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ORIGINAL PAPER

# Using forward Monte-Carlo simulation for the valuation of American barrier options

**Daniel Wei-Chung Miao[1] · Yung-Hsin Lee[2] · Jr-Yan Wang[3]**

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**Abstract** This paper extends the forward Monte-Carlo methods, which have been developed for the basic types of American options, to the valuation of American barrier options. The main advantage of these methods is that they do not require backward induction, the most time-consuming and memory-intensive step in the simulation approach to American options pricing. For these methods to work, we need to define the so-called *pseudo critical prices* which are used to determine whether early exercise should happen. In this study, we define a new and more flexible version of the pseudo critical prices which can be conveniently extended to all fourteen types of American barrier options. These pseudo critical prices are shown to satisfy the criteria of a *sufficient indicator* which guarantees the effectiveness of the proposed methods. A series of numerical experiments are provided to compare the performance between the forward and backward Monte-Carlo methods and demonstrate the computational advantages of the forward methods.

**Keywords** American barrier option · Forward Monte-Carlo method · Pseudo critical price · Sufficient indicator

## 1 Introduction

Using Monte-Carlo simulation to value American options has a relatively shorter history because of the complexity caused by their embedded rights of early exercise. Early studies (see Boyle et al. 1997 for a review) include the bundling algorithm (Tilley 1993), the

✉ Daniel Wei-Chung Miao
miao@mail.ntust.edu.tw

1 Graduate Institute of Finance, National Taiwan University of Science and Technology, 43, Sec. 4, Keelung Rd., Taipei 106, Taiwan

2 Industrial-Academic Research and Development Center, Lunghwa University of Science and Technology, Taoyuan City, Taiwan

3 Department of International Business, National Taiwan University, Taipei, Taiwan

stratified state aggregation algorithm (Barraquant and Martineau 1995), and the stochastic tree-based algorithm (Broadie and Glasserman 1997). More recently, the least-squares Monte-Carlo method (Longstaff and Schwartz 2001) has gained popularity and become a standard technique because of its wide applicability to many American-style options. There have since been many improvements based on this approach. See Areal et al. (2008) for a typical example.

In the aforementioned methods, we need to use backward induction because we do not know whether the stock price has entered the exercise region when it evolves forwardly. Nevertheless, backward induction is the most time-consuming and memory-intensive part of these pricing algorithms. However, it is shown in Miao and Lee (2013) that, for some basic types of American options, the forward Monte-Carlo method can be developed for their valuation without requiring backward induction. In the forward method (FM), a *pseudo* critical price is defined and used for the determination on whether early exercise should happen. The usefulness of this method is ensured by the fact that the *pseudo* critical price is a *sufficient indicator*, meaning that it provides the same information as the *real* critical price regarding early exercise decision. This FM is shown to outperform the least-squares method (LSM) for pricing American vanilla, chooser, and exchange options.

In this paper, we extend the forward Monte-Carlo method to value American barrier options. The challenge is that the pseudo critical price extended directly from Miao and Lee (2013) cannot be proved to satisfy the criteria of a sufficient indicator. To develop a more general forward method, we propose an alternative form of the pseudo critical price which is equally effective but more flexible. We show that it is not only workable for vanilla options but also extendable to barrier options. These pseudo critical prices are derived by taking advantage of the analytical approximations proposed in Barone-Adesi and Whaley (1987) and Chang et al. (2007), Chang et al. (2007). They are first applied to vanilla and some types of out barrier options. With proper adaptations, they can be applied to all fourteen types of American barrier options.

We provide a series of numerical experiments to compare the performance between the proposed FM and the standard LSM. The results show that our FM generally achieves better performance than the LSM, although the improvement varies from case to case. From the convergence analysis, we see that our FM shows a much better convergence pattern than the LSM. In particular, for some “hard” cases where the initial stock price is very close to the barrier and the LSM performs poorly, the proposed FM shows advantages in giving satisfactorily accurate results with reasonable computing times.

This paper proceeds as follows. Section 2 provides some background on the forward Monte-Carlo methods and American barrier options. Sections 3 and 4 discuss how the forward methods are respectively adapted to the American out and in barrier options. Section 5 provides numerical examples to compare the performance between the proposed FM and the LSM. Finally the paper is concluded in Sect. 6.

## 2 Some background on the forward method and American barrier options

This section presents the ideas behind the forward method and reviews its vanilla version. In addition, it gives a classification of all types of American barrier options to which we adapt our forward method.

### 2.1 Ideas behind the forward method

Consider an American option with maturity time $T$ and strike price $K$. Denote the current time by $t \in [0, T]$ and time to maturity by $\tau = T - t$. Let $S(\tau)$ (or $S_t$) represent the underlying asset price when time to maturity is $\tau$ (or at time $t$). For each $\tau$, early exercise can be determined by the *real* critical price $S^*(\tau)$ which is estimated by backward induction in the mainstream Monte-Carlo methods such as the LSM. The motivation of the FM is to eliminate the need for backward induction. However, calculating $S^*(\tau)$ during the forward evolution of simulation is computationally expensive and impractical. The central idea of the FM is to find its substitute, the *pseudo* critical price $\hat{S}(\tau)$. There are two conditions for the FM to be successful: (1) $\hat{S}(\tau)$ must be computationally convenient because it is calculated repeatedly (for each time step), and (2) $\hat{S}(\tau)$ must carry sufficient information regarding the determination of early exercise.

Below we introduce the definition of pseudo critical price $\hat{S}(\tau)$. It is motivated by the *value matching condition* for the real critical price $S^*(\tau)$ which can be expressed as $S^*(\tau) = f(S^*(\tau))$. For example, it can be written as $S^*(\tau) = K + C(S^*(\tau), \tau)$ for an American call option and $S^*(\tau) = K - P(S^*(\tau), \tau)$ for an American put option, where $C(S, \tau)$ and $P(S, \tau)$ are the corresponding American option prices. For notational convenience, the dependence of stock prices (e.g. $S(\tau)$, $S^*(\tau)$, $\hat{S}(\tau)$) on $\tau$ may also be shown as a function of $t$ if appropriate (e.g. $S_t$, $S_t^*$, $\hat{S}_t$), or may even be dropped when the dependence is not emphasized (e.g. $S$, $S^*$, $\hat{S}$).

**Definition 1** The pseudo critical price $\hat{S}(\tau) = f(S(\tau))$ is a function of stock price $S(\tau)$ where the function $f(\cdot)$ satisfies $f(S^*(\tau)) = S^*(\tau)$.

Note that $S^*(\tau)$ separates the stock price domain into the continuation and exercise regions. At each $\tau$, the determination on whether the option should be exercised early is equivalent to the determination on whether $S(\tau) > S^*(\tau)$ or $S(\tau) < S^*(\tau)$ is true (e.g., the American put option should be exercised if $S(\tau) < S^*(\tau)$). Given below are the conditions such that early exercise can be determined indirectly by $\hat{S}(\tau)$ without requiring $S^*(\tau)$.

**Definition 2** For all $\tau \in [0, T]$ and $S(\tau) \in \mathbb{R}^+$, the pseudo critical price $\hat{S}(\tau)$ is a *sufficient indicator* if the following properties hold: (1) $S(\tau) > \hat{S}(\tau)$ if and only if $S(\tau) > S^*(\tau)$, and (2) $S(\tau) < \hat{S}(\tau)$ if and only if $S(\tau) < S^*(\tau)$.

Figure 1 explains why we introduce the above definitions. Note that the *real* critical price $S^*(\tau)$ is fixed for each $\tau \in [0, T]$, but the *pseudo* critical price $\hat{S}(\tau)$ is a function of $S(\tau)$. Definition 1 ensures that $S(\tau) = \hat{S}(\tau)$ when $S(\tau)$ hits the real critical price $S^*(\tau)$ and triggers early exercise (see the hitting point in Fig. 1). The conditions in Definition 2 further ensure that $S(\tau) > \hat{S}(\tau)$ if the stock price has not yet hit the critical price (e.g. at $\tau_1$ when it remains in the continuation region) and $S(\tau) < \hat{S}(\tau)$ if the stock price has passed through the critical price (e.g. at $\tau_2$ when it enters the exercise region). We see that $\hat{S}(\tau)$ actually carries the same information as $S^*(\tau)$ regarding early exercise decision.

Based on these two definitions, the forward method can be developed using the following two steps: (1) define an easy-to-calculate $\hat{S}(\tau)$ from the value matching condition, and (2) prove that this $\hat{S}(\tau)$ is a sufficient indicator.

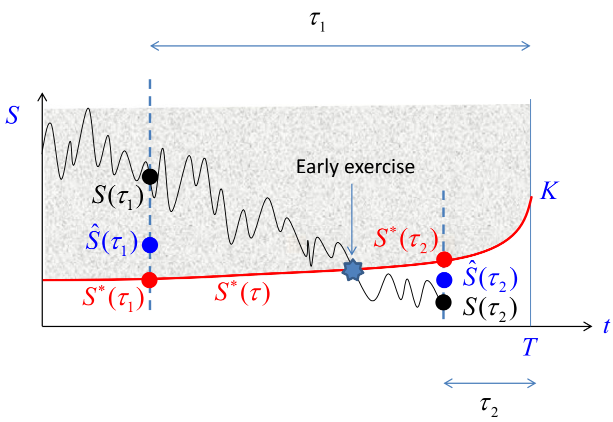


**Fig. 1** The relations between $S(\tau)$, $S^*(\tau)$, $\hat{S}(\tau)$ for an American vanilla put option. The grey area is the continuation region (where $S(\tau) > S^*(\tau)$) and the white area is the exercise region (where $S(\tau) < S^*(\tau)$)

## 2.2 Review of the forward method for American vanilla options

Since calls and puts are dual cases, we take vanilla puts for example. Consider the standard Black-Scholes model for the stock price (under the risk-neutral measure)

$$dS_t = (r - q)S_t dt + \sigma S_t dB_t, \ \ t \in [0, T],$$

where $r, q, \sigma$ represent respectively the interest rate, dividend yield, volatility, and $B_t$ is a standard Brownian motion. In Miao and Lee (2013), the value matching condition (derived from quadratic approximation) is expressed as

$$S^*(\tau) = \frac{Q_1\,(K - p(S^*(\tau), \tau))}{Q_1 - (1 + p'(S^*(\tau), \tau))},$$

where $Q_1 = \frac{-(n-1)-\sqrt{(n-1)^2+\frac{4m}{k}}}{2} > 0, m = \frac{2r}{\sigma^2}, n = \frac{2(r-q)}{\sigma^2}, k = 1-e^{-r\tau}, p(S, \tau)$ stands for the European put option price and $p'(S, \tau) = \frac{\partial p(S,\tau)}{\partial S}$ is its delta (their closed-form formulas are omitted here). Solving the above nonlinear equation to find $S^*(\tau)$ requires an iterative procedure which is unsuitable for repeated calculations. In contrast, the pseudo critical price, which is similarly defined as

$$\hat{S}(\tau) = f(S(\tau)) = \frac{Q_1\,(K - p(S(\tau), \tau))}{Q_1 - (1 + p'(S(\tau), \tau))}, \tag{1}$$

can be calculated much more efficiently because it is only a function valuation. For the vanilla put option, Theorem 2 of Miao and Lee (2013) provides conditions for this $\hat{S}(\tau)$ to be a sufficient indicator. It is subsequently proved in Proposition 2 that those conditions are indeed satisfied, ensuring the usefulness of the $\hat{S}(\tau)$ defined in (1). This provides theoretical support for the vanilla version of the FM algorithm as stated below.

**The Forward Monte-Carlo Algorithm (Vanilla Case)**

1. Generate $N$ paths of stock prices, where each path $i = 1, \cdots, N$ evolves in discrete time with index $j = 1, \cdots, M$ (time interval $\Delta t = \frac{T}{M}$) as follows:

$$S = S_{i,j} = S_{i,j-1} e^{(r-q-\frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t} Z_{i,j}}, \quad Z_{i,j} \sim N(0, 1).$$

2. If a given path $i$ is alive (option not yet exercised) at time index $j - 1 < M$, generate the price for time index $j$, denoted as $S = S_{ij}$. If $j = M$ (at maturity time $T$), the option is expired with value $V_i = e^{-rT}(K - S)^+$ and path $i$ is finished. If $j < M$ (prior to maturity time $T$), do the following:
   - 2.1 Calculate $\hat{S} = f(S)$ to be compared with $S$.
   - 2.2 If $S < \hat{S}$, the option is exercised with value $V_i = e^{-rt}(K - S)^+$ and path $i$ is stopped. Otherwise, the option is held and path $i$ continues to live to the next step $j + 1$.
3. When all the simulated paths are completed, the American option is valued by averaging the discounted payoff as $V = \frac{1}{N}\sum_{i=1}^{N} V_i$.

From the above algorithm, it is not difficult to see that, in the extension to American barrier options, it is Step 2 that must be modified to take into account the barrier features.

## 2.3 A classification of American barrier options

Because our forward method is meant to cover all types of American barrier options, we look first at how they are classified in the literature. According to their features, there are 16 combinations of barrier options formed by (let $H$ denote the barrier)

$$(\text{up, down}) \times (\text{in, out}) \times (\text{call, put}) \times (H < K, H > K)$$

as summarized in Table 1. Note that two out of the 16 cases are trivial cases (marked with $*$ in Table 1) known to have no value because they are out-of-the-money before being knocked out. Also note that each call option has a corresponding dual put option whose treatment is exactly the same. Ignoring these dual cases, the 14 cases are numbered in Table 1 as Cases 1–7 (Cases 1–3 are out barrier options whereas Cases 4–7 are in barriers). It is well known that the European options pricing formulas for all these cases are available in closed-form (see Haug 2006 for a summary of these formulas).

Note that the original forward method of Miao and Lee (2013) is based on the analytical formulas of Barone-Adesi and Whaley (1987) for American vanilla options. In this study, to extend the FM to price American out barrier options, we employ the analytical formulas

**Table 1** Classification of barrier options

| Case | Out barrier call | | Out barrier put | |
|---|---|---|---|---|
| 1<br>2 | Down-and-out call | $H < K$ (1c)<br>$H > K$ (2c) | Up-and-out put | $H > K$ (1p)<br>$H < K$ (2p) |
| –<br>3 | Up-and-out call | $H < K^*$<br>$H > K$ (3c) | Down-and-out put | $H > K^*$<br>$H < K$ (3p) |
| | In barrier call | | In barrier put | |
| 4<br>5 | Down-and-in call | $H < K$ (4c)<br>$H > K$ (5c) | Up-and-in put | $H > K$ (4p)<br>$H < K$ (5p) |
| 6<br>7 | Up-and-in call | $H < K$ (6c)<br>$H > K$ (7c) | Down-and-in put | $H > K$ (6p)<br>$H < K$ (7p) |

of Chang et al. (2007) which is an extension from Barone-Adesi and Whaley (1987). The main issue here is how to make use of these analytical results to define the pseudo critical prices and show that they provide sufficient information in the early exercise decision. As to American in barrier options, since they become vanilla options once they are knocked in, the main issue becomes how to handle the relations between the vanilla critical prices and the knock-in barriers. The FM is then developed from its vanilla version with some adaptations to take into account these relations.

## 3 Forward methods for American out barrier options

Although the $\hat{S}$ defined in (1) is appealing in that it is indeed a sufficient indicator and extendable to other (chooser and exchange) options, it faces some difficulties in its extension to barrier options. In this study we consider a simpler yet more flexible version of $\hat{S}$ (i.e. a redefined $f(\cdot)$ function) and show how it is applied to some types of out barrier options. The subsequent presentations are based on put options (their dual call options can be handled in a similar way). For convenience, some notations are redefined and may be slightly different from those used in the preceding review of the vanilla case.

### 3.1 American up-and-out put options (Cases 1 and 2)

Let $P_{uo}(S, \tau)$ denote the price of an American up-and-out put option (including Cases 1 and 2). It is known that when $S = S^*$, the value matching condition $K - S^* = P_{uo}(S^*, \tau)$ must hold, and can be written as

$$S^* = K - P_{uo}(S^*, \tau).$$

We define our pseudo critical price (referring to Definition 1) as the following simple formula

$$\hat{S} = f(S) = K - P_{uo}(S, \tau). \tag{2}$$

Note that this is different from the $\hat{S}$ defined in (1) since (2) comes directly from the value matching condition without further algebraic manipulation (e.g. collecting the $S$ terms as in (1)). In fact, it is rather difficult to define $\hat{S}$ based on (1) and prove the desired properties because of the complexity in the formula of $P_{uo}(S, \tau)$ (which will be seen later). As it turns out, the $\hat{S}$ defined in (2) is not only equally effective to the version defined in (1) for the vanilla cases, but is also effective for the barrier cases.

To examine whether the $\hat{S}$ defined in (2) is really a sufficient indicator, we define the function $h(S) = S - \hat{S}$ for the domain $S \in [0, H)$ (where the up-and-out option is alive) and discuss its properties in different cases ($H > K$ and $H < K$). The following theorem provides the conditions on the function $h(S)$ for $\hat{S}$ to be a sufficient indicator.

**Theorem 1** *Consider an American up-and-out barrier put option with barrier $H$. With $\hat{S}$ defined in (2), if the function $h(S) = S - \hat{S}$ satisfies the following three properties: (1) $h(0) < 0$, (2) $h(H^-) > 0$ ($H^- = H - \delta$ where $\delta > 0$ is an infinitesimal amount), and (3) $h(S) = 0$ has only one root over $S \in [0, H)$, then $\hat{S}$ is a sufficient indicator of early exercise, i.e.,*

$$S > \hat{S}\ (S < \hat{S}) \quad \textit{if and only if} \quad S > S^*\ (S < S^*).$$

*Proof* According to Definition 1, we have $h(S^*) = S^* - f(S^*) = 0$. Because there is only one root over $[0, H)$, the root must be $S^*$. Since $h(S)$ is a continuous function of $S$ and

$h(0) < 0$, $h(H^-) > 0$, we have $h(S) < 0$ for $S \in [0, S^*)$ and $h(S) > 0$ for $S \in (S^*, H)$. This means

$$S > \hat{S}\ (S < \hat{S}) \quad \textit{iff} \quad h(S) > 0\ (h(S) < 0) \quad \textit{iff} \quad S > S^*\ (S < S^*)$$

as claimed. □

It is clear that the pricing formula of $P_{uo}(S, \tau)$ plays a major role in the valuation of $\hat{S}$. Below we introduce the pseudo critical prices for the first two cases based on the analytical results of Chang et al. (2007).

**Case 1. $H > K$**
Let $p_0(S, \tau)$, $P_0(S, \tau)$ denote the analytical pricing formulas of European and American vanilla options, and $p_1(S, \tau)$, $P_1(S, \tau)$ denote the analytical pricing formulas of the corresponding up-and-out put option. (The subscript shows the case index: 0 is the vanilla case, 1 represents Case 1, and so on.) It is well known that under the Black-Scholes model, the European barrier option pricing formula is related to the vanilla formula as (see Haug 2006)

$$p_1(S, \tau) = p_0(S, \tau) - \left(\frac{H}{S}\right)^{2\lambda-2} p_0\left(\frac{H^2}{S}, \tau\right), \quad \lambda = \frac{r-q}{\sigma^2} + \frac{1}{2}.$$

In Chang et al. (2007), the quadratic approximation approach is employed to express the American barrier option price for $S \geq S_1^*$ as a sum of the European price and the early exercise premium, i.e.

$$P_1(S, \tau) = p_1(S, \tau) + \alpha(S^{\beta_-} - H^{\beta_- - \beta_+} S^{\beta_+}) \tag{3}$$

where $\alpha$ is a constant to be determined, and $\beta_- < 0$, $\beta_+ > 0$ are given by $\beta_\pm = \frac{-(N-1)\pm\sqrt{(N-1)^2+4M'}}{2}$, where $N = \frac{2(r-q)}{\sigma^2}$, $M = \frac{2r}{\sigma^2}$, $M' = M\left[1 + u\left(\frac{1}{1-e^{-ur\tau}} - 1\right)\right]$. (Note that the $u$ in $M'$ is proposed by Chang et al. (2007) to improve accuracy; it is suggested to use $u = \frac{S+5H}{H}$.)

When $S = S_1^*$, the value matching condition ($P_1(S_1^*, \tau) = K - S_1^*$) and smooth pasting condition ($\frac{\partial}{\partial S} P_1(S_1^*, \tau) = -1$) must hold. These two conditions are used to find the two unknown constants $\alpha$ and $S_1^*$. Solving these two equations, we see that $\alpha$ must be related to $S_1^*$ as follows

$$\alpha = \frac{-1 - \frac{\partial}{\partial S} p_1(S_1^*, \tau)}{\beta_-(S_1^*)^{\beta_- - 1} - \beta_+ H^{\beta_- - \beta_+}(S_1^*)^{\beta_+ - 1}}.$$

Replacing $S_1^*$ by $S$ in the above formula, we define a new function

$$W_1(S) = \frac{-1 + e^{-q\tau} N(-d_1(S)) + \left(\frac{H}{S}\right)^{2\lambda} \left[e^{-q(T-t)} N\left(-d_1\left(\frac{H^2}{S}\right)\right) - (2\lambda - 2) p_0\left(\frac{H^2}{S}, \tau\right) \frac{S}{H^2}\right]}{\beta_- S^{\beta_- - 1} - \beta_+ H^{\beta_- - \beta_+} S^{\beta_+ - 1}}$$

and clearly $\alpha = W_1(S_1^*)$.

It is worth comparing Case 1 (with finite $H$) to its special vanilla case (Case 0, with $H = \infty$). Mathematically, when $H \to \infty$, it is easy to check that (3) reduces to its vanilla version (the analytical pricing formula as given in Barone-Adesi and Whaley 1987). Their relation can be seen more clearly from Fig. 2. Comparing both plots, we see that the finite barrier $H < \infty$ limits the continuation region and in turn pulls the pricing function $P_1(S, \tau)$ (see Fig. 2b) slightly downward from $P_0(S, \tau)$ (see Fig. 2a). The downward moving pricing

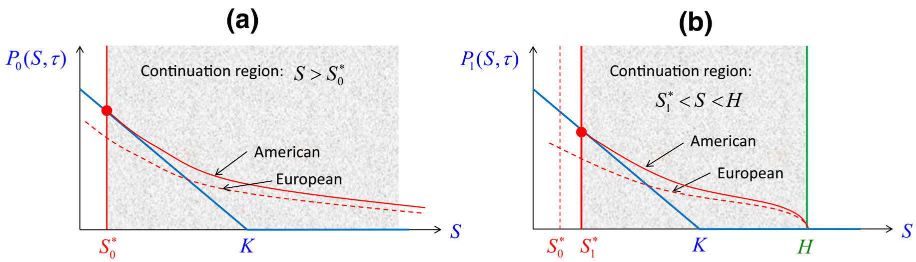


**Fig. 2** Pricing function ($P(S, \tau)$ versus $S$) for American up-and-out put option when $H > K$ (Case 1): **a** $H = \infty$ (the special vanilla case, i.e., Case 0); **b** $H < \infty$ and $S_1^* > S_0^*$. Shaded areas are the continuation regions

function then causes the critical price $S_1^*$ to move toward the right from $S_0^*$ because the value matching and smooth pasting conditions must still hold.

From (3), we rewrite the value matching condition at $S_1^*$ as

$$S_1^* = K - p_1(S_1^*, t) - W_1(S_1^*)(S_1^{*\beta_-} - H^{\beta_- - \beta_+} S_1^{*\beta_+}).$$

Following (2), we may define the pseudo critical price for Case 1 as (change $S_1^*$ to $S$)

$$\hat{S}_1 = f_1(S) = K - p_1(S, \tau) - W_1(S)(S^{\beta_-} - H^{\beta_- - \beta_+} S^{\beta_+}). \tag{4}$$

In order to show that $\hat{S}_1$ is a sufficient indicator, we investigate the properties of the function $h(S) = S - \hat{S}_1$.

**Lemma 1** *Consider an American up-and-out put option with $H > K$ (Case 1) for which $\hat{S}_1$ is defined in* (4). *The function $h(S)$ has the following properties*

$$h(0) < 0, \quad h(H) = H - K > 0.$$

*Proof* Note from (4) that

$$h(S) = S - \hat{S}_1 = S - K + p_1(S, \tau) + W_1(S)(S^{\beta_-} - H^{\beta_- - \beta_+} S^{\beta_+}).$$

It is clear that $h(0) < 0$ (since $p_1(0, \tau) < K$) and $h(H) = H - K > 0$ (because of the boundary condition $p_1(H, \tau) = 0$ and $H > K$). □

The above lemma implies that the first two conditions in Theorem 1 are satisfied. It remains to check the third condition. We first consider the simpler vanilla case with $H = \infty$ (Case 0). In this case, $\alpha = W_1(S_1^*)$ becomes

$$\alpha = W_0(S_0^*) = \frac{-1 + e^{-q\tau} N(-d_1(S_0^*))}{\beta_-(S_0^*)^{\beta_- - 1}},$$

and the pseudo critical price reduces to

$$\hat{S}_0 = f_0(S) = K - P_0(S, \tau) = K - p_0(S, \tau) - \frac{-1 + e^{-q\tau} N(-d_1(S))}{\beta_-} S. \tag{5}$$

The following result follows from the fact that $h(S)$ is an increasing function.

**Proposition 1** *In the limiting case $H = \infty$ of an American up-and-out put option with $H > K$ (i.e. American vanilla put option), $\hat{S}_1 = \hat{S}_0$ is a sufficient indicator of early exercise.*

*Proof* By observing that

$$\frac{\partial h(S)}{\partial S} = 1 + \frac{\partial P_0(S,\tau)}{\partial S} = \left[1 - e^{-q\tau} N(-d_1(S))\right]\left(1 - \frac{1}{\beta_-}\right) - \frac{e^{-q\tau} n(d_1(S))}{\beta_- \sigma \sqrt{\tau}} > 0,$$

where $\beta_- < 0$ and $n(x) = N'(x)$ is the normal pdf, we see that $h(S)$ is a strictly increasing function for $S \in [0, \infty)$. This implies that the third condition in Theorem 1 is satisfied and hence the claimed result. $\square$

However, for $H < \infty$ (Case 1), we cannot claim that $h(S)$ is a strictly increasing function over $[0, H)$ as above. As seen in Fig. 2b, we are not sure if $-1 < \frac{\partial P_1(S,\tau)}{\partial S} < 0$ is true for all $S \in [0, H)$, particularly for $S$ close to $H$ (although it appears to be true for $S$ away from $H$). This is because the boundary condition $P_1(H, \tau) = 0$ forces the pricing function to zero as $S \to H$ which may ruin the desired property. Fortunately, this property is not necessary in proving the third condition of Theorem 1. As seen in the following proposition, what we need is an idea behind the quadratic approximation formula.

**Proposition 2** *For an American up-and-out put option with $H > K$ (Case 1), if $S_1^*$ is the unique stock price that satisfies the value matching and smooth pasting conditions in the quadratic approximation, then $\hat{S}_1$ is a sufficient indicator of early exercise.*

*Proof* By Lemma 1, the equation $h(S) = 0$ has at least one root in $[0, H)$. It remains to show that this root is unique. To this end, let $\tilde{S}_1$ be a root of the equation (i.e. $h(\tilde{S}_1) = 0$) and we intend to show $\tilde{S}_1 = S_1^*$. From (4), $h(\tilde{S}_1) = 0$ can be expressed back to the following form

$$K - \tilde{S}_1 = p_1(\tilde{S}_1, \tau) + W_1(\tilde{S}_1)(\tilde{S}_1^{\beta_-} - H^{\beta_- - \beta_+}\tilde{S}_1^{\beta_+}).$$

By the definition of the function $W_1(S)$, we also have

$$-1 = \tfrac{\partial}{\partial S} p_1(\tilde{S}_1, \tau) + W_1(\tilde{S}_1)(\beta_- \tilde{S}_1^{\beta_- - 1} - \beta_+ H^{\beta_- - \beta_+}\tilde{S}_1^{\beta_+ - 1}).$$

If we replace $\tilde{S}_1$ by $S_1^*$, the above two equations can be written as

$$\begin{cases} K - S_1^* = p_1(S_1^*, \tau) + W_1(S_1^*)((S_1^*)^{\beta_-} - H^{\beta_- - \beta_+}(S_1^*)^{\beta_+}), \\ -1 \quad\;\; = \tfrac{\partial}{\partial S} p_1(S_1^*, \tau) + W_1(S_1^*)(\beta_-(S_1^*)^{\beta_- - 1} - \beta_+ H^{\beta_- - \beta_+}(S_1^*)^{\beta_+ - 1}). \end{cases}$$

These are the value matching and smooth pasting conditions that must hold at the critical stock price. Since $S_1^*$ is the unique stock price that satisfies these two conditions, we must have $\tilde{S}_1 = S_1^*$. Therefore, all three conditions in Theorem 1 are satisfied, and thus $\hat{S}_1$ is a sufficient indicator. $\square$

*Remark.* While Proposition 2 covers Proposition 1, it is worth discussing their differences. In the limiting vanilla case, the fact that $h(S)$ is a strictly increasing function can be proved mathematically. For $H < \infty$, however, it is difficult to check $\frac{\partial h(S)}{\partial (S)}$ directly because the formula is too complex to analyze. However, the proof of Proposition 2 does not rely on the increasing property of $h(S)$ but on the uniqueness of the critical stock price which solves the value matching and smooth pasting conditions simultaneously. In fact, this uniqueness is an implicit premise of the quadratic approximation approach (such as Barone-Adesi and Whaley 1987; Chang et al. 2007).

Figure 3 shows the curves of the function $h(S)$ under different $H$ from $\infty$ down to $K$. We see that the three conditions of Theorem 1 are indeed satisfied. As seen in Fig. 3a, $h(S)$ is a strictly increasing function for $H = \infty$. As $H$ decreases, the increasing property seems to remain true for most part of $[0, H)$. This indicates that the boundary condition ($P_1(H, \tau) = 0$

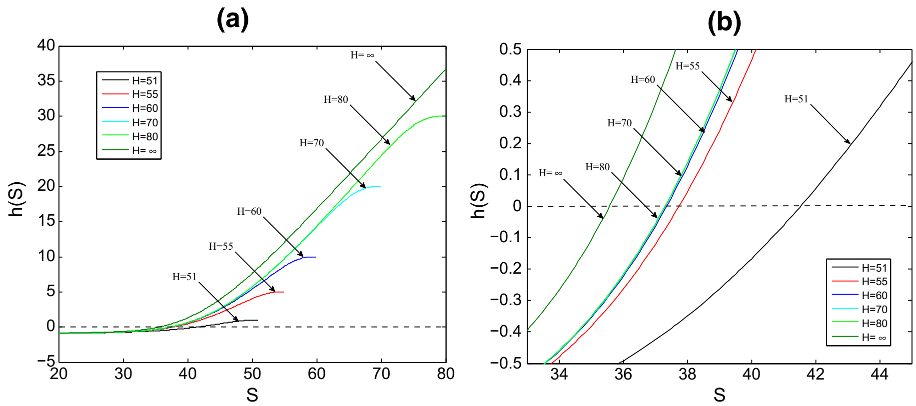


**Fig. 3** The $h(S)$ function for an American up-and-out put option (Case 1) with the barrier $H$ decreasing from $\infty$ toward $K$. The parameters are $K = 50$, $H = \infty, 80, 70, 60, 55, 51$, $r = 0.05$, $q = 0.03$, $\sigma = 0.3$, $\tau = 0.5$: **a** a normal view to show the curves moving downward as $H$ decreases; **b** a close-up view to show $S_1^*$ moving toward the right as $H$ decreases

for all $\tau$) brings little impact on the increasing property when $S$ is away from $H$. But for $S \approx H$, the barrier causes the curve to stop increasing or even start decreasing. In addition, the fact that $h(H) = H - K$ pulls down the curves as $H$ decreases, moving the root $S_1^*$ (recall $h(S_1^*) = 0$) toward the right. This is seen more clearly from the close-up view in Fig. 3b.

Note that other option parameters also influence the function curve of $h(S)$ and the root $S_1^*$. Figure 4 shows the curves when the barrier is fixed at $H = 55$ but other parameters $r, q, \sigma, \tau$ are varying. It is observed that the curve shape has the same style under these typical parameter sets and, more importantly, all three conditions in Theorem 1 are indeed satisfied.

With $\hat{S}_1$ well defined and proved to have the desired properties, the FM for Case 1 American barrier option can be developed. It requires only a modification to Step 2 of its vanilla version, which is stated as below:

**The Step 2 of the FM Algorithm (Case 1)**

2.1 If $S > H$, then the option is knocked out with no value.
2.2 If $S < H$, calculate $\hat{S}_1$. If $S < \hat{S}_1$, then the option is exercised early with a value of $e^{-rt}(K - S)$. Otherwise, the option is held to the next time step.

**Case 2. *H* < *K***

The development of the FM for $H < K$ follows a similar idea to $H > K$. Because the barrier $H$ is in-the-money in this case, when the stock price is about to hit the barrier, the option must be exercised just before the barrier is hit. We may see this as the option giving a rebate at the barrier. Chang et al. (2007) (p.51) pointed out the following decomposition:

$$\begin{aligned}&\textit{American Up-and-Out Put Option}\\&\quad = \textit{European Up-and-Out Put Option With Rebate } R = K\text{-}H\\&\qquad + \textit{Early Exercise Premium}.\end{aligned}$$

Let $p_2(S, \tau)$ represent the above European up-and-out put price. According to Haug (2006), it can be expressed as $p_2(S, \tau) = p_2^{(1)} + p_2^{(2)} + p_2^{(3)}$, where $p_2^{(1)}$ is the European vanilla

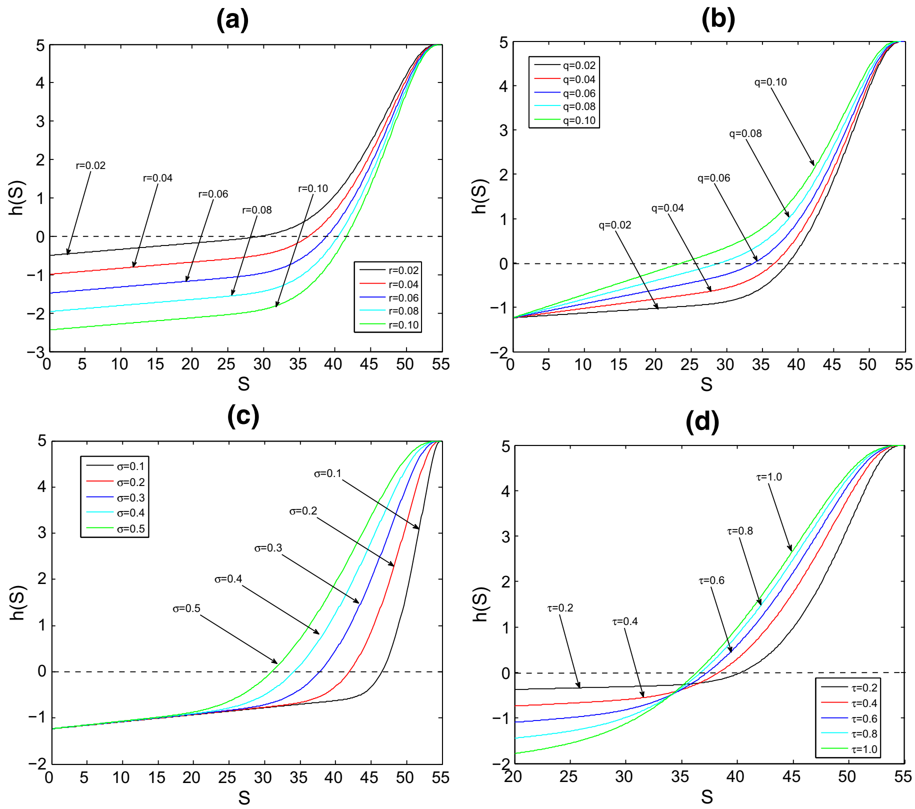


**Fig. 4** The $h(S)$ function for the American up-and-out put option with $H > K$ (Case 1), where the barrier $H$ is fixed and other parameters are varying: **a** varying $r$; **b** varying $q$; **c** varying $\sigma$; **d** varying $\tau$. Except for the varying parameter, other fixed option parameters are $K = 50$, $H = 55$, $r = 0.05$, $q = 0.03$, $\sigma = 0.3$, $\tau = 0.5$

put price, $p_2^{(2)}$ is the (negative) value contributed from knock-out barrier, and $p_2^{(3)}$ is the (positive) value contributed from rebate. They are respectively given by

$$
\begin{aligned}
p_2^{(1)} &= Ke^{-r\tau}N(-x_2) - Se^{-q\tau}N(-x_1),\\
p_2^{(2)} &= -Ke^{-r\tau}\left(\frac{H}{S}\right)^{2\lambda-2}N(-y_2) + Se^{-q\tau}\left(\frac{H}{S}\right)^{2\lambda}N(-y_1),\\
p_2^{(3)} &= R\left[\left(\frac{H}{S}\right)^{\lambda-1+\nu}N(-z_1) + \left(\frac{H}{S}\right)^{\lambda-1-\nu}N(-z_2)\right],
\end{aligned}
$$

where

$$
x_1 = \frac{\ln(\frac{S}{H})}{\sigma\sqrt{\tau}} + \lambda\sigma\sqrt{\tau}, \quad x_2 = x_1 - \sigma\sqrt{\tau}, \quad y_1 = \frac{\ln(\frac{H}{S})}{\sigma\sqrt{\tau}} + \lambda\sigma\sqrt{\tau}, \quad y_2 = y_1 - \sigma\sqrt{\tau},
$$

$$
\nu = \sqrt{(\lambda-1)^2 + \frac{2r}{\sigma^2}}, \quad z_1 = \frac{\ln(\frac{H}{S})}{\sigma\sqrt{\tau}} + \nu\sigma\sqrt{\tau}, \quad z_2 = z_1 - 2\nu\sigma\sqrt{\tau}.
$$

To follow the approach of Chang et al. (2007), it is useful to calculate its delta $\Delta_2 = \frac{\partial p_2(S,\tau)}{\partial S}$, which can also be decomposed as $\Delta_2 = \Delta_2^{(1)} + \Delta_2^{(2)} + \Delta_2^{(3)}$, where

$$\Delta_2^{(1)} = -\frac{e^{-q\tau}n(x_1)}{\sigma\sqrt{\tau}}\left(\frac{K}{H}-1\right) - e^{-q\tau}N(-x_1),$$

$$\Delta_2^{(2)} = \left(\frac{H}{S}\right)^{2\lambda}\left[\left(\frac{KS}{H^2}\right)e^{-r\tau}(2\lambda-2)N(-y_2) + e^{-q\tau}(1-2\lambda)N(-y_1) - \frac{e^{-q\tau}n(y_1)}{\sigma\sqrt{\tau}}\left(\frac{K}{H}-1\right)\right],$$

$$\Delta_2^{(3)} = \frac{R}{S}\left[\left(\frac{H}{S}\right)^{\lambda-1+\nu}\left(\frac{n(z_1)}{\sigma\sqrt{\tau}} - (\lambda-1+\nu)N(-z_1)\right) + \left(\frac{H}{S}\right)^{\lambda-1-\nu}\left(\frac{n(z_2)}{\sigma\sqrt{\tau}} - (\lambda-1-\nu)N(-z_2)\right)\right].$$

Using the quadratic approximation, the American and European prices $P_2(S,\tau)$ and $p_2(S,\tau)$ may be related in the same way as (3), but the corresponding $W_2(S)$ function for Case 2 should be changed to

$$W_2(S) = \frac{-1-\left(\Delta_2^{(1)}+\Delta_2^{(2)}+\Delta_2^{(3)}\right)}{\beta_- S^{\beta_- -1} - \beta_+ H^{\beta_- - \beta_+} S^{\beta_+ -1}}$$

according to the value matching and smooth pasting conditions.

Figure 5 shows how the pricing function $P_2(S,\tau)$ is influenced by the barrier $H$. As seen in Fig. 5a, for a fixed $\tau$, the function $P_2(S,\tau)$ for $S \in [S_2^*, H]$ must satisfy the value matching and smooth pasting conditions at $S = S_2^*$ and the boundary condition $P_2(H,\tau) = K - H$ (rebate value) at $S = H$. (As $S \to H$, the early exercise premium goes to zero, making $P_2(S,\tau)$ converge to the European rebate value $p_2(S,\tau) = p_2^{(3)} = K - H$; the corresponding

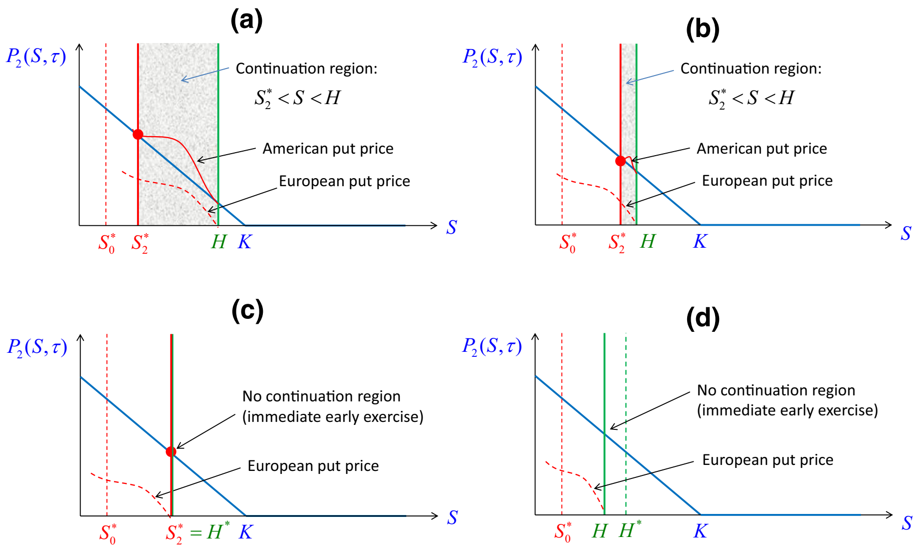


**Fig. 5** The pricing function for the American up-and-out put option when $H < K$ (Case 2): **a** $H$ is slightly less than $K$; **b** $H$ is even smaller and the continuation region narrows; **c** $H = H^*$ and the continuation region disappears; **d** $H < H^*$. The European prices shown in the plots do not include the value from rebate (i.e. $\tilde{p}_2(S,\tau) = p_2^{(1)} + p_2^{(2)}$)

European option with no rebate becomes worthless, i.e., its price $\tilde{p}_2(S,\tau) = p_2^{(1)} + p_2^{(2)} \to 0$). Comparing Fig. 5a with Fig. 5b, we see that as $H$ becomes smaller, $S_2^*$ moves toward the right and farther away from $S_0^*$, making the continuation region even smaller. As $H$ decreases further, there is a particular value of barrier $H^*$ such that the continuation region starts to disappear (see Fig. 5c, where $H = H^*$ is coinciding with $S_2^*$), indicating that the option should be exercised immediately. This is also true for $H < H^*$ (see Fig. 5d). It is clear that $S_2^*$ cannot be found when the barrier $H$ is lower than the particular threshold value $H^*$.

Following the treatment for Case 1, the pseudo critical price for Case 2 is defined as

$$\hat{S}_2 = f_2(S) = K - p_2(S,\tau) - W_2(S)(S^{\beta_-} - H^{\beta_- - \beta_+} S^{\beta_+}). \tag{6}$$

To examine whether $\hat{S}_2$ is a sufficient indicator, again we must look at the properties of the function $h(S) = S - \hat{S}_2$.

**Lemma 2** *For an American up-and-out put option with $H < K$ (Case 2), the function $h(S)$ has the following properties*

$$h(0) < 0, \quad h(H) = 0.$$

*Proof* Note from (2) that

$$h(S) = S - \hat{S}_2 = S - K + p_2(S,\tau) + W_2(S)(S^{\beta_-} - H^{\beta_- - \beta_+} S^{\beta_+}).$$

It is clear that $h(0) < 0$ (since $p_2(0,\tau) < K$) and $h(H) = 0$ (this is because of the boundary condition $p_2(H,\tau) = p_2^{(3)} = R = K - H$). □

From Lemma 2, we see that a major difference between the $h(S)$ functions in Cases 1 and 2 is on the right end $S \to H$ (note that $h(H) = 0$ in Case 2 but $h(H) > 0$ in Case 1). Unlike Case 1, in Case 2 we cannot guarantee the existence of a root of $h(S) = 0$ over $S \in [0, H)$. Fortunately, $\hat{S}_2$ still provides useful information for the early exercise decision regardless of whether there is a root.

**Proposition 3** *For an American up-and-out put option with $H < K$ (Case 2), suppose that $S_2^*$ is the unique stock price satisfying the value matching and smooth pasting conditions in the quadratic approximation. Then we have:*

1. *When there is a root of $h(S) = 0$ for $S \in [0, H)$, the root is unique and must be $S_2^*$, and $\hat{S}_2$ defined in (6) is a sufficient indicator of early exercise.*
2. *When there is no root of $h(S) = 0$ for $S \in [0, H)$, i.e., $S_2^*$ does not exist in $[0, H)$, $\hat{S}_2$ defined in (6) is still a sufficient indicator of early exercise.*

*Proof* From Lemma 2, there may or may not be a root of the equation $h(S) = 0$ for $S \in [0, H)$. We discuss these two cases as follows:

(1) When there is a root, we may use the argument in Proposition 2 to deduce that the root is unique and must be $S_2^*$. This means $S_2^* \in [0, H)$ and implies $h(H^-) > 0$ (recall $H^- = H - \delta$). Therefore, all three conditions in Theorem 1 are satisfied, and hence $\hat{S}_2$ is a sufficient indicator.

(2) When there is no root, we have $h(S) < 0$ (i.e., $S < \hat{S}_2$) for all $S \in [0, H)$. In this case, we cannot find $S_2^* \in [0, H)$ (since $S_2^*$ must satisfy $h(S_2^*) = 0$). Note that the continuation region consists of stock prices $S > S_2^*$. Since we cannot find $S \in [0, H)$ such that $S > S_2^*$, all stock prices $S \in [0, H)$ are in the exercise region. This shows that merely using $S < \hat{S}_2$ is sufficient to determine early exercise. Namely, $\hat{S}_2$ is a sufficient indicator. □

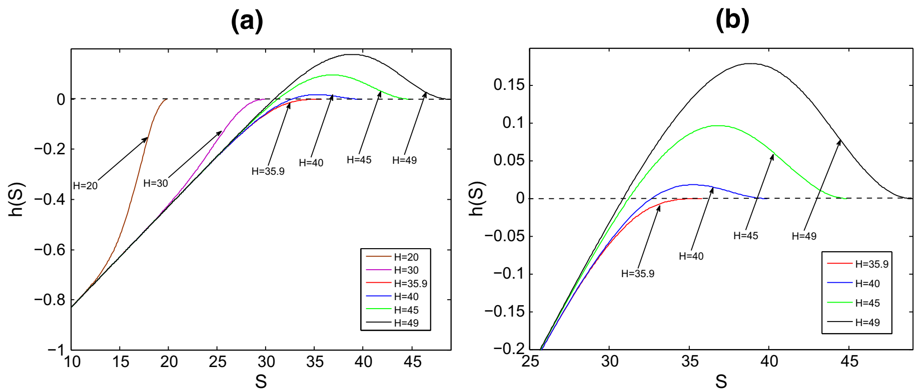


**Fig. 6** The $h(S)$ function for an American up-and-out put option (Case 2) with $H$ decreasing from $K$. The parameters are $K = 50$, $H = 49, 45, 40, 35.9\ (= H^*), 30, 20$: **a** a normal view to show the curves moving downward as $H$ decreases; **b** a close-up view to show that $S_2^*$ moves toward the right as $H$ decreases

In fact, the two cases in the above proof correspond to Fig. 5a, b where $H > H^*$ (one root) and Fig. 5c, d where $H \leq H^*$ (no root). This is seen more clearly in Fig. 6a, which shows how the curves move as $H$ decreases. Note that the left end $h(0) = p_2(0, \tau) - K$ is fixed for different $H$. To keep $h(H) = 0$, as $H$ decreases, the whole curve moves downward (similar to Fig. 3), making the root $S_2^*$ move toward the right (see the close-up view in Fig. 6b). For $H \leq H^* = 35.9$, the root $S_2^*$ does not exist as $h(S) < 0$ holds true for all $S \in [0, H)$.

Figure 7 further shows how the curves of $h(S)$ vary with other parameters $r, q, \sigma, \tau$ when $H$ is fixed (in parallel to Fig. 4 for Case 1). Basically, the curves show patterns similar to those in Fig. 6. The equation $h(S) = 0$ may have either one root or no root in $S \in [0, H)$, corresponding to the aforementioned two cases in the proof of Proposition 3.

Below we describe the adaptation of Step 2 of our FM algorithm.

**The Step 2 of the FM Algorithm (Case 2)**

2.1 If $S > H$, then the option is knocked out with a discounted rebate value of $e^{-rt}(K - H)$.
2.2 If $S < H$, calculate $\hat{S}_2$. If $S < \hat{S}_2$, then the option is exercised early with a value of $e^{-rt}(K - S)$. Otherwise, the option is held to the next time step.

## 3.2 American down-and-out put option with $H < K$ (Case 3)

Although the analytical approximation for American down-and-out put option has also been developed (see Chang et al. 2007), our forward method does not require these results. Instead, we directly apply the FM for the vanilla option (Case 0) to this down-and-out option (Case 3) with some adaptation. The reason behind this is the following relation between the critical prices of Cases 3 and 0 (see Gao et al. 2000, Theorem 6, p. 1798):

$$S_3^*(\tau) = \max(H, S_0^*(\tau)),\ \ \tau \in [0, T]. \tag{7}$$

A graphical illustration is given in Fig. 8. The relation in (7) can be easily understood: the critical price $S_3^*(\tau)$ is either $H$ or $S_0^*(\tau)$, whichever is reached (from above) first. In case $H$

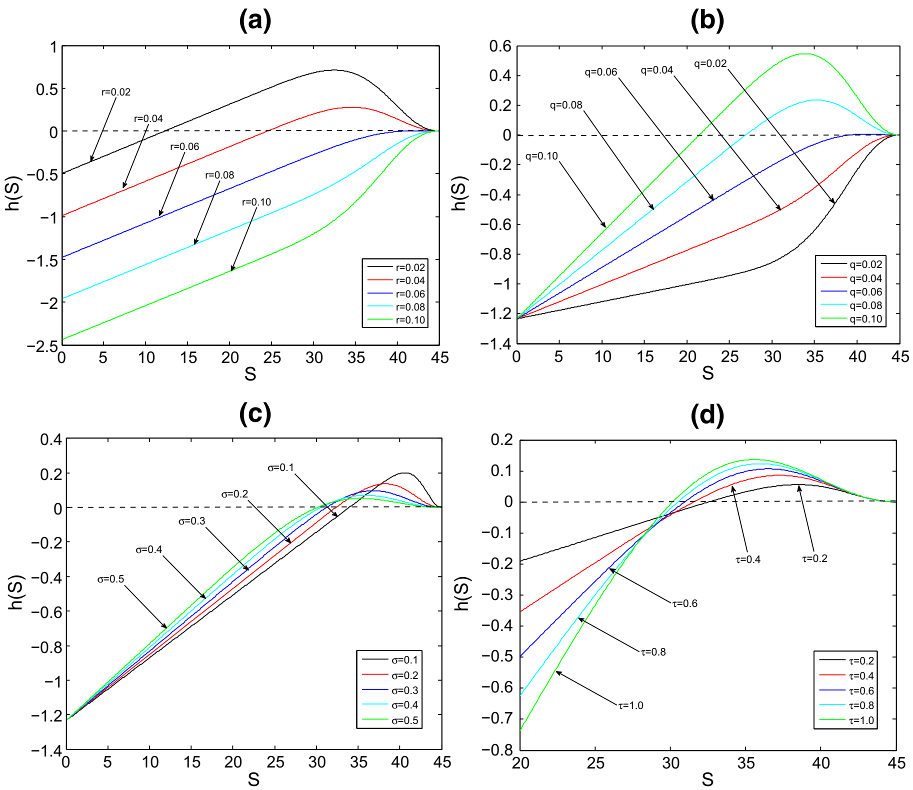


**Fig. 7** The $h(S)$ function for the American up-and-out put option with $H < K$ (Case 2), where the barrier $H$ is fixed and other parameters are varying: **a** varying $r$; **b** varying $q$; **c** varying $\sigma$; **d** varying $\tau$. Except for the varying parameter, other fixed option parameters are $K = 50$, $H = 45$, $r = 0.05$, $q = 0.07$, $\sigma = 0.3$, $\tau = 0.5$

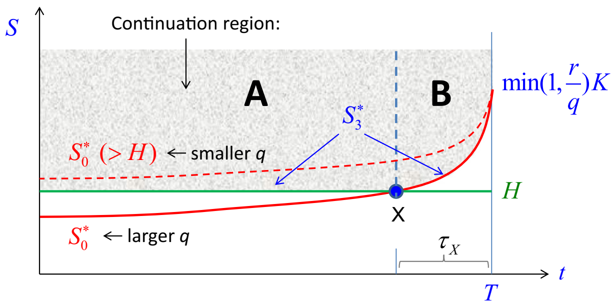


**Fig. 8** The continuation region for an American down-and-out option (Case 3) is bounded from below by $S_3^*$, which is the maximum of $H$ and $S_0^*$ (see the solid line for larger $q$). In case the dividend yield $q$ is smaller, the $S_0^*$ may be heightened above the barrier (the dashed line). In this case, $S_3^* = S_0^*$ for all $\tau \in [0, T]$

is higher and is reached first, as the stock price approaches $H$, the option holder will exercise the option to avoid being knocked out and obtain the rebate value.

When the dividend yield $q$ is small ($q \approx 0$ or $q \ll r$) such that $S_0^*(\tau) > H$ for all $\tau \in [0, T]$, we have $S_3^*(\tau) = S_0^*(\tau)$ (see the dashed line in Fig. 8). In this case, this American down-and-out put option can be treated as an American vanilla put option because $S_0^*(\tau)$ is always hit before the option is knocked out. However, when $q$ is large enough (or $H$ is high enough) such that $H > \min(1, \frac{r}{q})K = S^*(0)$ and therefore $S_0^*(\tau) < H$ for all $\tau \in [0, T]$, we have $S_3^*(\tau) = H$. In this case, the option is knocked out before reaching $S_0^*(\tau)$. In other words, early exercise only happens when the price is about to hit the barrier, and therefore this option is essentially a European down-and-out put option with rebate.

The nontrivial case is that $H$ and $S_0^*(\tau)$ (the solid line in Fig. 8) have an intersection point $X$ at $\tau_X \in [0, T]$. The point X separates the continuation region into two parts, labeled A and B. Below is a summary of the exercise rules for both regions:

(1) For region A (the earlier life of option with $\tau > \tau_X$) where $S_3^*(\tau) = H$, the rule is: if $H$ is hit, then the option is knocked out and a rebate $R = K - H$ is paid.
(2) For region B (the later life of option with $\tau < \tau_X$) where $S_3^*(\tau) = S_0^*(\tau)$, the rule becomes: the option is treated as if it is a vanilla put option.

From the above discussion we see that, for Case 3, as long as the region (A or B) is known, there is no need to define $\hat{S}_3(\tau)$ because its vanilla counterpart $\hat{S}_0(\tau)$ has provided sufficient information for early exercise decision. Either the version defined in (5) or in (1) may be used. But it remains to determine which region the stock price is in when simulation is in progress. This can be easily checked by comparing $H$ to $\hat{H} = f(H)$ (where $f(\cdot)$ is given by (5) or (1)), i.e. it is in region A if $H > \hat{H}$ (implying $H > S_0^*(\tau)$, according to the property of an sufficient indicator function $f(\cdot)$ as described in Definition 2) and it is in region B if $H < \hat{H}$ (implying $H < S_0^*(\tau)$). Based on these observations, we have the following adaptation of Step 2 of our FM algorithm.

**The Step 2 of the FM Algorithm (Case 3)**

2.1 Calculate $\hat{S}_0 = f(S)$ and $\hat{H} = f(H)$ where $f(\cdot)$ is given in (5) or (1).
2.2 If $H > \hat{H}$, it is in region A. If $S < H$, then the option is knocked out with discounted rebate value $e^{-rt}(K - H)$. Otherwise, the option is held to the next time step.
2.3 If $H < \hat{H}$, it is in region B. If $S < \hat{S}_0$, then the option is exercised early with discounted exercise value $e^{-rt}(K - S)$. Otherwise, the option is held to the next time step.

## 4 Forward methods for American in barrier options

Prior to our discussion on in barrier options, let us stress that the *in-out parity* relation is not applicable to American barrier options. This relation only holds for their European versions for which one can obtain the in barrier price from the vanilla and out barrier prices (see the discussion in Haug 2001, p.358). For the American versions, since the early exercise time point of an in barrier option may not be the same as that of the corresponding out barrier option, the parity relation does not hold. This justifies the purpose of this section.

Compared to American out barrier options (Cases 1–3), the forward method is less involved in pricing American in barrier options (Cases 4–7). The situation is similar to Case 3 where only the pseudo critical price for the vanilla option ($\hat{S}_0$) is required. This is because an in barrier option becomes a vanilla option once it is knocked in. The adaptation only needs to take account of the time before the option being knocked in; this is easily handled by the forward method as described below.

**Table 2** Further classification of American in barrier options (Cases 4–7)

| Case | In barrier call | In barrier put |
|---|---|---|
| 4, 6 | $H < K$ | $H > K$ |
| 5, 7 | $H > K$ | $H < K$ |
| (a) | $K < H < \max(1, \frac{r}{q})K$ (if $r > q$) | $\min(1, \frac{r}{q})K < H < K$ (if $r < q$) |
| (b) | $\max(1, \frac{r}{q})K < H < S_0^*(\infty)$ | $S_0^*(\infty) < H < \min(1, \frac{r}{q})K$ |
| (c) | $H > S_0^*(\infty)$ | $H < S_0^*(\infty)$ |

The classification of subcases (a) (b) (c) is according to the relation between the barrier $H$ and the curve of $S_0^*(\tau)$

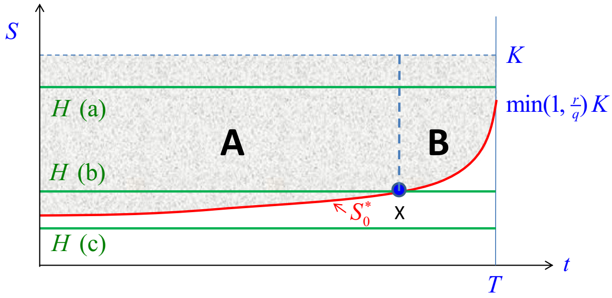


**Fig. 9** The subcases (a), (b), (c) for the American in barrier put options in Table 2. The in-the-money continuation region is shown in gray. The barrier is located in the continuation region in subcase (a) and region A of subcase (b); it is located in the exercise region otherwise

**The Forward Monte-Carlo Algorithm (Cases 4–7)**

1. Simulate the stock price as usual until the barrier $H$ is hit and the option is knocked in.
2. Continue with the simulation as if it is an American vanilla option, i.e., calculate $\hat{S}_0$ from (5) or (1), and early exercise happens if $S < \hat{S}_0$.
3. Average the discounted payoffs from all the paths to obtain the option price.

Depending on the relation between the barrier $H$ and the vanilla critical price $S_0^*(\tau)$, there are different scenarios before the barrier is hit (similar to Case 3). Table 2 (a continuation of Table 1) provides a summary of these scenarios for the in barrier cases. Cases 4 and 6 are simpler because the barrier $H$ is located in the out-of-the-money region and there is no intersection between $H$ and $S_0^*(\tau)$. The FM waits for the option to be knocked in and treats it as a vanilla option.

Cases 5 and 7 are more complicated because the barrier $H$ may or may not cross the curve of $S_0^*(\tau)$. Here we divide them into three subcases (see Fig. 9; also refer to Dai and Kwok 2004), where there is no intersection in subcases (a) and (c) but there is one in subcase (b). Similar to Fig. 8, Fig. 9 shows that $H$ intersects with $S_0^*(\tau)$ at point $X$ which (again) separates the whole in-the-money continuation region into regions A and B. The barrier is sometimes located in the exercise region of the vanilla option (e.g. subcase (c) and the region B of subcase (b)). In these cases, once the option is knocked in and becomes a vanilla option, it should be exercised immediately.

It is worth mentioning that the forward method does not need to identify these subcases (i.e. (a) (b) (c) in Table 2) in advance. Take the up-and-in put for example. When the option is knocked-in in the continuation region (subcase (a) and region A of subcase (b)), the FM algorithm checks at the barrier and finds that $S(=H) > \hat{S}_0$ (implying $S > S_0^*$) and lets the simulation continue. But if it is knocked-in in the exercise region (region B of subcase (b) and subcase (c)), the FM algorithm immediately finds that $S(=H) < \hat{S}_0$ (implying $S < S_0^*$) and stops this path to let the option be exercised.

# 5 Numerical results

In this section, we provide numerical examples of using the FM for the valuation of all seven types of American barrier put options. The FM performance is evaluated in terms of accuracy and computing time and is compared to the performance of the standard LSM. Accuracy is measured by relative error (RE) for each single parameter set and by root mean squared error (RMSE) for a group of parameter sets. The benchmark is obtained from the lattice method proposed by Ritchken (1995) with 10,000 time steps per year. Unless otherwise stated, for both FM and LSM, each simulation uses $N = 100{,}000$ stock price paths and the option is exercisable at equal-distance $M = 200$ time points in option life. As exercisable time points are discrete, a continuity correction (see Broadie et al. 1997, 1999; Kou 2003) is applied to adjust the barrier in both methods. In addition, to improve simulation accuracy, the moment matching technique suggested by Duan and Simonato (1998) and Glasserman (2004) is also used in the generation of random paths. All the pricing methods (implemented in Matlab) were run on a PC with an Intel Core2 Quad processor (Q8400, 2.67GHz) and 4GB RAM.

## 5.1 Out barrier options

**Case 1. American up-and-out put options with $H > K$**

Under the following parameters: time to maturity $T = (0.25, 0.5, 0.75, 1.0)$, volatility $\sigma = (0.2, 0.3, 0.4)$, and $K = 45$, $H = 50$, $r = 0.0488$, $q = 0.0$, we consider two initial stock price cases: $S = 40$ which is away from $H$ and $S = 49.5$ which is very close to $H$.

Table 3(a) shows the results for $S = 40$. As regards computing time, we see that our FM is no more efficient than the LSM. These results seem counter-intuitive, as our FM uses no backward induction and is expected to take less time. (By contrast, in the vanilla case studied in Miao and Lee (2013), the FM is clearly more efficient than the LSM.) This is attributed to the complex calculation of the pseudo critical price $\hat{S}_1$ as defined in (4) (because calculating $W_1(S)$ can be time-consuming). Although the computing time of the two methods is roughly at the same level, the proposed FM outperforms the LSM in terms of accuracy. We observe that the FM yields even higher accuracy with RMSE reduced to a third of that of the LSM.

Table 3(b) shows the results for $S = 49.5$. These cases are prone to errors and harder to handle. With $S = 49.5$ and $H = 50$, the simulation based on $M = 200$ exercisable time points (as in Table 3(a)) generally provides unsatisfactory results. The data presented in Table 3(b) are produced with $M \geq 1000$ over the option life $T$ (i.e. each time step $\Delta t = T/M$). The actual number of $M$ varies among parameter sets: if the RE from $M = 1000$ is significantly larger than 1%, we increase $M$ by 200 each time until the RE is below or close to 1%. However, this is not possible in some LSM cases because memory space has run out (the cases with superscript $*$). In these cases, the results with the largest possible $M$ are reported. Actually, when comparing with the FM, one weakness of the LSM is precisely the fact that it is memory-intensive and thus reaches its memory limit sooner.

From Table 3(b), we see the significant improvements of our FM over the LSM. In our FM, the RE for all the parameter sets can be made less than 1% by increasing $M$. But in the

**Table 3** American up-and-out put prices when $H > K$ (Case 1)

| Parameters | | | Benchmark | LSM | | | | FM | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $S$ | $\sigma$ | $T$ | Price | Price | (s.e.) | RE (%) | Time | Price | (s.e.) | RE (%) | Time |
| (a) $S = 40$ | | | | | | | | | | | |
| 40.0 | 0.2 | 0.25 | 5.0358 | 5.0353 | (0.0054) | −0.01 | 2.7 | 5.0328 | (0.0048) | −0.06 | 1.6 |
| 40.0 | 0.2 | 0.50 | 5.1881 | 5.1946 | (0.0081) | 0.12 | 5.1 | 5.1839 | (0.0063) | −0.08 | 4.5 |
| 40.0 | 0.2 | 0.75 | 5.3084 | 5.3101 | (0.0096) | 0.03 | 7.5 | 5.3065 | (0.0063) | −0.04 | 6.7 |
| 40.0 | 0.2 | 1.00 | 5.3861 | 5.3843 | (0.0105) | −0.03 | 9.7 | 5.3833 | (0.0059) | −0.05 | 8.5 |
| 40.0 | 0.3 | 0.25 | 5.4640 | 5.4568 | (0.0106) | −0.13 | 2.5 | 5.4600 | (0.0093) | −0.07 | 3.5 |
| 40.0 | 0.3 | 0.50 | 5.8526 | 5.8426 | (0.0136) | −0.17 | 4.8 | 5.8477 | (0.0098) | −0.08 | 6.4 |
| 40.0 | 0.3 | 0.75 | 6.0453 | 6.0335 | (0.0158) | −0.20 | 6.9 | 6.0410 | (0.0091) | −0.07 | 9.2 |
| 40.0 | 0.3 | 1.00 | 6.1455 | 6.1393 | (0.0173) | −0.10 | 9.5 | 6.1443 | (0.0081) | −0.02 | 10.1 |
| 40.0 | 0.4 | 0.25 | 5.9774 | 5.9651 | (0.0143) | −0.21 | 2.5 | 5.9736 | (0.0117) | −0.06 | 3.8 |
| 40.0 | 0.4 | 0.50 | 6.4285 | 6.4056 | (0.0185) | −0.36 | 4.6 | 6.4262 | (0.0117) | −0.04 | 6.4 |
| 40.0 | 0.4 | 0.75 | 6.6163 | 6.5998 | (0.0210) | −0.25 | 6.6 | 6.6150 | (0.0106) | −0.02 | 8.2 |
| 40.0 | 0.4 | 1.00 | 6.7055 | 6.6918 | (0.0227) | −0.20 | 8.8 | 6.7027 | (0.0091) | −0.04 | 9.4 |
| Average | | | | | RMSE = 0.18% | | 5.9 | | RMSE = 0.06% | | 6.5 |
| (b) $S = 49.5$ | | | | | | | | | | | |
| 49.5 | 0.2 | 0.25 | 0.1103 | 0.1120 | (0.0019) | 1.61 | 4.7 | 0.1100 | (0.0052) | −0.24 | 1.2 |
| 49.5 | 0.2 | 0.50 | 0.1613 | 0.1599 | (0.0027) | −0.88 | 9.8 | 0.1614 | (0.0071) | 0.04 | 2.4 |
| 49.5 | 0.2 | 0.75 | 0.1828 | 0.1844 | (0.0032) | 0.85 | 14.9 | 0.1829 | (0.0075) | 0.08 | 3.4 |
| 49.5 | 0.2 | 1.00 | 0.1936 | 0.1930 | (0.0034) | −0.30 | 19.6 | 0.1932 | (0.0072) | −0.19 | 4.2 |
| 49.5 | 0.3 | 0.25 | 0.1990 | 0.1980 | (0.0031) | −0.49 | 7.6 | 0.1990 | (0.0084) | 0.00 | 2.8 |
| 49.5 | 0.3 | 0.50 | 0.2439 | 0.2461 | (0.0044) | 0.90 | 15.7 | 0.2440 | (0.0097) | 0.02 | 4.9 |
| 49.5 | 0.3 | 0.75 | 0.2606 | 0.2617 | (0.0048) | 0.41 | 23.6 | 0.2629 | (0.0091) | 0.88 | 6.4 |
| 49.5 | 0.3 | 1.00 | 0.2684 | 0.2723* | (0.0052) | 1.43 | 27.6 | 0.2697 | (0.0088) | 0.47 | 7.6 |
| 49.5 | 0.4 | 0.25 | 0.2563 | 0.2571 | (0.0043) | 0.33 | 10.6 | 0.2580 | (0.0109) | 0.67 | 4.0 |
| 49.5 | 0.4 | 0.50 | 0.2930 | 0.2936 | (0.0054) | 0.19 | 23.5 | 0.2950 | (0.0115) | 0.67 | 6.9 |
| 49.5 | 0.4 | 0.75 | 0.3059 | 0.3169* | (0.0062) | 3.59 | 26.6 | 0.3081 | (0.0107) | 0.72 | 9.1 |
| 49.5 | 0.4 | 1.00 | 0.3117 | 0.3308* | (0.0067) | 6.13 | 27.2 | 0.3145 | (0.0091) | 0.92 | 10.6 |
| Average | | | | | RMSE = 2.20% | | 17.6 | | RMSE = 0.53% | | 5.3 |

Other parameters are $K = 45$, $H = 50$, $r = 0.0488$, $q = 0.0$. The benchmark is Ritchken's method with 10,000 time steps. The number of exercisable times $M = 200$ in (a) but $M \geq 1000$ in (b). The superscript $*$ indicates that the largest possible $M$ is reached due to memory limitation

LSM, for some parameter sets with $S$ close to $H$, when the memory limit is reached, the (best possible) RE is still greater than 1%. Since the FM is by nature much less memory-intensive, its highest reachable $M$ is much higher than that of the LSM, but the actual computing time of the FM with higher $M$ is still less than that of the LSM with lower $M$. Overall, for these more challenging cases, the forward Monte-Carlo method performs much more efficiently without loss of accuracy.

To further investigate the relation between $M$ and the performance measures, Table 4 provides a convergence analysis for Table 3. We first notice that, in this case, the maximal achievable $M$ for the LSM is around 1500, which reflects memory limitations as discussed above. In contrast, in the proposed FM, we may increase $M$ up to 10,000 (or even larger) to

**Table 4** Convergence analysis of Table 3 (Case 1)

| $(H, S, \sigma)$ | | (50, 40, 0.2) | | | (50, 40, 0.3) | | | (50, 40, 0.4) | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Method | $M$ | Price | RE (%) | Time | Price | RE (%) | Time | Price | RE (%) | Time |
| Benchmark | | 5.3861 | | | 6.1455 | | | 6.7055 | | |
| (a) $S = 40$ ($S$ is away from $H$) | | | | | | | | | | |
| LSM | 50 | 5.3720 | −0.26 | 2.3 | 6.1346 | −0.18 | 2.2 | 6.7004 | −0.08 | 2.0 |
| | 100 | 5.3834 | −0.05 | 5.0 | 6.1251 | −0.33 | 4.4 | 6.6729 | −0.49 | 4.1 |
| | 200 | 5.3843 | −0.03 | 9.7 | 6.1393 | −0.10 | 8.8 | 6.6918 | −0.20 | 8.2 |
| | 500 | 5.3664 | −0.37 | 24.4 | 6.1534 | 0.13 | 22.1 | 6.7105 | 0.08 | 20.6 |
| | 1000 | 5.3812 | −0.09 | 48.8 | 6.1431 | −0.04 | 45.3 | 6.7235 | 0.27 | 42.6 |
| FM | 50 | 5.3748 | −0.21 | 2.3 | 6.1359 | −0.16 | 2.7 | 6.6973 | −0.12 | 2.5 |
| | 100 | 5.3811 | −0.09 | 4.5 | 6.1444 | −0.02 | 5.2 | 6.7009 | −0.07 | 4.9 |
| | 200 | 5.3831 | −0.06 | 8.6 | 6.1452 | −0.01 | 10.1 | 6.7051 | −0.01 | 9.6 |
| | 500 | 5.3842 | −0.04 | 20.9 | 6.1441 | −0.02 | 24.9 | 6.7033 | −0.03 | 23.7 |
| | 1000 | 5.3845 | −0.03 | 41.1 | 6.1440 | −0.02 | 49.4 | 6.7038 | −0.03 | 47.0 |
| | 2000 | 5.3858 | −0.01 | 80.7 | 6.1450 | −0.01 | 98.0 | 6.7049 | −0.01 | 93.6 |
| | 5000 | 5.3869 | 0.01 | 199.4 | 6.1461 | 0.01 | 243.3 | 6.7052 | 0.00 | 233.1 |
| | 10,000 | 5.3844 | −0.03 | 398.0 | 6.1448 | −0.01 | 486.1 | 6.7054 | 0.00 | 466.4 |
| $(H, S, \sigma)$ | | (50, 49.5, 0.2) | | | (50, 49.5, 0.3) | | | (50, 49.5, 0.4) | | |
| Method | M | Price | RE (%) | Time | Price | RE (%) | Time | Price | RE (%) | Time |
| Benchmark | | 0.1936 | | | 0.2684 | | | 0.3117 | | |
| (b) $S = 49.5$ ($S$ is close to $H$) | | | | | | | | | | |
| LSM | 50 | 0.2936 | 51.7 | 1.0 | 0.5286 | 96.9 | 1.0 | 0.7649 | 145.4 | 1.0 |
| | 100 | 0.2484 | 28.3 | 2.0 | 0.4254 | 58.5 | 2.0 | 0.5910 | 89.6 | 2.0 |
| | 200 | 0.2180 | 12.6 | 3.9 | 0.3509 | 30.8 | 3.9 | 0.4765 | 52.9 | 3.9 |
| | 500 | 0.1983 | 2.4 | 9.5 | 0.3032 | 13.0 | 9.6 | 0.3850 | 23.5 | 9.6 |
| | 1000 | 0.1930 | −0.3 | 19.7 | 0.2795 | 4.2 | 19.6 | 0.3407 | 9.3 | 19.7 |
| FM | 50 | 0.2987 | 54.3 | 0.3 | 0.5404 | 101.3 | 0.4 | 0.7758 | 148.9 | 0.4 |
| | 100 | 0.2502 | 29.3 | 0.5 | 0.4291 | 59.9 | 0.6 | 0.6022 | 93.2 | 0.6 |
| | 200 | 0.2194 | 13.3 | 0.9 | 0.3560 | 32.6 | 1.0 | 0.4814 | 54.5 | 1.0 |
| | 500 | 0.1989 | 2.7 | 2.1 | 0.2990 | 11.4 | 2.1 | 0.3840 | 23.2 | 2.1 |
| | 1000 | 0.1922 | −0.7 | 4.2 | 0.2779 | 3.6 | 3.9 | 0.3422 | 9.8 | 3.8 |
| | 2000 | 0.1926 | −0.5 | 8.1 | 0.2703 | 0.7 | 7.6 | 0.3199 | 2.6 | 7.1 |
| | 5000 | 0.1925 | −0.6 | 20.2 | 0.2677 | −0.3 | 18.9 | 0.3118 | 0.0 | 17.2 |
| | 10,000 | 0.1934 | −0.1 | 40.1 | 0.2687 | 0.1 | 37.3 | 0.3111 | −0.2 | 34.7 |

Other parameters: $K = 45$, $r = 0.0488$, $q = 0.00$, $T = 1.00$

improve its accuracy. In general, the computing time grows linearly with $M$ in both methods (as expected), but it is of interest to inspect the case of $M = 1000$. For $S = 40$, our FM provides more accurate results with a similar amount of computing time. In contrast, for $S = 49.5$, the accuracy is similar in both methods, but our FM uses notably much less computing time than the LSM (the results from both methods are unsatisfactory with RE typically at 3–10%). The reason for the FM using less time is that the random path hits the barrier easily in the option's early life and the simulation tends to finish much sooner.

**Table 5** American up-and-out put prices when $H < K$ (Case 2)

| Parameters | | | Benchmark | LSM | | | | FM | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $S$ | $\sigma$ | $T$ | Price | Price | (s.e.) | RE (%) | Time | Price | (s.e.) | RE (%) | Time |
| 35.0 | 0.2 | 0.5 | 15.0000 | 15.0027 | (0.0026) | 0.02 | 16.9 | 15.0000 | (0.0009) | 0.00 | 0.5 |
| 40.0 | | | 10.0132 | 10.0347 | (0.0090) | 0.21 | 16.9 | 10.0131 | (0.0130) | 0.00 | 32.8 |
| 45.0 | | | 5.0552 | 5.0782 | (0.0138) | 0.45 | 13.2 | 5.0541 | (0.0148) | −0.02 | 36.8 |
| 48.0 | | | 2.0268 | 2.0354 | (0.0080) | 0.42 | 7.4 | 2.0274 | (0.0091) | 0.03 | 12.9 |
| 48.5 | | | 1.5147 | 1.5297 | (0.0057) | 0.99 | 6.1 | 1.5234 | (0.0067) | 0.57 | 7.3 |
| 35.0 | 0.2 | 1 | 15.0000 | 15.0088 | (0.0051) | 0.06 | 32.4 | 15.0000 | (0.0009) | 0.00 | 0.9 |
| 40.0 | | | 10.0197 | 10.0418 | (0.0142) | 0.22 | 31.4 | 10.0147 | (0.0166) | −0.05 | 54.5 |
| 45.0 | | | 5.0644 | 5.0895 | (0.0138) | 0.50 | 23.4 | 5.0651 | (0.0171) | 0.01 | 51.3 |
| 48.0 | | | 2.0295 | 2.0416 | (0.0087) | 0.59 | 13.5 | 2.0305 | (0.0102) | 0.05 | 17.6 |
| 48.5 | | | 1.5160 | 1.5277 | (0.0056) | 0.77 | 11.4 | 1.5242 | (0.0074) | 0.54 | 10.3 |
| 35.0 | 0.4 | 0.5 | 15.0000 | 15.0245 | (0.0092) | 0.16 | 65.7 | 15.0000 | (0.0015) | 0.00 | 2.7 |
| 40.0 | | | 10.0053 | 10.0475 | (0.0162) | 0.42 | 58.1 | 10.0011 | (0.0190) | −0.04 | 70.9 |
| 45.0 | | | 5.0166 | 5.0541 | (0.0144) | 0.75 | 40.8 | 5.0098 | (0.0175) | −0.13 | 54.4 |
| 48.0 | | | 2.0075 | 2.0377 | (0.0093) | 1.50 | 24.4 | 2.0007 | (0.0072) | −0.34 | 11.5 |
| 48.5 | | | 1.5040 | 1.5268 | (0.0058) | 1.51 | 21.4 | 1.5073 | (0.0048) | 0.22 | 6.7 |
| 35.0 | 0.4 | 1 | 15.0000 | 15.0045 | (0.0041) | 0.03 | 92.1 | 15.0000 | (0.0056) | 0.00 | 11.9 |
| 40.0 | | | 10.0056 | 10.0000 | (0.0040) | −0.06 | 82.2 | 10.0000 | (0.0194) | −0.06 | 78.3 |
| 45.0 | | | 5.0169 | 5.0030 | (0.0033) | −0.28 | 65.5 | 5.0063 | (0.0152) | −0.21 | 45.5 |
| 48.0 | | | 2.0076 | 2.0012 | (0.0021) | −0.32 | 52.6 | 2.0000 | (0.0015) | −0.38 | 3.8 |
| 48.5 | | | 1.5041 | 1.5081 | (0.0017) | 0.27 | 52.1 | 1.5013 | (0.0012) | −0.18 | 1.5 |
| Average | | | | | RMSE = 0.64% | | 36.4 | | RMSE = 0.23% | | 25.6 |

Other parameters are $K = 50$, $H = 49$, $r = 0.0488$, $q = 0.06$. The table is arranged in the same way as Table VII of Chang et al. (2007)

For the FM with even larger $M \geq 2000$, we see different convergence behavior in both cases. For $S = 40$, increasing $M$ does not significantly improve accuracy (i.e. the error has converged). But for $S = 49.5$, we observe that RE continues to converge and can be reduced to a level less than 1%. This indicates that, for these hard cases, compared with the LSM, our FM's advantage is that it uses a larger $M$ to achieve satisfactory results within a reasonable computing time.

**Case 2. American up-and-out put options with $H < K$**

In this case, since the option is exercised immediately if $q$ is small ($q = 0$ or $q \ll r$), we consider a relatively larger $q = 0.06$ to avoid trivial cases. The results are given in Table 5, where the "away from $H$" and "close to $H$" cases are reported together. We clearly observe the improvements (in both accuracy and time) of the proposed FM over the LSM. Note that when our FM is used in the hard cases with $S = 48.5 \approx H = 49$, the RE tends to be slightly larger than the other cases but all the results seem satisfactory. Also note that these hard cases take the least computing time. This is again attributed to the highest probability for the stock price to hit the barrier and finish the simulation sooner.

**Case 3. American down-and-out put options with $H < K$**

In the preceding two cases, our FM is based on either $\hat{S}_1$ or $\hat{S}_2$ which requires complex calculations and in turn results in longer computing times. However, from Case 3 onward,

**Table 6** American down-and-out put prices (Case 3)

| Parameters | | | Benchmark | LSM | | | | FM | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $S$ | $\sigma$ | $T$ | Price | Price | (s.e.) | RE (%) | Time | Price | (s.e.) | RE (%) | Time |
| 50.0 | 0.2 | 0.5 | 2.3379 | 2.3328 | (0.0089) | −0.22 | 3.6 | 2.3319 | (0.0089) | −0.26 | 1.6 |
| 45.0 | | | 5.3424 | 5.3467 | (0.0092) | 0.08 | 4.4 | 5.3411 | (0.0095) | −0.03 | 1.3 |
| 43.0 | | | 7.0431 | 7.0477 | (0.0067) | 0.07 | 3.8 | 7.0412 | (0.0067) | −0.03 | 0.7 |
| 41.0 | | | 9.0000 | 9.0000 | (0.0019) | 0.00 | 2.5 | 9.0000 | (0.0019) | 0.00 | 0.2 |
| 40.5 | | | 9.5000 | 9.5000 | (0.0017) | 0.00 | 2.1 | 9.5000 | (0.0016) | 0.00 | 0.2 |
| 50.0 | 0.2 | 1.0 | 3.0634 | 3.0544 | (0.0112) | −0.29 | 6.7 | 3.0519 | (0.0112) | −0.37 | 3.0 |
| 45.0 | | | 5.7642 | 5.7579 | (0.0119) | −0.11 | 7.4 | 5.7692 | (0.0121) | 0.09 | 2.7 |
| 43.0 | | | 7.2552 | 7.2708 | (0.0107) | 0.22 | 6.4 | 7.2571 | (0.0107) | 0.03 | 2.0 |
| 41.0 | | | 9.0160 | 9.0377 | (0.0064) | 0.24 | 4.4 | 9.0269 | (0.0061) | 0.12 | 0.8 |
| 40.5 | | | 9.5009 | 9.5000 | (0.0047) | −0.01 | 3.8 | 9.5000 | (0.0040) | −0.01 | 0.5 |
| 50.0 | 0.4 | 0.5 | 4.9118 | 4.9105 | (0.0147) | −0.03 | 18.5 | 4.9144 | (0.0147) | 0.05 | 6.6 |
| 45.0 | | | 7.1798 | 7.1698 | (0.0133) | −0.14 | 18.3 | 7.1813 | (0.0134) | 0.02 | 5.5 |
| 43.0 | | | 8.2499 | 8.2460 | (0.0112) | −0.05 | 15.9 | 8.2541 | (0.0114) | 0.05 | 4.1 |
| 41.0 | | | 9.4000 | 9.3938 | (0.0070) | −0.07 | 11.8 | 9.4050 | (0.0071) | 0.05 | 2.1 |
| 40.5 | | | 9.6982 | 9.6883 | (0.0050) | −0.10 | 10.6 | 9.6841 | (0.0052) | −0.14 | 1.5 |
| 50.0 | 0.4 | 1.0 | 6.1140 | 6.1024 | (0.0146) | −0.19 | 32.3 | 6.1100 | (0.0147) | −0.07 | 9.5 |
| 45.0 | | | 7.8924 | 7.8707 | (0.0122) | −0.27 | 30.5 | 7.8944 | (0.0123) | 0.03 | 7.6 |
| 43.0 | | | 8.6981 | 8.6843 | (0.0101) | −0.16 | 26.7 | 8.6978 | (0.0102) | 0.00 | 5.7 |
| 41.0 | | | 9.5543 | 9.5480 | (0.0062) | −0.07 | 21.7 | 9.5541 | (0.0062) | 0.00 | 3.2 |
| 40.5 | | | 9.7758 | 9.7667 | (0.0044) | −0.09 | 20.7 | 9.7662 | (0.0046) | −0.10 | 2.4 |
| Average | | | | | RMSE = 0.15% | | 12.6 | | RMSE = 0.12% | | 3.1 |

Other parameters are $K = 50$, $H = 40$, $r = 0.0488$, $q = 0$. The simulation is based on 100,000 paths for the stock-price process and the option is exercisable 200 times and 1000 times for $\sigma = 0.2$ and $\sigma = 0.4$ respectively

the FM algorithm is based on the vanilla pseudo critical price $\hat{S}_0$ which can be evaluated more quickly. It is expected that the computing time of the FM will be greatly reduced, making it a better-performing method for these cases.

The results are given in Table 6. Here two values of $\sigma = 0.2, 0.4$ are considered, and more exercisable time points are used for larger $\sigma$ cases in both methods (see the notes below the table). This is because higher volatility generally causes larger RE, and using more exercisable time points helps to control the RE to a desired level. Clearly, the proposed FM shows advantages in both RE and computing time (particularly the latter). The average computing time is only 3.1 (sec), which is less than one-fourth of the computing time used by the LSM. For the FM, the computing time is significantly less in the hard cases ($S = 40.5$) as the barrier is expected to be hit much sooner.

## 5.2 In barrier options

For the American in barrier options, Table 7 presents the results for Cases 4–7 following the classification of Dai and Kwok (2004). In Table 7(a) (Table 7(b)), there are four categories of $H$ corresponding to Case 4 and subcases (a) (b) (c) of Case 5 (Case 6 and subcases of Case 7). Overall, we see that the forward Monte-Carlo method performs more efficiently, but

**Table 7** American knock-in put price (Cases 4–7)

(a) Up-and-in put options (Cases 4, 5)

| Case | Location of | Parameters | Benchmark | LSM | | | | FM | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Index | the barrier $H$ | $(H, S)$ | Price | Price | (s.e.) | RE (%) | Time | Price | (s.e.) | RE (%) | Time |
| 4 | $H > K$ | (110, 109.5) | 7.8947 | 7.4738 | (0.0333) | −5.33 | 12.7 | 7.4994 | (0.0343) | −5.01 | 3.7 |
| | | (110, 105) | 6.1798 | 6.1345 | (0.0303) | −0.73 | 12.1 | 6.1596 | (0.0317) | −0.33 | 3.6 |
| | | (110, 100) | 4.5769 | 4.5603 | (0.0269) | −0.36 | 10.9 | 4.5597 | (0.0278) | −0.37 | 3.3 |
| 5(a) | $\frac{r}{q}K < H < K$ | (90, 89.5) | 16.2487 | 15.6242 | (0.0418) | −3.84 | 16.2 | 15.6107 | (0.0434) | −3.93 | 4.7 |
| | | (90, 85) | 12.7736 | 12.7522 | (0.0403) | −0.17 | 14.1 | 12.8265 | (0.0423) | 0.41 | 4.1 |
| | | (90, 80) | 9.4337 | 9.4572 | (0.0370) | 0.25 | 12.2 | 9.4554 | (0.0388) | 0.23 | 3.6 |
| 5(b) | $S_0^*(\infty) < H < \frac{r}{q}K$ | (70, 69.5) | 29.8991 | 29.1654 | (0.0374) | −2.45 | 17.1 | 29.1483 | (0.0405) | −2.51 | 2.9 |
| | | (70, 65) | 22.7926 | 22.8624 | (0.0484) | 0.31 | 14.6 | 22.7780 | (0.0498) | −0.06 | 2.6 |
| | | (70, 60) | 15.8488 | 15.9121 | (0.0510) | 0.40 | 10.9 | 15.8724 | (0.0511) | 0.15 | 2.4 |
| 5(c) | $H < S_0^*(\infty)$ | (40, 39.5) | 57.2408 | 56.6111 | (0.0415) | −1.10 | 15.8 | 56.6013 | (0.0416) | −1.12 | 0.4 |
| | | (40, 35) | 34.5035 | 34.5454 | (0.0906) | 0.12 | 11.4 | 34.5603 | (0.0906) | 0.16 | 1.1 |
| | | (40, 30) | 15.4545 | 15.4226 | (0.0801) | −0.21 | 6.7 | 15.3579 | (0.0800) | −0.62 | 1.9 |
| | | | Average | | RMSE = 2.27% | | 12.9 | | RMSE = 2.02% | | 2.9 |

(b) Down-and-in put options (Cases 6, 7)

| Case | Location of | Parameters | Benchmark | LSM | | | | FM | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Index | the barrier $H$ | $(H, S)$ | Price | Price | (s.e.) | RE (%) | Time | Price | (s.e.) | RE (%) | Time |
| 6 | $H > K$ | (110, 110.5) | 7.9475 | 7.9370 | (0.0339) | −0.13 | 9.5 | 7.9251 | (0.0349) | −0.28 | 3.9 |
| | | (110, 115) | 6.6715 | 6.6612 | (0.0314) | −0.15 | 8.4 | 6.6544 | (0.0325) | −0.26 | 3.6 |
| | | (110, 120) | 5.4685 | 5.4678 | (0.0288) | −0.01 | 7.6 | 5.4745 | (0.0299) | 0.11 | 3.4 |
| 7(a) | $\frac{r}{q}K < H < K$ | (90, 90.5) | 16.3814 | 16.3435 | (0.0410) | −0.23 | 13.0 | 16.3751 | (0.0436) | −0.04 | 4.9 |
| | | (90, 95) | 14.0179 | 13.9837 | (0.0402) | −0.24 | 10.5 | 14.0244 | (0.0423) | 0.05 | 4.3 |
| | | (90, 100) | 11.7100 | 11.6984 | (0.0385) | −0.10 | 9.0 | 11.7256 | (0.0404) | 0.13 | 3.9 |

**Table 7** continued

(b) Down-and-in put options (Cases 6, 7)

| Case | Location of | Parameters | Benchmark | LSM | | | | FM | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Index | the barrier $H$ | $(H, S)$ | Price | Price | (s.e.) | RE (%) | Time | Price | (s.e.) | RE (%) | Time |
| 7(b) | $S_0^*(\infty) < H < \frac{r}{q}K$ | (70, 70.5) | 30.2140 | 30.3028 | (0.0349) | 0.29 | 13.9 | 30.3100 | (0.0375) | 0.32 | 2.9 |
| | | (70, 75) | 25.7344 | 25.7644 | (0.0443) | 0.12 | 11.1 | 25.7323 | (0.0455) | −0.01 | 2.5 |
| | | (70, 80) | 21.1972 | 21.2483 | (0.0484) | 0.24 | 8.9 | 21.2263 | (0.0492) | 0.14 | 2.4 |
| 7(c) | $H < S_0^*(\infty)$ | (40, 40.5) | 58.3116 | 58.0423 | (0.0326) | −0.46 | 12.8 | 58.0586 | (0.0325) | −0.43 | 0.3 |
| | | (40, 45) | 43.9431 | 43.9832 | (0.0799) | 0.09 | 9.6 | 43.9574 | (0.0799) | 0.03 | 0.9 |
| | | (40, 50) | 30.5565 | 30.5382 | (0.0908) | −0.06 | 6.8 | 30.5387 | (0.0908) | −0.06 | 1.4 |
| | | | Average | | RMSE = 0.21% | | 10.1 | | RMSE = 0.20% | | 2.9 |

The other parameters are $K = 100$, $r = 0.1$, $q = 0.12$, $\sigma = 0.3$, $T = 1$. Here $q > r$ is chosen such that it is meaningful to discuss the case $\min(1, \frac{r}{q})K < H < K$. For the corresponding vanilla put option, $S_0^*(0) = \min(1, \frac{r}{q})K = 83.33$, $S_0^*(\infty) = 48.30$

**Table 8** Convergence analysis of Table 7(a) (Cases 4, 5)

(a) $S$ is away from $H$

| $(H, S)$ | | (110,100) | | | (90,80) | | | (70,60) | | | (40,30) | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Method | $M$ | Price | RE (%) | Time | Price | RE (%) | Time | Price | RE (%) | Time | Price | RE (%) | Time |
| Benchmark | | 4.5769 | | | 9.4337 | | | 15.8488 | | | 15.4545 | | |
| LSM | 50 | 4.5840 | 0.15 | 1.8 | 9.5019 | 0.72 | 2.0 | 15.8996 | 0.32 | 1.9 | 15.5263 | 0.46 | 1.3 |
| | 100 | 4.5720 | −0.11 | 4.5 | 9.4706 | 0.39 | 4.6 | 15.9519 | 0.65 | 4.3 | 15.4288 | −0.17 | 2.8 |
| | 200 | 4.5603 | −0.36 | 11.9 | 9.4572 | 0.25 | 11.7 | 15.9121 | 0.40 | 10.8 | 15.4226 | −0.21 | 6.6 |
| | 500 | 4.5804 | 0.08 | 51.1 | 9.4134 | −0.22 | 47.6 | 15.8357 | −0.08 | 43.0 | 15.4615 | 0.05 | 23.9 |
| | 1000 | 4.5480 | −0.63 | 175.1 | 9.4012 | −0.34 | 157.6 | 15.8164 | −0.20 | 140.7 | 15.3905 | −0.41 | 72.7 |
| FM | 50 | 4.5881 | 0.24 | 0.9 | 9.4663 | 0.35 | 0.9 | 15.8864 | 0.24 | 0.7 | 15.5076 | 0.34 | 0.5 |
| | 100 | 4.5905 | 0.30 | 1.7 | 9.4069 | −0.28 | 1.8 | 15.8618 | 0.08 | 1.2 | 15.5107 | 0.36 | 0.9 |
| | 200 | 4.5597 | − 0.37 | 3.5 | 9.4554 | 0.23 | 3.6 | 15.8724 | 0.15 | 2.4 | 15.3579 | −0.62 | 1.8 |
| | 500 | 4.6130 | 0.79 | 8.7 | 9.4677 | 0.36 | 8.9 | 15.8151 | −0.21 | 5.7 | 15.3632 | −0.59 | 4.5 |
| | 1000 | 4.5999 | −0.50 | 17.3 | 9.4463 | 0.13 | 17.7 | 15.9162 | 0.42 | 11.3 | 15.5013 | 0.30 | 9.0 |
| | 2000 | 4.5811 | 0.09 | 34.3 | 9.4032 | −0.32 | 35.3 | 15.8143 | −0.22 | 22.5 | 15.3819 | −0.47 | 18.0 |
| | 3000 | 4.5608 | −0.35 | 51.2 | 9.4599 | 0.28 | 52.9 | 15.9125 | 0.40 | 33.5 | 15.4787 | 0.16 | 27.0 |
| | 5000 | 4.5697 | −0.16 | 85.1 | 9.4560 | 0.24 | 87.8 | 15.8177 | −0.20 | 56.0 | 15.4918 | 0.24 | 44.8 |
| | 10,000 | 4.5753 | −0.04 | 169.1 | 9.4270 | −0.07 | 174.8 | 15.7984 | −0.32 | 111.2 | 15.4729 | 0.12 | 88.5 |

(b) $S$ is close to $H$

| $(H, S)$ | | (110,109.5) | | | (90,89.5) | | | (70,69.5) | | | (40,39.5) | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Method | $M$ | Price | RE (%) | Time | Price | RE (%) | Time | Price | RE (%) | Time | Price | RE (%) | Time |
| Benchmark | | 7.8946 | | | 16.2487 | | | 29.8991 | | | 57.2408 | | |
| LSM | 50 | 6.9133 | −12.43 | 2.4 | 14.7371 | −9.30 | 2.9 | 27.9320 | −6.58 | 2.9 | 54.8748 | −4.13 | 2.7 |
| | 100 | 7.2393 | −8.30 | 5.1 | 15.2781 | −5.97 | 6.4 | 28.6953 | −4.03 | 6.7 | 55.8996 | −2.34 | 6.3 |
| | 200 | 7.4738 | −5.33 | 13.8 | 15.6242 | −3.84 | 15.9 | 29.1654 | −2.45 | 16.7 | 56.6111 | −1.10 | 15.6 |

**Table 8** continued

(b) $S$ is close to $H$

| $(H, S)$ | | (110,109.5) | | | (90,89.5) | | | (70,69.5) | | | (40,39.5) | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Method | $M$ | Price | RE (%) | Time | Price | RE (%) | Time | Price | RE (%) | Time | Price | RE (%) | Time |
| Benchmark | | 7.8946 | | | 16.2487 | | | 29.8991 | | | 57.2408 | | |
| | 500 | 7.6755 | −2.78 | 60.6 | 15.9797 | −1.66 | 63.5 | 29.5478 | −1.18 | 65.8 | 57.0685 | −0.30 | 62.4 |
| | 1000 | 7.8050 | −1.14 | 208.5 | 16.0890 | −0.98 | 205.6 | 29.7677 | −0.44 | 206.3 | 57.2128 | −0.05 | 202.5 |
| FM | 50 | 6.8888 | −12.74 | 1.1 | 14.7230 | −9.39 | 1.2 | 27.8496 | −6.85 | 0.8 | 54.8231 | −4.22 | 0.2 |
| | 100 | 7.2007 | −8.79 | 1.8 | 15.2631 | −6.07 | 2.4 | 28.6622 | −4.14 | 1.6 | 55.9211 | −2.31 | 0.2 |
| | 200 | 7.4994 | −5.01 | 3.7 | 15.6107 | −3.93 | 4.8 | 29.1483 | −2.51 | 2.9 | 56.6013 | −1.12 | 0.4 |
| | 500 | 7.6634 | −2.93 | 9.2 | 15.9734 | −1.69 | 11.9 | 29.5741 | −1.09 | 7.0 | 57.0569 | −0.32 | 0.8 |
| | 1000 | 7.8007 | −1.19 | 18.4 | 16.0861 | −1.00 | 23.7 | 29.7728 | −0.42 | 13.8 | 57.2100 | −0.05 | 1.5 |
| | 2000 | 7.8448 | −0.63 | 36.7 | 16.1682 | −0.50 | 47.2 | 29.8608 | −0.13 | 27.2 | 57.2359 | −0.01 | 2.9 |
| | 3000 | 7.8642 | −0.39 | 54.9 | 16.2148 | −0.21 | 70.8 | 29.8993 | 0.00 | 40.5 | 57.2349 | −0.01 | 4.3 |
| | 5000 | 7.8884 | −0.08 | 91.4 | 16.2455 | −0.02 | 117.9 | 29.8844 | −0.05 | 67.1 | 57.2227 | −0.03 | 7.1 |
| | 10,000 | 7.8851 | −0.12 | 186.6 | 16.2750 | 0.16 | 235.4 | 29.9134 | 0.05 | 133.5 | 57.2404 | 0.00 | 14.0 |

The other parameters for the up-and-in put are $K = 100$, $r = 0.1$, $q = 0.12$, $\sigma = 0.3$, $T = 1$

the accuracy of both methods seems to be at a similar level (unlike the observations from the out barrier options). The better efficiency is attributed to the fact that the FM is a simple adaptation from its vanilla version with a less complex formula of $\hat{S}$. From Table 7(a), it is also observed that for both methods, the RE is significantly larger than 1% when $S \approx H$ but is below 1% when $S$ is away from $H$. This indicates that $S \approx H$ is problematic for up-and-in put options. As seen in Table 7(b), however, this problem is less serious for down-and-in put options. This is because for an up-and-in put, the stock price must first go up for the option to be knocked in and subsequently go down to trigger early exercise. Therefore, the results are sensitive to the barrier hitting time (which is error-prone in simulation when $S \approx H$). But for a down-and-in put, the knock-in barrier $H$ and exercise barrier $S_0^*$ are in the same direction (i.e. the stock price must go down to be knocked in and exercised). For a stock price moving towards $H$ and $S_0^*$ and about to trigger early exercise, the exact time of crossing $H$ is relatively less important, posing less of the problem.

To further investigate the advantages of the proposed FM over the LSM, we examine their convergence behaviors. Table 8 provides a convergence analysis for the up-and-in barrier put option from Table 7(a) (the analysis for Table 7(b) is similar and omitted). Again, in our FM, the possibility of using $M \geq 2000$ helps to reduce the RE to a level below 1%, but this is not achievable in the LSM (the maximal reachable $M$ is around 1500). When a larger $M$ is used in our FM, the computing time increases linearly but remains at an acceptable level. (The time used by our FM with $M = 5000 \sim 10{,}000$ is roughly at the same level as the time used by the LSM with $M = 1000$.) For the hard cases shown in Table 8(b), the possibility of using larger $M$ is important because the RE has not yet converged at a moderate $M$ (e.g. 500 or 1000). By comparing the results from both methods, we see that the proposed FM produces nearly converged results (with $M = 3000 \sim 5000$) with higher accuracy than the LSM (with $M = 1000$).

## 6 Conclusions

This paper discusses how the forward Monte-Carlo method can be applied to the valuation of American barrier options. Instead of directly extending the original approach developed for American vanilla options, we propose a more flexible version of the pseudo critical price which can be easily extended to American barrier options. Based on this new and more general pseudo critical price, the forward methods are successfully developed for the pricing of all fourteen types of American barrier options.

In the first two cases of out barrier put options, the forward method relies on the analytical approximation formulas of these options. We prove that the proposed pseudo critical prices are sufficient indicators that guarantee the usefulness of the forward methods. In the third case of out barrier put options, as well as all four cases of in barrier put options, the forward method can be developed with an adaptation from its vanilla version. In these cases, only the vanilla pseudo critical price is required; this implies that the computing time can be greatly reduced.

Our numerical experiments compare the proposed FM to the LSM for all seven types of American barrier put options. It is observed that our FM outperforms the LSM in a trade-off between accuracy and efficiency, although the improvements vary among different cases. Moreover, our FM shows a better convergence pattern compared to the LSM. In particular, for the more challenging and error-prone cases where the initial stock price is close to the barrier, our FM performs particularly well and yields sufficiently accurate results. This is not achievable by the LSM given the same computing resources.

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## References

Areal, N., Rodrigues, A., & Armada, M. R. (2008). On improving the least squares Monte Carlo option valuation method. *Review of Derivatives Research*, *11*, 119–151.
Barone-Adesi, G., & Whaley, R. (1987). Efficient analytic approximation of American option values. *Journal of Finance*, *42*, 301–320.
Barraquant, J., & Martineau, D. (1995). Numerical valuation of high dimensional multivariate American securities. *Journal of Financial and Quantitative Analysis*, *30*, 383–405.
Boyle, P., Broadie, M., & Glasserman, P. (1997). Monte Carlo methods for security pricing. *Journal of Economic Dynamics and Control*, *21*, 1267–1321.
Broadie, M., & Glasserman, P. (1997). Pricing American-style securities using simulation. *Journal of Economic Dynamics and Control*, *21*, 1323–1352.
Broadie, M., Glasserman, P., & Kou, S. (1997). A continuity correction for discrete barrier options. *Mathematical Finance*, *7*, 325–348.
Broadie, M., Glasserman, P., & Kou, S. (1999). Connecting discrete and continuous path-dependent options. *Finance and Stochastics*, *3*, 55–82.
Chang, G., Kang, J., Kim, H.-S., & Kim, I. J. (2007). An efficient approximation method for American exotic options. *Journal of Futures Markets*, *27*, 29–59.
Dai, M., & Kwok, Y. K. (2004). Knock-in American options. *Journal of Futures Markets*, *24*, 172–192.
Duan, J. C., & Simonato, J. G. (1998). Empirical martingale simulation for asset prices. *Management Science*, *44*, 1218–1233.
Gao, B., Huang, J., & Subrahmanyam, M. (2000). The valuation of American barrier options using the decomposition techniques. *Journal of Economic Dynamics & Control*, *24*, 1783–1827.
Glasserman, P. (2004). *Monte Carlo methods in financial engineering*. New York: Springer.
Haug, E. G. (2001). Closed form valuation of American barrier options. *International Journal of Theoretical and Applied Finance*, *4*, 355–359.
Haug, E. G. (2006). *The complete guide to option pricing formulas* (2nd ed.). New York: McGraw-Hill.
Kou, S. G. (2003). On pricing of discrete barrier options. *Statistica Sinica*, *13*, 955–964.
Longstaff, F. A., & Schwartz, E. S. (2001). Valuing American options by simulation: A simple least-squares approach. *Review of Financial Studies*, *14*, 113–147.
Miao, D. W.-C., & Lee, Y.-H. (2013). A forward Monte Carlo method for American options pricing. *Journal of Futures Markets*, *33*, 369–395.
Ritchken,. (1995). On pricing barrier options. *Journal of Derivatives, Winter 1995*, 19–28.
Tilley, J. A. (1993). Regression methods for pricing complex American-style options. *Transactions of the Society of Actuaries*, *45*, 249–266.