales proceeds to purchase the SSHP, whose value is denoted as $\text{SSHP}_{t_0}(S(t_0), S_{\max}(t_0))$.[13] The remaining part (could be negative), that is, $\eta_{t_0} = P_{t_0} - \text{SSHP}_{t_0}(S(t_0), S_{\max}(t_0))$, is invested to earn the risk-free interest rate r.

Step 2. On any rebalancing time point τ, the hedging position value evolves to be $\pi_\tau = \text{SSHP}_\tau(S(\tau), S_{\max}(\tau - \Delta t)) + \eta_\tau$,[14] where $\eta_\tau = \eta_{\tau-\Delta t}e^{r\Delta t}$. If the early exercise condition is triggered, that is, $u(\tau) \geq u^*(\tau)$, the hedging process is terminated early at time τ and the hedging error (HE), defined as the negative present value of the profit and loss from writing a lookback option and hedging it with either the DHP or the SSHP, is calculated as[15]

$$
HE(\tau) = -e^{-r\tau}(\pi_\tau - (S_{\max}(\tau) - S(\tau))^+).
\tag{19}
$$

Otherwise, if $S_{\max}(\tau) = S_{\max}(\tau - \Delta t)$, that is, the realized maximum asset price does not change from time $\tau - \Delta t$ to time τ, we hold the same portfolio, that is, $\text{SSHP}_\tau(S(\tau), S_{\max}(\tau)) = \text{SSHP}_\tau(S(\tau), S_{\max}(\tau - \Delta t))$; else, $S_{\max}(\tau) > S_{\max}(\tau - \Delta t)$, that is, a new maximum asset price is observed from time $\tau - \Delta t$ to time τ, rebalancing

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[11]One special feature of the SSHP(n) is that the calls and puts of the newly rebalanced portfolio have different strike prices depending on the newly realized maximum of the stock price but with the same portfolio weights.

[12]The hedging error for the DHP approach is similar and presented in Appendix A.

[13]We include parameter $S_{\max}(t)$ is because this information is needed to determine the strike prices of the options in the SSHP, as shown in Equation (18).

[14]Here Δt denotes the time interval between two rebalancing time points, which is different from the time step to implement Babbs's (2000) tree method.

[15]Note that the issuer is concerned more with losses than profits. Thus we define the hedging error in such a way that it is positive (negative) when the issuer has a loss (profit) from writing a lookback option and hedging it with either the DHP or SSHP.

the SSHP by selling the original portfolio, $SSHP_\tau(S(\tau), S_{max}(\tau - \Delta t))$, and purchasing a new portfolio, $SSHP_\tau(S(\tau), S_{max}(\tau))$, with the rebalancing cost

$$RC = SSHP_\tau(S(\tau), S_{max}(\tau)) - SSHP_\tau(S(\tau), S_{max}(\tau - \Delta t)). \quad (20)$$

Meanwhile, the balance of the cash account is updated as $\eta_\tau = \eta_{\tau-\Delta t} e^{r\Delta t} - RC$.

Step 3. If Step 2 is repeated until the maturity T, the hedging position value evolves to be $\pi_T = SSHP_T(S(T), S_{max}(T - \Delta t)) + \eta_T$, where $\eta_T = \eta_{T-\Delta t} e^{r\Delta t}$ and the present value of the hedging error is

$$HE(T) = -e^{-rT}(\pi_T - (S_{max}(T) - S(T))^+). \quad (21)$$

In Figure 2, we show dynamically the replication mismatches of SSHP and DHP with daily rebalancing ($m = 1$) for the 15th contract ($S(t_0) = 50$, $u(t_0) = S_{max}(t_0)/S(t_0) = 1.02$, $\sigma = 0.2$, $T - t_0 = 0.5$, $r = 0.05$, and $q = 0.05$) of Table 1. We arbitrarily choose as examples two paths from the 2,000 simulated paths. Panel A presents the mismatches under the path where the AFSLPO is not exercised early, and Panel B presents the results when the American option is exercised early. It is apparent from Figure 2 that the SSHP values are generally very close to the benchmark values of the AFSLPO as time passes and the asset price changes, especially when the option is exercised early (see Panel B). In contrast, the DHP replication mismatches are usually larger than those of the SSHP and the maximum hedging error that occurs in Panel A is nearly 20% of the option value. Other unreported paths have similar results regardless of whether the AFSLPO is exercised early or not in each path. Therefore we conclude that the dynamic hedging performance of the SSHP approach is superior to that of the DHP approach.

The profit and loss distributions of the above 2,000 paths for the SSHP and the DHP methods with daily rebalancing are shown in Figure 3a. We further simulate another 2,000 paths, repeat the procedure above, and then present the results in Figure 3b for the 33rd contract of Table 1. Figure 3 shows that the SSHP profit and loss distributions have smaller standard deviations than those of the DHP. In other words, the risk associated with writing an AFSLPO and hedging it with the SSHP is far smaller than that with the DHP.

To further assess the hedging effectiveness concisely, we employ four risk measures used in Siven and Poulsen (2009) to estimate the hedging performance of the SSHP and the DHP approaches. In addition, to reduce the sampling error from using only 2,000 simulated paths, we calculate four risk measures based on the demeaned hedging error (HE^*), where $HE^* = HE - E[HE]$. The first risk measure, the 5% value-at-risk, is defined as $VaR^*_{0.05} = \inf\{z \in \Re; \Pr(HE^* \geq z) \leq 0.05\}$. Because of the Basel accords for banking regulations, this risk measure is one of the most popular risk measures in practice. The second risk measure, the expected shortfall (also known as conditional value-at-risk), is the mean loss beyond the 5% value-at-risk, that is, $ES^*_{0.05} = E[HE^*|HE^* \geq VaR^*_{0.05}]$. The third risk measure, $E[HE^{*2}]$, represents the quadratic hedging error. By defining HE^{*+} as $\max(0, HE^*)$, the fourth risk measure, $E[HE^{*+}]$, represents the expected loss of the issuer and is thus a one-sided risk measure.

To generate Table 2, we simulate 2,000 paths, calculate the demeaned hedging errors of the SSHP ($n = 6$) and the DHP approaches, and then compute the four risk measures for the 13th and 31st contracts of Table 1. There are several points worth discussing from our numerical results. First of all, Table 2 shows that the risk associated with the SSHP approach is smaller than that with the DHP approach, regardless of the risk measure used. For example, when $m = 4$ (i.e., rebalancing the hedge portfolio every 4 days), the SSHP has only 62%, 63%, 81%, and 56% of the risk of the DHP for hedging the 13th contract of Table 1 in terms of $VaR^*_{0.05}$, $ES^*_{0.05}$, $E[HE^{*2}]$, and $E[HE^{*+}]$, respectively. Second, as expected, the risk decreases when the rebalancing frequency increases (m decreases) for both the SSHPs and the DHPs. For example, the $VaR^*_{0.05}$ of the DHP with daily rebalancing for hedging the 31st contract is only 63% of the $VaR^*_{0.05}$ of the DHP with $m = 4$. Finally, as m decreases, the risk of the SSHP approach improves more than that of the DHP approach for all the risk measures considered in this paper. For instance, when the risk measure $VaR^*_{0.05}$ is used, the risk of the DHP for hedging the 13th contract reduces by only 59% (from 1.1978 to 0.4936) when the rebalancing frequency increases 40 times (from $m = 4$ to 0.1). In contrast, the $VaR^*_{0.05}$ of the SSHP decreases by 96% (from 0.7451 to 0.0285) when the rebalancing frequency increases 40 times. To sum up, Table 2 indicates that the hedging performance of an SSHP is far less risky than that of a delta-hedged portfolio for all risk measures and rebalancing frequencies.
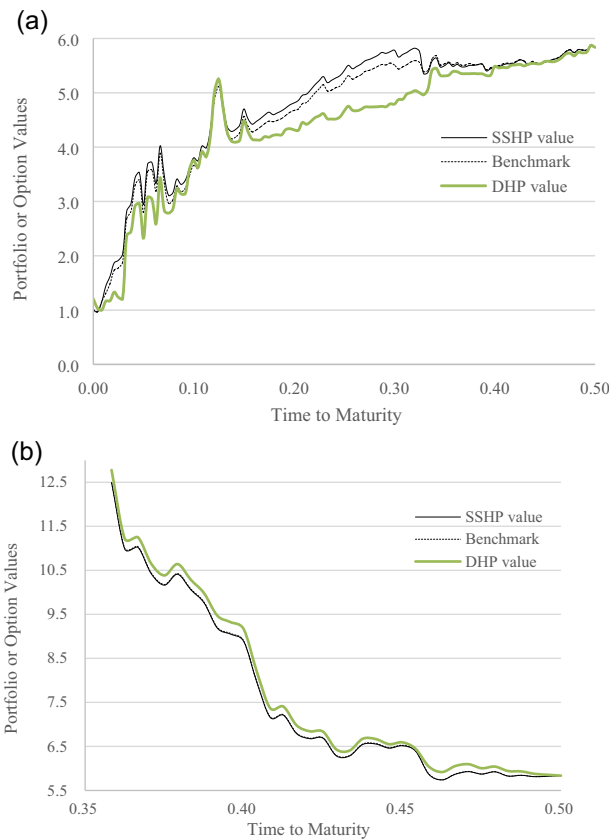
FIGURE 2 Replication mismatches of SSHP and DHP with daily rebalancing ($m = 1$). It shows replication mismatches of SSHP and DHP for an AFSLPO under two simulated paths. (a) presents the mismatches under a path where the AFSLPO is not exercised early, whereas (b) presents the results for another case in which the American option is exercised early. These paths are arbitrarily selected from two of the 2,000 simulated asset price paths for the 15th contract ($S(t_0) = 50$, $u(t_0) = S_{max}(t_0)/S(t_0) = 1.02$, $\sigma = 0.2$, $T - t_0 = 0.5$, $r = 0.05$, and $q = 0.05$) of Table 1. The benchmark value at each time point is obtained from Equation (17). AFSLPO: American floating strike lookback put option; DHP: delta-hedged portfolio; SSHP: semistatic hedging portfolio [Color figure can be viewed at wileyonlinelibrary.com]

4 | DISCUSSION OF MODEL RISK

Since our SSHP approach is developed under BS dynamics, a natural question is how sensitive the proposed semistatic hedge is to model risk, that is, how bad the AFSLPO issuer is off when it turns out that the real-world option prices and underlying dynamics are more complicated than those described in the BS model.[16]

In this section, we discuss the impact of model risk on the hedging performance of our SSHP approach. Specifically, we concentrate on the model risk in a situation where the option issuer believes that the asset price follows the BS model but in fact its market dynamics follow Heston's (1993) SV model.[17] The reason to focus on examining hedging performance rather than the pricing accuracy of our approach under the SV model is due to the unavailability of an option value benchmark. To the best of our knowledge, there is no feasible pricing method for the AFSLPO under the SV model in the literature.

Following Heston (1993), the underlying stock price $S(t)$ and its variance $V(t)$ under the risk-neutral measure are, for any t, given by

$$\frac{dS(t)}{S(t)} = (r - q)dt + \sqrt{V(t)}\,dW_1(t), \tag{22}$$

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[16]We are grateful to the anonymous reviewer for pointing out this important issue.

[17]We know that model risk emerges as long as the underlying asset price dynamics do not follow geometric Brownian motion. However, here we only examine the SV model, which is one of the most popular alternatives of the BS model in practice, since it is impossible to exhaust all different models for the underlying asset price process.
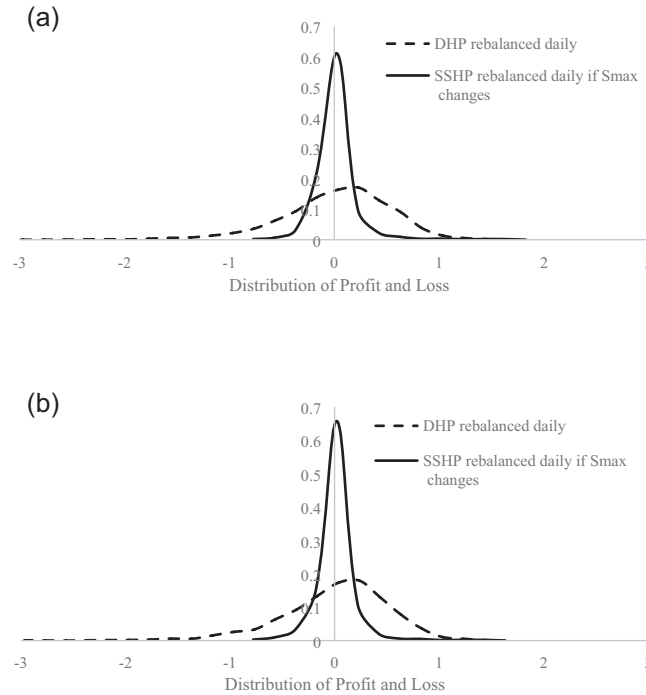
(a)

(b)

FIGURE 3 Profit and loss distributions for SSHP and the DHP with daily rebalancing ($m = 1$). It shows the distributions of profit and loss from selling an AFSLPO and hedging it using either SSHP or DHP with daily rebalancing. The distributions are obtained based on 2,000 simulated stock price paths, with 1,000 time steps per day. (a) presents the result for the 15th contract ($S(t_0) = 50$, $u(t_0) = S_{\max}(t_0)/S(t_0) = 1.02$, $\sigma = 0.2$, $T - t_0 = 0.5$, $r = 0.05$, and $q = 0.05$) and (b) presents the outcome for the 33rd contract ($S(t_0) = 50$, $u(t_0) = S_{\max}(t_0)/S(t_0) = 1.02$, $\sigma = 0.2$, $T - t_0 = 0.5$, $r = 0.05$, and $q = 0$) of Table 1, respectively. AFSLPO: American floating strike lookback put option; DHP: delta-hedged portfolio; SSHP: semistatic hedging portfolio

$$dV(t) = [\theta_v - \kappa_v V(t)]dt + \sigma_v \sqrt{V(t)}\left(\rho dW_1(t) + \sqrt{(1-\rho)}\, dW_2(t)\right), \tag{23}$$

where $W_1(t)$ and $W_2(t)$ are two independent standard Brownian motions, r is the interest rate, q is the dividend yield, κ_v is the speed of adjustment, θ_v/κ_v is the long-run mean of $V(t)$, σ_v is the variation coefficient of the diffusion volatility of $V(t)$, and ρ is the correlation coefficient of stock price and volatility. For determining the parameters in the $V(t)$ process, we refer to the work of Bakshi, Cao, and Chen (1997). We adopt their empirical estimations based on at-the-money S&P 500 index options: $\kappa_v = 0.99$, $\theta_v = 0.04$, $\sigma_v = 0.4$, and $\rho = -0.7$.[18] These parameter values are chosen because we aim to examine the impact of introducing stochastic features of the volatility (rather than the influence caused by using different magnitudes of the volatility level) on the hedging performance. Note that $\kappa_v = 0.99$ and $\theta_v = 0.04$ implies the square root of long-run mean of $V(t)$ is close to the constant volatility of the BS model used to generate Table 2, that is, $\sqrt{0.04/0.99} = 0.2010 \approx 0.2$. Moreover, the initial variance of the SV model, $V(t_0)$, is determined such that the implied volatility of the SV model is comparable to the constant volatility of the BS model.[19] With the above settings, it becomes meaningful to examine the model risk by comparing the hedging performance under the SV model with that under the original BS model.

In the paper, we take the 13th contract of Table 1 as an example to conduct the hedging performance analysis for our SSHP approach with the presence of model risk. Specifically, the parameter values are $S(t_0) = 50$, $u(t_0) = S_{\max}(t_0)/S(t_0) = 1.02$, $T - t_0 = 0.1$, $r = 0.05$, $q = 0.05$, $\kappa_v = 0.99$, $\theta_v = 0.04$, and $\sigma_v = 0.4$. In addition to $\rho = -0.7$, we also examine the other two cases with $\rho = 0$ and $\rho = 0.7$. Moreover, we choose the rebalancing frequency $m = 0.25$ (rebalancing the hedging portfolios four times a day) to alleviate discretization errors.

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[18]Refer to Table III of Bakshi et al. (1997).

[19]Specifically, we determine the initial variance $V(t_0)$ such that the European put price under the SV model, $P^E_{SV}(S(t_0), S_{\max}(t_0), \sqrt{V(t_0)}, r, q, t_0, T)$ (its pricing formula is introduced later), is equal to the European put price under the original BS model, $P^E(S(t_0), S_{\max}(t_0), \sigma, r, q, t_0, T)$, in Equation (18) given $\sigma = 0.2$, which is the volatility parameter used in Table 2.

Since we assume that the option issuer believes the asset price follows the BS model, he thus determines the early exercise boundary $u^*(t)$ by equalizing the benchmark option value in Equation (17) with the early exercise value $(S_{max}(t)-S(t))^+ = S(t)(u(t)-1)^+$ and solves the weights w_i and $\hat{w}_i$ and the critical boundary u_i^* at each time point $(i = n - 1, n - 2,...,0)$ based on Equations 8–14 under the BS model. Equipped with the obtained early exercise boundary, w_i, $\hat{w}_i$, and u_i^*, the option issuer implements the three-step hedging procedures described in Section 3 of this paper. However, when the option issuer initially constructs or later rebalances the SSHP through trading the corresponding European calls and puts from the market, the market prices of those options under the assumed market dynamics (SV model) are determined by Heston's closed-form option pricing formulas. For a European call with a strike price X, its pricing formula is

$$
\begin{aligned}
C_{SV}^E(S(t), X, \sqrt{V(t)}, r, q, t, T) &\equiv C_{SV}^E(S(t), X, \sqrt{V(t)}, r, q, t, T|\kappa_v, \theta_v, \sigma_v, \rho) \\
&= \frac{1}{2}J(T - t; -i) - \frac{1}{\pi}\int_0^\infty \frac{\text{Im}[J(T - t;-i - u)e^{iu\ln X}]}{u}du \\
&\quad - K\left[\frac{1}{2}J(T - t;0) - \frac{1}{\pi}\int_0^\infty \frac{\text{Im}[J(T - t;- u)e^{iu\ln X}]}{u}du\right],
\end{aligned} \quad (24)
$$

where $J(\tau;\phi)$ represents the characteristic function expressed as

$$J(\tau;\phi) = e^{A(\tau;\phi)+B(\tau;\phi)V(t)}S(t)^{i\phi},$$

$$A(\tau;\phi) = [\phi i(r - q) - r]\tau - \frac{\theta_v}{\sigma_v^2}\left[(\varepsilon + i\phi\sigma_v\rho - \kappa_v)\tau + 2\ln\left(1-\frac{(\varepsilon + i\phi\sigma_v\rho - \kappa_v)(1 - e^{-\varepsilon\tau})}{2\varepsilon}\right)\right],$$

$$B(\tau;\phi) = \frac{i\phi(i\phi - 1)(1 - e^{-\varepsilon\tau})}{2\varepsilon - (\varepsilon + i\phi\sigma_v\rho - \kappa_v)(1 - e^{-\varepsilon\tau})},$$

$$\varepsilon = \sqrt{(i\phi\sigma_v\rho - \kappa_v)^2 - i\phi(i\phi - 1)\sigma_v^2}.$$

As for the European put, its price can be obtained via the put–call parity, that is,

$$
\begin{aligned}
P_{SV}^E(S(t), X, \sqrt{V(t)}, r, q, t, T) &\equiv P_{SV}^E(S(t), X, \sqrt{V(t)}, r, q, t, T|\kappa_v, \theta_v, \sigma_v, \rho) = C_{SV}^E(S(t), X, V(t), r, q, t, T) \\
&\quad - S(t)e^{-q(T-t)} + Xe^{-r(T-t)}.
\end{aligned} \quad (25)
$$

The two integrals in Equation (24) are evaluated numerically using the standard rectangle method with $du = 0.01$.[20]

In the above analysis framework, model risk arises due to the incorrect weights w_i and $\hat{w}_i$, critical boundary u_i^*, and early exercise boundary,[21] all of which are solved based on the BS model. Similar to Table 2, the four risk measures of hedging performance are calculated based on the demeaned hedging error HE^*, which is used to mitigate the sampling error in our analysis of hedging performance. It should be noted that the sampling error is even more serious when the SV model is examined, as we find that 2,000 simulated paths are sometimes not enough to represent the actual asset

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[20]Since it is impossible to use an infinite upper bound in a numerical integration method, we examine the effect of different values of the upper bound in the integrals of Heston's option pricing formulas by investigating whether the option prices and hedging performance under the degenerated SV model with a very small volatility of volatility (e.g., $\sigma_v = 0.000001$) converge to those of the BS model. At the beginning of our analysis, we simply set the upper bound to be 100 which we believed to be a conservative assumption. However, the hedging performance of the degenerated SV model would not converge to that of the BS model. We finally found the problem: Under the extremely short time to maturity (e.g., 0.001 years), a large upper-bound value (e.g., 1,000) is needed to ensure the pricing accuracy of Heston's option pricing formulas. To the best of our knowledge, this accuracy issue for short-maturity options has never been discussed in the literature. Consequently, we set the upper bound of integrals in Heston's option pricing formulas to 1,000 when generating numerical results associated with the model risk in Section 4.

[21]We assume that option holder exercises the AFSLPO when the early exercise boundary $u^*(t) = S_{max}(t)/S^*(t)$ is reached. In our analysis, we have no choice but to adopt this assumption because as we know, there is no feasible pricing method for the AFSLPO in the literature, and without that, one cannot identify the actual early exercise boundary, which we expect to be a function of not only $S_{max}(t)$ and $S^*(t)$ but also $V(t)$ in the SV model.

TABLE 2 Hedging performance of SSHP and DHP for AFSLPO

Panel A: 13th contract ($S(t_0) = 50$, $u(t_0) = S_{max}(t_0)/S(t_0) = 1.02$, $\sigma = 0.2$, $T - t_0 = 0.1$, $r = 0.05$, and $q = 0.05$)

	DHP					
Risk measures	**$m = 4$**	**$m = 2$**	**$m = 1$**	**$m = 0.5$**	**$m = 0.25$**	**$m = 0.1$**
$VaR^*_{0.05}$	1.1978	0.9684	0.7366	0.6383	0.6082	0.4936
$ES^*_{0.05}$	1.6070	1.3024	1.0447	0.8993	0.8289	0.7329
$E[HE^{*2}]$	0.4229	0.2585	0.1683	0.1211	0.0938	0.0682
$E[HE^{*+}]$	0.5457	0.4320	0.3502	0.3074	0.2813	0.2401
	SSHP ($n = 6$)					
Risk measures	**$m = 4$**	**$m = 2$**	**$m = 1$**	**$m = 0.5$**	**$m = 0.25$**	**$m = 0.1$**
$VaR^*_{0.05}$	0.7451	0.4553	0.3117	0.1809	0.1042	0.0285
$ES^*_{0.05}$	1.0193	0.6713	0.4641	0.2857	0.1734	0.0431
$E[HE^{*2}]$	0.3440	0.1402	0.0594	0.0238	0.0099	0.0012
$E[HE^{*+}]$	0.3058	0.1314	0.0882	0.0558	0.0371	0.0148

Panel B: 31st contract ($S(t_0) = 50$, $u(t_0) = S_{max}(t_0)/S(t_0) = 1.02$, $\sigma = 0.2$, $T - t_0 = 0.1$, $r = 0.05$, and $q = 0$)

	DHP					
Risk measures	**$m = 4$**	**$m = 2$**	**$m = 1$**	**$m = 0.5$**	**$m = 0.25$**	**$m = 0.1$**
$VaR^*_{0.05}$	1.2047	0.9846	0.7546	0.6298	0.5848	0.5091
$ES^*_{0.05}$	1.6381	1.3332	1.0519	0.8929	0.807	0.7237
$E[HE^{*2}]$	0.4414	0.2667	0.1728	0.1186	0.0906	0.0651
$E[HE^{*+}]$	0.5559	0.4336	0.3661	0.3029	0.2708	0.2413
	SSHP ($n = 6$)					
Risk measures	**$m = 4$**	**$m = 2$**	**$m = 1$**	**$m = 0.5$**	**$m = 0.25$**	**$m = 0.1$**
$VaR^*_{0.05}$	0.7677	0.4679	0.3234	0.1852	0.1107	0.0335
$ES^*_{0.05}$	1.0416	0.696	0.4767	0.2936	0.1804	0.0485
$E[HE^{*2}]$	0.3539	0.1436	0.0601	0.0234	0.009	0.0009
$E[HE^{*+}]$	0.2502	0.1525	0.1019	0.0593	0.0358	0.0124

Note. The hedging performance for the SSHP and DHP approaches for the 13th and 31st contracts of Table 1 is compared. Denote m as the time interval (measured in days) to rebalance the portfolio. We simulate 2,000 stock price paths, with 1,000 time steps per day. We calculate four risk measures used in Siven and Poulsen (2009) with the demeaned hedging error (HE^*) to assess the hedging performance, where $HE^* = HE - E[HE]$. The first risk measure, 5% value-at-risk, is defined as $VaR^*_{0.05} = \inf\{z \in \Re; \Pr(HE^* \geq z) \leq 0.05\}$. The second risk measure, the expected shortfall, is the mean loss beyond the 5% value-at-risk, that is, $E_{0.05} = E[HE^*|HE^* \geq VaR^*_{0.05}]$. The third risk measure, $E[HE^{*2}]$, represents the quadratic hedging error. The fourth risk measure is the expected loss $E[HE^{*+}]$. AFSLPO: American floating strike lookback put option; DHP: delta-hedged portfolio; SSHP: semistatic hedging portfolio.

price dynamics. Another reason to adopt the demeaned hedging error HE^* rather than the raw hedging error HE is that due to the unavailability of benchmark values for AFSLPOs under the SV model, we assume the issuance price of the AFSLPO is equal to the cost of constructing the initial SSHP, whose value is calculated based on Heston's closed-form option pricing formulas in Equations 24 and 25. As a result, the initial value of the cash part η_{t_0} in the first step of the SSHP hedging procedure is always zero for the option issuer. In contrast, under the BS model, we assume the option issuer sells the AFSLPO at its benchmark value and thus η_{t_0}, which captures the difference between the benchmark value and the value of the initial SSHP, is usually nonzero. The demeaned hedging error HE^* effectively eliminates this initial difference of η_{t_0} under the SV and BS models.

Table 3 shows the model risk for the SSHP ($n = 6$) with the rebalancing frequency $m = 0.25$ for the 13th contract of Table 1. The second to fourth columns present the results of the four risk measures under the SV model given $\rho = -0.7$, 0, and 0.7, respectively. The last column, copied from Table 2a for comparison, presents the hedging performance under

TABLE 3 Model risk for proposed SSHP approach under Heston's SV model

Risk measures	SV model			BS model
	$\rho = -0.7$	$\rho = 0$	$\rho = 0.7$	
$VaR^*_{0.05}$	0.1304	0.1110	0.2470	0.1042
$ES^*_{0.05}$	0.1734	0.1915	0.3668	0.1734
$E[HE^{*2}]$	0.0274	0.0114	0.0212	0.0099
$E[HE^{*+}]$	0.0978	0.0393	0.1242	0.0371

Note. The degree of the model risk for the proposed SSHP approach under Heston's SV model is analyzed. The results are based on the hedging performance of the SSHP ($n = 6$) with the rebalancing frequency $m = 0.25$ (measured in days) for the 13th contract of Table 1. The second to fourth columns present the hedging performance under the SV model with different levels of the correlation coefficient ($\rho = -0.7$, 0, and 0.7). The other parameters are $S(t_0) = 50$, $u(t_0) = S_{max}(t_0)/S(t_0) = 1.02$, $\kappa_v = 0.99$, $\theta_v = 0.04$, $\sigma_v = 0.4$, $T - t_0 = 0.1$, $r = 0.05$, and $q = 0.05$, respectively. Moreover, the initial variance of the SV model, $V(t_0)$, is determined such that the implied volatility of the SV model is comparable to the constant volatility of the BS model. For comparison, the last column shows the hedging performance under the original BS model, which are the results reported in Table 2. We use the four risk measures in Siven and Poulsen (2009) to evaluate the hedging performance based on the demeaned hedging error $HE^* = HE - E[HE]$. BS: Black–Scholes; SSHP: semistatic hedging portfolio; SV: stochastic volatility.

the original BS model. The differences across columns represent the model risk considered in our analysis. There are three points worth discussing from Table 3.

First of all, Table 3 demonstrates the existence of the model risk where all of the four risk measures are larger than those in the original BS model whether ρ is -0.7, 0, or 0.7. Taking $VaR^*_{0.05}$ for $\rho = -0.7$ as an example, this 5% value-at-risk increases by 0.0262 (= 0.1304 – 0.1042) dollars, which represents approximately 1% of 2.6906 dollars, the AFSLPO option value of the 13th contract in Table 1. It seems an acceptable model risk level in practice because when financial institutions issue options, the offer prices are usually marked up by 10–20% of their theoretical prices to absorb risks such as unpredictable change in volatility. Secondly, the hedging performance of the case with $\rho = 0$ does not deteriorate that much and behaves analogously to that of the original BS model. For example, the $VaR^*_{0.05}$ for $\rho = 0$ increases by only 0.0068 dollars. We conjecture that the smallest model risk for $\rho = 0$ is because the asset price dynamics in the SV model with $\rho = 0$ (compared to $\rho = \pm0.7$) is closest to that of the original BS model due to the independence of the asset price and its volatility. Thirdly, comparing with the case of $\rho = -0.7$, the case of $\rho = 0.7$ in general shows a higher degree of model risk. For three out of the four risk measures used, including $VaR^*_{0.05}$, $ES^*_{0.05}$, and $E[HE^{*+}]$, their values for $\rho = 0.7$ (0.2470, 0.3668, and 0.1242, respectively) are significantly higher than those for $\rho = -0.7$ (0.1304, 0.1734, and 0.0978, respectively). Note also that this asymmetric impact on the hedging performance becomes more pronounced especially for the first two risk measures, $VaR^*_{0.05}$ and $ES^*_{0.05}$, which are tail risk measures representing extreme hedging costs. This phenomenon is probably due to the impact of the correlation between $S(t)$ and $V(t)$ on the rebalance cost for our SSHP approach when a new S_{max} occurs. For $\rho = 0.7$ ($\rho = -0.7$), the positive (negative) correlation implies that higher (lower) volatilities follow higher asset prices. As a result, the volatility must be relatively high (low) in scenarios when the rising asset price paths cause a new S_{max}, where our SSHP needs rebalancing. Consequently, at rebalancing time points, the market (SV model) prices of individual European options and thus the rebalance costs for the SSHP are more expensive (cheaper) for $\rho = 0.7$ ($\rho = -0.7$) because the volatility is relatively high (low). The accumulated hedging costs are thus amplified, due to the positive correlation between $S(t)$ and $V(t)$, which explains the larger values of $VaR^*_{0.05}$ and $ES^*_{0.05}$ given $\rho = 0.7$.

5 | CONCLUSION

In this paper, we extend the static hedging approach developed in Derman et al. (1995) and Carr et al. (1998) to price and hedge AFSLPOs under the BS model. We first obtain the AFSLPO values by constructing an HSHP to match the complicated boundary and terminal conditions. Compared with the tree method, when the asset price changes and/or time passes, the recalculation of the HSHP for various AFSLPOs is easier and faster. The numerical accuracy of HSHP is generally superior to the tree method of Babbs (2000) and it exhibits a pattern of monotonic convergence.

In addition, it is difficult to statically hedge American floating strike lookback options because their payoffs depend on the realized maximum or minimum of the stock price. Due to the fact that European call and put options are homogeneous of degree one in stock price and strike price, and as the put–call duality holds for European options, we propose an SSHP for the AFSLPO by transforming the HSHP into an SSHP. The SSHP is rebalanced only when a new maximum of the stock price is observed and the calls and puts of the rebalanced portfolio have different strike prices depending on the new realized maximum of the stock price but with the same weights.

We also compare the hedging performance of SSHP versus DHP for an AFSLPO. The numerical analyses indicate that the hedging performance of the SSHP is much better than that of the DHP. Furthermore, the model risk for SSHP under Heston's SV model is investigated. Model risk indeed exists and deteriorates the hedging performance of our SSHP approach developed under the BS model. The influence is within an acceptable level given practical parameter values, and it becomes more serious especially in the case when the asset price and its volatility are positively correlated.

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APPENDIX A

HEDGING ERRORS OF DHP APPROACH BASED ON Babbs's (2000) TREE MODEL

For the DHP approach, the hedging strategy follows the common delta-neutral strategy in practice. From the viewpoint of the issuer of an AFSLPO, the hedging strategy can be described as the following three-step method.

Step 1. Suppose the issuer sells the AFSLPO at the benchmark option value P_{t_0} and uses the sales proceeds to invest $\Delta_{t_0} S(t_0)$ in the underlying asset, where $\Delta_t \equiv \Delta_t(S(t), S_{\max}(t))$ is the delta value at time t calculated with the Babbs's tree model. Finally, the remaining funds (could be negative), that is, $\eta_{t_0} = -\Delta_{t_0} S(t_0) + P_{t_0}$, are invested to earn the risk-free interest rate r.

Step 2. On any rebalance time point τ, the dynamic hedging portfolio value just before rebalance evolves to be $\Pi_\tau = \Delta_{\tau-\Delta t} S(\tau) e^{q\Delta t} + \eta_{\tau-\Delta t} e^{r\Delta t}$. If $u(\tau) \geq u^*(\tau)$, that is, the early exercise condition is triggered, the present value of the hedging error is calculated as follows, and the hedging process is terminated:

$$HE = -e^{-r\tau}[-(S_{\max}(\tau) - S(\tau))^+ + \Pi_\tau].$$

Otherwise, calculate the new delta value according to the prevailing $S(\tau)$ and $S_{\max}(\tau)$, that is, $\Delta_\tau = \Delta_\tau(S(\tau), S_{\max}(\tau))$, using the Babbs's tree model. Invest $\Delta_\tau S(\tau)$ in the underlying asset and the remaining funds $\eta_\tau = -\Delta_\tau S(\tau) + \Pi_\tau$ in the risk-free asset.

Step 3. If Step 2 is repeated until the maturity T, the dynamic hedging portfolio value evolves to be $\Pi_T = \Delta_{T-\Delta t} S(T) e^{q\Delta t} + \eta_{T-\Delta t} e^{r\Delta t}$ and the present value of the hedging error is

$$HE = -e^{-rT}[-(S_{\max}(T) - S(T))^+ + \Pi_T].$$