

A Modified Reduced-Form Model with Time-Varying Default and Recovery Rates and Its Applications in Pricing Convertible Bonds

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Due to the lack of proper credit derivatives and opaque financial status of a reference entity, it is difficult to accurately estimate recovery rates in reduced-form models. In addition, most reduced-form models adopt a constant recovery rate assumption that fails to capture the time-varying dynamics and inverse relationships between recovery and default rates found in empirical studies. Most revisions that incorporate this inverse relationship require strict calibration procedures or limit the wide applicability of reduced-form models due to introducing complexity models involving stochastic processes. The authors propose a notion of the expected recovery rate conditional on the default rate by combining a regression relationship between these two rates and a transformation between default rates under the physical and risk-neutral measures. This notion can be simply incorporated into any reduced-form model to simultaneously produce reliable time-varying recovery and default rates. To demonstrate their idea, they revise Jarrow and Turnbull's [1995] reduced-form model used in Chambers and Lu [2007] for pricing convertible bonds. The resulting tree structure is also adjusted to alleviate the infeasible branching probability problem.

The evaluation of default risk has become more important, especially after the financial crisis of 2008. Credit risk can be modeled

by structural models or reduced-form models. This article proposes a feasible notion that can be incorporated into any reduced-form model to produce reliable recovery and default rates.

The structural models pioneered by Merton [1974] and Black and Cox [1976] simulate the evolution of a firm's value and debt level and specify the conditions leading to default. When a default event occurs, debt holders receive only part of the total debt obligation, equal to the current firm value minus the bankruptcy cost. The ratio between the amount recovered through the bankruptcy procedure and the amount of total debt obligation is defined as the recovery rate. Altman, Resti, and Sironi [2001] show that in a structural model, a low (high) firm value implies a high (low) default rate and a low (high) recovery rate in the default event. That is, structural models generally imply an inverse relationship between the recovery and default rates. In fact, this inverse relationship is widely confirmed in many empirical studies, including those of Altman et al. [2005]; Hu and Perraudin [2006]; Acharya, Bharath, and Srinivasan [2007]; and Hamilton et al. [2007].

In contrast, without modeling firm value and its debt level, reduced-form models employ certain intensity-based approaches to

model the likelihood of default events (default rate) and the percentage recovered from default (recovery rate) to match observable market variables such as credit spreads. Most reduced-form models, such as those of Jarrow and Turnbull [1995] and Jarrow, Lando, and Turnbull [1997], assume debt holders receive an exogenously constant recovery rate in default events and calibrate the default rate to match expected default losses. However, the constant recovery rate assumption fails to produce the aforementioned inverse relationship between the recovery and default rates.

A common way to resolve this issue is to introduce stochastic recovery rates into reduced-form models. Some in this stream of literature even consider a functional relationship between the recovery rate and other variables. For instance, Karoui [2007], Gaspar and Slinko [2008], and Chiang and Tsai [2010] model both recovery and default rates to depend on a set of state variables, which may represent related macroeconomic or firm-specific factors. However, incorporating complicated stochastic processes into reduced-form models may significantly raise the complexity of computation and calibration. Moreover, specific relationships between the recovery rate and the default rate or other variables introduce potential errors in model identification and parameter estimation. Both Bakshi, Madan, and Zhang [2006] and Das and Hanouna [2009] explicitly specify recovery rates as inverse functions of default rates. The former approach adopts a heuristic assumption on the relationship between default and recovery rates and derives a closed-form model for straight bonds. The latter approach is essentially a jump-to-default model based on the binomial tree, in which the default and recovery rates are properly related to the stock price to capture their stylized inverse relationship. However, the constant interest rate assumption also makes it difficult for the latter approach to evaluate interest-rate-option-embedded securities. In contrast to the above models, Schäfer and Uhrig-Homburg [2014] take advantage of the fact that credit default swaps (CDSs) on a reference entity's debts of different seniorities face identical default risk but different recovery rates to separate the default and recovery rates and thus obtain recovery rate distributions. Lastly, it is worth noting that outstanding CDSs concentrate on firms with BBB or BB credit ratings. For firms with other credit ratings, the availability and liquidity of CDSs are serious problems and limit the applicability of most stochastic recovery

rate models that require market prices of CDS spreads for parameter calibration.

In contrast to stochastic recovery rate models, which increase substantially the complexity of reduced-form models and thus damage the simplicity and applicability of reduced-form models, in our article we propose a novel notion that can be incorporated into all reduced-form models to endogenously determine the recovery and default rates and to preserve their inverse relationship. To achieve this, we introduce a relationship between these two rates by converting an empirical regression equation between the recovery and default rates under the physical probability measure in the literature (e.g., Altman et al. [2005]) to the risk-neutral one with a transformation of default rates under different measures (e.g., Hull, Predescu, and White [2005]). The resulting relationship can be expressed as an expected recovery rate conditional on the risk-neutral default rate. If accurate estimates of recovery rates are not available, this conditional expected recovery rate can be incorporated into any reduced-form model to endogenously determine time-varying recovery and default rates with an inverse relationship.

Our approach also offers the following advantages. First, it no longer requires exogenous estimation of a constant recovery rate as in traditional reduced-form models or the calibration for parameters as in stochastic recovery rate models. This is especially important if the reference entity does not have CDSs on its debts or if its financial condition is not transparent enough to reliably estimate recovery rates. Second, our approach properly reflects the market consensus of recovery rates by exactly calibrating the prevailing term structure of credit spreads and thus obtains time-varying dynamics of recovery rates.¹ It is analogous to no-arbitrage interest rate models, such as Hull and White's [1990] model, which generate time-varying interest rates to calibrate the prevailing term structure of interest rates. In contrast, stochastic recovery rate reduced-form models use best-fit methods for parameter tuning and do not completely calibrate the prevailing term structure of credit spreads—which are analogous to equilibrium interest rate models. Third, reduced-form models revised by incorporating our regression-based conditional expected recovery rate are expected to perform better when pricing a portfolio of defaultable claims or credit derivatives with multiple reference entities because the regression's prediction errors can be averaged out in large samples.

Consequently, the proposed notion is particularly useful for financial institutions that hold large portfolios of loan assets.

To demonstrate how our core idea of the conditional expected recovery rate works, we integrate it into the flexible tree-based reduced-form model proposed by Jarrow and Turnbull [1995], hereafter the JT reduced-form model or simply the JT model. We choose this model because the tree-based structure can be easily extended to price option-embedded defaultable claims. In addition, the unreliability issues for the recovery and default rates of reduced-form models influence option-embedded defaultable claims more severely than straight corporate bonds. For a straight corporate bond, the expected loss due to the default risk is code-termined by the recovery and default rates. Even if the default rate is poorly calibrated due to the misestimated recovery rate, the combinations of the poorly estimated default and recovery rates are still sufficient to capture the expected default loss of the straight corporate bond. However, this is not the case for option-embedded defaultable claims. Due to the nonlinear dependence of the option payoffs on the default and recovery rates, the exercise decisions of the embedded options can be significantly influenced by the default or recovery rates. Therefore, inaccurate estimates of default or recovery rates could significantly misprice embedded options in defaultable claims.²

Then we demonstrate how our revised JT reduced-form model is employed to price convertible bonds (CBs), which are popular, frequently traded defaultable claims with embedded options on interest rates and issuers' stock prices. Traditionally, the default risk for pricing CBs can be modeled by the structural model (e.g., Brennan and Schwartz [1977, 1980]; Ingersoll [1977]). Tsiveriotis and Fernandes [1998] decompose a CB value into equity and debt components and evaluate the latter component by discounting the corresponding payoff with a risky interest rate to reflect the potential default risk. Alternatively, Takahashi, Kobayashi, and Nakagawa [2001] use the default-adjusted interest rate proposed by Duffie and Singleton [1999] to capture the default risk. The defaultable interest tree of the JT model is first employed by Hung and Wang [2002] to price CBs. They develop a bivariate tree CB pricing model that combines the JT model with the binomial stock price tree of Cox, Ross, and Rubinstein [1979], hereafter referred to as the CRR model.

Hung and Wang [2002] adopt the interest rate model of Black, Derman, and Toy [1990], the BDT model, as a foundation for implementing the JT model. In line with the JT model, Hung and Wang assume the recovery rate to be an exogenously specified constant. Chambers and Lu [2007], hereafter the CL model, propose a modified tree model that incorporates the correlation between the interest rate and the stock price into Hung and Wang's model.

We further improve the CL model in the following two aspects. First, we substitute our conditional expected recovery rate for the constant recovery rate to allow for the revision of the JT reduced-form model. Second, we resolve the infeasible branching probability problem in the CL model, whose bivariate tree model could be invalidated by branching probabilities outside the range of $[0, 1]$. Specifically, a high interest rate simulated by the BDT model or a large default rate would result in an unexpectedly high growth rate for the stock price, thus causing the CRR model to produce infeasible branching probabilities. Moreover, their adjustment method to calibrate the correlation between the interest rate and the stock price could also result in infeasible branching probabilities. To solve the infeasible branching probability problem, which occurs frequently in the CL model, we exploit the mean-tracking method proposed by Dai [2009] and Dai and Lyuu [2010] and modify the adjustment method to calibrate the correlation between the interest rate and the stock price.

The rest of this article is organized as follows. The second section proposes an explicit equation for the expected recovery rate conditional on the default rate under the risk-neutral measure. This equation is then incorporated into our revision of the JT model to simultaneously determine the default and recovery rates. The third section proposes a modified CB pricing model to rectify the infeasible branching probability problem of the CL model. The fourth section reports the pricing results of a real CB contract issued by the Danaher Corporation, as well as the results of sensitivity analyses based on this real case. The fifth section illustrates a possible extension of our model for determining the expected recovery rate conditional on extra macroeconomic factors in addition to the default rate. We summarize the findings and conclude the article in the sixth section.

MODIFIED REDUCED-FORM MODEL

Expected Recovery Rate Conditional on the Default Rate

Numerous studies assess the empirical relationship between the recovery and default rates, such as Altman et al. [2005]; Hu and Perraudin [2006]; Hamilton et al. [2007]; and Acharya, Bharath, and Srinivasan [2007]. Our core idea is that default and recovery rates can be endogenously determined in reduced-form models by inserting a proper relationship (between these two rates) which fits empirical market conditions or model requirements. For example, this article exploits the regression result between the recovery rate and the logarithmic default rate from 1982 to 2001 in Altman et al. [2005] to implement our model:

$$\delta = a + b \ln(DR^P) + \nu, \quad (1)$$

where δ denotes the recovery rate, DR^P is the default rate per year under the physical probability measure P (i.e., the annual default rate in the real world), and ν is standard white noise. The least-squares regression results are $a = 0.0022$, $b = -0.1133$, and $R^2 = 0.63$. The negative value of b indicates an inverse relationship between the recovery rate and the default rate under the physical probability measure. We use this regression relationship for several reasons. First, we focus on the explicit relationship between the recovery and default rates, which are studied only in Altman et al. [2005] and Hamilton et al. [2007].³ Second, the R -squared value of the linear-log regression in Altman et al. [2005], illustrated in Equation (1), is comparatively higher than other studies.⁴ Third, our model can easily integrate this linear-log regression equation and the default intensity analysis in Hull, Predescu, and White [2005], which is discussed in detail in the following.

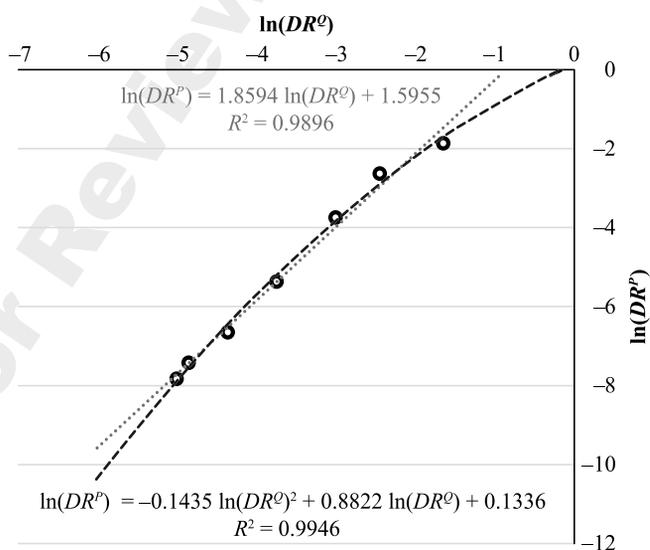
Pricing derivatives with the risk-neutral valuation method requires the default rate under the risk-neutral probability measure. To transfer the default rates in the physical probability measure mentioned in Equation (1), we take advantage of previous work that analyzes default intensities under the physical and the risk-neutral probability measures, such as Hull, Predescu, and White [2005], to derive the transformation between the default rates in the real and the

EXHIBIT 1

Proposed Regression Relationships between Physical and Risk-Neutral Default Rates

Rating	λ^P (bps)	λ^Q (bps)	DR^P	DR^Q
AAA	4	67	0.04%	0.67%
AA	6	78	0.06%	0.78%
A	13	128	0.13%	1.27%
BBB	47	238	0.47%	2.35%
BB	240	507	2.37%	4.94%
B	749	902	7.22%	8.63%
CCC-C	1,690	2,130	15.55%	19.18%

Linear vs. Quadratic Regression



Notes: This exhibit compares the linear and quadratic regressions between $\ln DR^P = \ln(1 - e^{-\lambda^P})$ and $\ln DR^Q = \ln(1 - e^{-\lambda^Q})$. The data of λ^P and λ^Q are inherited from Table 1 in Hull, Predescu, and White [2005]. Using either the linear or quadratic regression, we obtain a positive-sloping, extremely high R -squared regression relationship between $\ln DR^P$ and $\ln DR^Q$.

risk-neutral worlds. They estimate the physical default intensities as the average annual default rates over the seven-year cumulative default rates published by Moody's and the risk-neutral default intensities based on Merrill Lynch bond indexes. Based on their Table 1 (also shown in Exhibit 1), we conduct the following regression for the default rates under the physical probability measure (DR^P) and the risk-neutral one (DR^Q) in different credit rating classes:

$$\ln(DR^P) = \alpha + \beta \ln(DR^Q) + \gamma (\ln(DR^Q))^2 + \eta, \quad (2)$$

where η denotes white noise. Note that the default rates can be expressed in terms of the default intensities as follows: $DR^P = 1 - e^{-\lambda^P}$ and $DR^Q = 1 - e^{-\lambda^Q}$. The least-squares regression results for Equation (2) are $\alpha = 0.1336$, $\beta = 0.8822$, and $\gamma = -0.1435$, with an R-squared of 0.9946. The extremely high R-squared value implies that Equation (2) precisely maps the default rates (or equivalently, default intensities) between the real and the risk-neutral worlds. Consequently, replacing the term $\ln(DR^P)$ in Equation (1) with Equation (2) yields

$$\delta = a + b(\alpha + \beta \ln(DR^Q) + \gamma(\ln(DR^Q))^2 + \eta) + v.$$

We next take the expectation for the above composite regression equation conditional on DR^Q . The expectation of η is zero since it is, by definition, independent of DR^Q in Equation (2). We further assume that the white noise v and DR^Q are independent such that the expectation of v is also zero. This assumption is likely to be true since the white noise v is independent of DR^P in Equation (1), and DR^P and DR^Q are almost perfectly related, as indicated by the R-squared value almost being one in Equation (2). As a result, the expected recovery rate conditional on the risk-neutral default rate, denoted as $\hat{\delta}$, can be expressed as

$$\begin{aligned} \hat{\delta} &= a + b[\alpha + \beta \ln(DR^Q) + \gamma(\ln(DR^Q))^2] \\ &= a + b[\alpha + \beta \ln(1 - e^{-\lambda^Q}) + \gamma(\ln(1 - e^{-\lambda^Q}))^2], \end{aligned} \quad (3)$$

where the last equation is derived by replacing DR^Q with $1 - e^{-\lambda^Q}$.

We also examine the linear relationship between $\ln(DR^P)$ and $\ln(DR^Q)$ based on the same dataset in Hull, Predescu, and White [2005] for deriving Equations (2) and (3). There are two reasons for selecting the quadratic relationship instead of the linear one even though the linear regression line and the quadratic regression curve illustrated in Exhibit 1 are highly similar (almost coinciding for a large portion). First, the R-squared value of the quadratic regression is even higher and thus more suitable to obtain the conditional expected recovery rate $\hat{\delta}$ in Equation (3). Second, the quadratic regression is superior to the linear one in maintaining the widely-observed fact of positive default risk premiums, proxied by the risk-neutral default rate DR^Q minus the physical one DR^P . For the quadratic (linear) relationship, the difference $(DR^Q - DR^P)$ remains positive when DR^Q

is smaller than 23% (15%).⁵ Since the DR^Q of CCC-C rating is around 19% as shown in Exhibit 1, the quadratic regression maintains the property of positive default risk premiums for almost all credit rating classes while the linear regression fails for lower credit rating classes. In addition, the positive slope of the quadratic regression curve in Exhibit 1 also implies an inverse relationship between the conditional expected recovery rate and the risk-neutral default rate, which is analogous to the inverse relationship between the recovery rate and the physical default rates widely observed in past empirical studies.

We emphasize that the conditional expected recovery rate in Equation (3) is a feasible notion rather than a rigid equation. Since different lending policies in individual financial institutions can yield different relationships between recovery and default rates, financial institutions can take advantage of our concept to derive their own relationships, such as those in Equations (1) and (2), based on their latest internal data. Therefore, they can obtain for themselves a more appropriate and up-to-date conditional expected recovery rate equation. Furthermore, when using our model, financial institutions can estimate the relationships between the recovery and default rates for different industries according to their private data to enhance the accuracy performance. In addition, the relationship between physical and risk-neutral default rates (e.g., Equation (2)) can be derived by classifying the creditworthiness of borrowers in other ways. For example, financial institutions can classify borrowers according to internal credit scores or the information on the distance to default, which is argued by Berndt et al. [2011] to be a more effective measurement for classifying borrowers.⁶ However, we adopt the credit rating classification of Hull, Predescu, and White [2005] instead of internal credit scores or distances to default in order to classify borrowers, since the former data are public and easily obtained, while the latter require detailed and even unpublicized borrower financial data for inference.

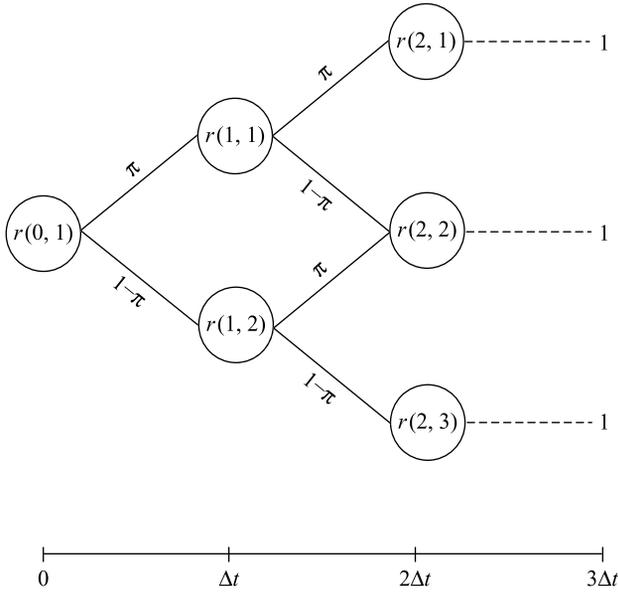
Revising the JT Model

To demonstrate our core idea, we revise the JT reduced-form model by incorporating the conditional expected recovery rate in Equation (3) into their model. To achieve this goal efficiently, this article is the first to develop recursive formulae to express the prices

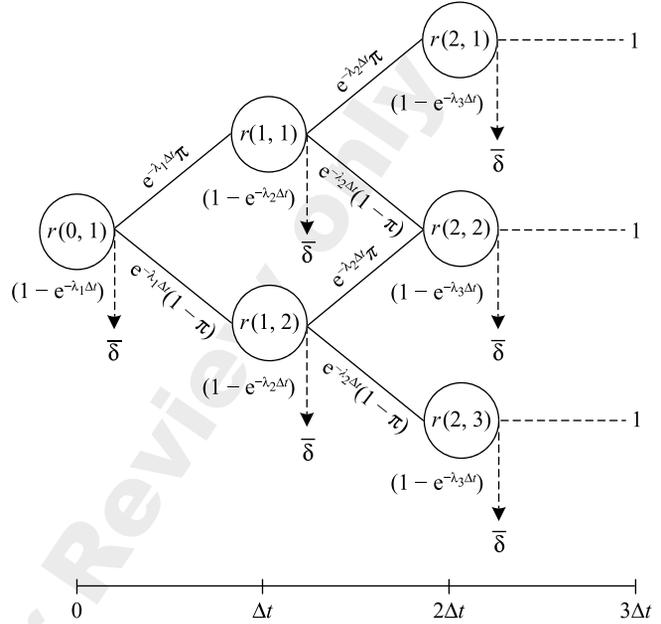
EXHIBIT 2

Illustration of the JT Model

Panel A: Three-Time-Period Default-Free Binomial Interest Rate Tree



Panel B: Three-Time-Period Defaultable Binomial Interest Rate Tree



Notes: Panels A and B illustrate the default-free and defaultable binomial interest rate trees of the JT model, respectively, in a three-time-period scenario. The notation $r(i, j)$ represents the interest rate of the j -th node (counted from the uppermost position) at time $i\Delta t$. The branching probabilities to move upward or downward are listed next to the branches. Each downward-pointing arrow in Panel B denotes a default event occurring in the i -th time period with probability $(1 - e^{-\lambda_i \Delta t})$, where λ_i denotes the default intensity. The notation $\bar{\delta}$ represents the constant recovery rate in the event of default. The face values of both default-free and defaultable bonds are normalized at one dollar.

of the riskless and risky zero-coupon bonds matured in different time periods in the tree-based JT model. We then propose a bootstrap method based on these recursive formulae to derive the risk-neutral default intensity and the corresponding conditional expected recovery rate for each time period of the JT model.

JT reduced-form model with a constant recovery rate. The main idea in the JT model is to derive default rates by applying observable credit spreads and an exogenous constant recovery on a no-arbitrage binomial interest rate tree, say the BDT model. Panels A and B in Exhibit 2 illustrate the three-time-period default-free and defaultable binomial interest rate trees, respectively, that begin at time zero with the length of each time period to be Δt . For the default-free interest rate tree, the interest rate $r(i, j)$ located at the j -th position at time $i\Delta t$ moves either upward to $r(i + 1, j)$ or downward to $r(i + 1, j + 1)$ with probabilities π and $1 - \pi$, respectively. For the defaultable interest rate tree,

the default event occurring in the i -th time period, i.e., in time interval $((i - 1)\Delta t, i\Delta t]$, is represented by vertical downward arrows with the probability $1 - e^{-\lambda_i \Delta t}$, where λ_i is the default intensity under the risk-neutral measure Q in that time period. For simplicity, the superscript Q is removed from λ_i since the following discussions are all under the risk-neutral probability measure. The probabilities that the prior-to-default interest rate move upward and downward are $e^{-\lambda_i \Delta t} \pi$ and $e^{-\lambda_i \Delta t} (1 - \pi)$, respectively.⁷ Once a default occurs in the i -th time period, the bondholders can recover δ_i (i.e., the recovery rate in the i -th time period) portion of the face value at time $i\Delta t$.⁸ The recovery rate δ_i is assumed to be a constant $\bar{\delta}$ for all time periods in the JT model. They employ the market prices of riskless and risky zero-coupon bonds with different times to maturity (denoted as $P(T)$ and $V(T)$, respectively) to calibrate both the interest rate and the default intensity for each time period of the tree.

Calibrating the BDT tree to match $P(T)$. The BDT model simulates the following lognormal interest rate process with the tree structure in Panel A of Exhibit 2:

$$d\ln r_t = \theta(t)dt + \sigma_r dW_t, \quad (4)$$

where σ_r is a constant volatility for the interest rate process, dW_t is a standard Wiener process under the risk-neutral measure, and $\theta(t)$ is a function of time to match the prevailing term structure of interest rates.

When constructing the binomial interest rate tree, the BDT assumes that the branching probability π is fixed at 0.5 and all interest rates $r(i, j)$ at the same time $i\Delta t$ satisfy

$$r(i, j) = r(i, 1)e^{-(j-1)2\sigma_r\sqrt{\Delta t}}, \text{ for } 1 \leq j \leq i+1, \quad (5)$$

in order to match the volatility term σ_r . The interest rate for each node defined in Equation (5) can then be calibrated to match the present market prices of riskless zero-coupon bonds with different times to maturity. To systematically express the prices for riskless long-term zero-coupon bonds, we introduce the notation $C(i, j)$ to represent the expected present value of one dollar received at node (i, j) . By definition, $C(0, 1) = 1$ for the root node, and $C(i, j)$ (for $i = 1, 2, \dots$, and $j = 1, \dots, i+1$) can be iteratively defined by the following equations:

$$C(i, j) = \begin{cases} C(i-1, j)\pi e^{-r(i-1, j)\Delta t} & \text{if } j = 1 \\ C(i-1, j)\pi e^{-r(i-1, j)\Delta t} \\ + C(i-1, j-1)(1-\pi)e^{-r(i-1, j-1)\Delta t} & \text{if } 1 < j < i+1. \\ C(i-1, j-1)(1-\pi)e^{-r(i-1, j-1)\Delta t} & \text{if } j = i+1 \end{cases} \quad (6)$$

Consequently, the market price of the riskless zero-coupon bond that matures at time $(i+1)\Delta t$ can be represented in terms of $C(i, j)$ by the following equation:

$$P((i+1)\Delta t) = \sum_{j=1}^{i+1} C(i, j)e^{-r(i, j)\Delta t}. \quad (7)$$

Equations (6) and (7) can be used alternately to calibrate the interest rate at each node of the tree. Specifically, one can employ the interest rates calculated at time $(i-1)\Delta t$ (i.e., $r(i-1, j)$) to determine $C(i, j)$ with Equation (6). The interest rates at time $i\Delta t$

(i.e., $r(i, j)$) can then be solved by substituting the determined $C(i, j)$ and the prevailing market price $P((i+1)\Delta t)$ into Equation (7).

Calibrating the defaultable tree to match $V(T)$. The default intensities can be calibrated to match the market prices of a series of risky zero-coupon bonds with different times to maturity. In order to simplify the equations for the prices for risky long-term zero-coupon bonds, we define K_i and SV_i as the accumulated expected present value of the recoveries up to time $i\Delta t$ and the survival rate at time $i\Delta t$, respectively. The variables K_0 and SV_0 are set to 0 and 1, respectively, since the firm is assumed to be solvent at time 0. Hence, K_{i+1} and SV_{i+1} can be expressed in terms of K_i and SV_i using the following recursive equations:

$$\begin{cases} SV_{i+1} = SV_i e^{-\lambda_{i+1}\Delta t} \\ K_{i+1} = K_i + SV_i \cdot P((i+1)\Delta t) \cdot (1 - e^{-\lambda_{i+1}\Delta t}) \bar{\delta} \end{cases} \quad (8)$$

Finally, we derive the following equation to express the value of the risky zero-coupon bond that matures at time $(i+1)\Delta t$:

$$V((i+1)\Delta t) = K_i + SV_i \cdot P((i+1)\Delta t) \cdot [(1 - e^{-\lambda_{i+1}\Delta t}) \bar{\delta} + e^{-\lambda_{i+1}\Delta t}]. \quad (9)$$

We therefore propose a bootstrap method to calibrate the default intensity for each time period by using Equations (8) and (9) alternately. Specifically, one can employ the known information of K_i and SV_i up to time $i\Delta t$ and the market values of $V((i+1)\Delta t)$ and $P((i+1)\Delta t)$ to calibrate λ_{i+1} using Equation (9). Once the default intensity λ_{i+1} is solved, K_{i+1} and SV_{i+1} can be derived by evaluating Equation (8).

Replacing the constant recovery rate with the conditional expected recovery rate. The conditional expected recovery rate $\hat{\delta}$ proposed in Equation (3) can be incorporated into any reduced-form model, says the aforementioned JT model, by substituting $\hat{\delta}$ for the constant recovery rate $\bar{\delta}$ in Equation (9). The default intensity λ_{i+1} is then calibrated to satisfy

$$V((i+1)\Delta t) = K_i + SV_i \cdot P((i+1)\Delta t) \cdot [(1 - e^{-\lambda_{i+1}\Delta t}) \hat{\delta}_{i+1} + e^{-\lambda_{i+1}\Delta t}], \quad (10)$$

where $\hat{\delta}_{i+1} = a + b[\alpha + \beta \ln(1 - e^{-\lambda_{i+1}}) + \gamma(\ln(1 - e^{-\lambda_{i+1}}))^2]$ denotes the conditional expected recovery rate for the $(i + 1)$ -th time period derived in Equation (3). By iteratively calibrating the default intensity with Equation (10) under the constraint that $0 \leq \hat{\delta}_{i+1} \leq 1$, the risk-neutral default intensity (λ_i) and the corresponding conditional expected recovery rate ($\hat{\delta}_i$) for each time period can be endogenously solved. Note that the constant recovery rate $\bar{\delta}$ in Equation (8) should be replaced by $\hat{\delta}_{i+1}$ during calibration.

Empirical Tests for Our Modified Reduced-Form Model

To demonstrate the performance of our revised JT model, we study the recovery and default rates implied from the zero rates of AA, A, BBB, BB, and B ratings (collected from Bloomberg) on January 22 in 2010, 2011, 2012, 2013, and 2014.⁹ Our experiments focus only on these five credit ratings because the zero rates for other credit ratings are not available from Bloomberg or other databases that we can access. To avoid the missing data problem for longer times to maturity, we limit the longest time to maturity to 15 years. We choose January 22 because we focus on pricing CBs on these dates in the empirical study in the fourth section. To estimate the volatility of the risk-free interest rate process in Equation (4), we collect from the website of the U.S. Department of the Treasury the daily data on the six-month¹⁰ Treasury yield over the three-year period prior to each examined date. Exhibit 3 reports the default rates and the conditional expected recovery rates according to the revised JT model on these dates. In addition, the default rates of the traditional JT model with constant recovery rates are presented for comparison. Hamilton et al. [2007] calculate the long-term average recovery rates for different credit ratings based on the actual default events in the period between 1982 and 2006. The constant recovery rates for AA, A, BBB, BB, and B ratings are retrieved from their Exhibit 20, with values of 57.04%, 49.54%, 45.90%, 42.48%, and 38.34%, respectively. The traditional and revised JT tree models are constructed with each time period to be one year. The reported default rates are the conditional default rates in each year, given that default events did not occur in prior years, and the reported conditional expected recovery rate ($\hat{\delta}_i$) is the recovery rate for default events during that year.

The results in Exhibit 3 verify that our revised JT model endogenously determines recovery and default rates reasonably by calibrating the prevailing zero-rate curves. In addition, empirically verified phenomena that fail to be produced by the traditional JT model, such as significant changes in recovery rates over time and the inverse relationship between recovery and default rates, are reasonably constructed in our revised model. For example, the conditional expected recovery rates for an AA rating on January 22, 2014, range from 77.49% to 41.24%, reflecting the dynamics of the prevailing term structures of zero rates. The levels of the default rates essentially react inversely with the magnitudes of the default rates. The highest recovery rate (in the first year) is accompanied by the lowest default rate, 1.07%, while the lowest recovery rate (in the 15th year) is accompanied by the highest default rate, 5.54%. Similar phenomena occur for all other ratings over these five years. In addition, by examining the means of the recovery and default rates over these five different ratings, our revised JT model generates higher (lower) default rates and lower (higher) recovery rates for lower (higher) credit rating reference entities, which is also consistent with the empirical evidence in the Moody's Investors Service report (Hamilton et al. [2007]).

While the constant recovery rates $\bar{\delta}$ retrieved by Hamilton et al. [2007] reflect the historical average of 1982–2006, the conditional expected recovery rates $\hat{\delta}_i$ are endogenously determined by calibrating the prevailing credit spreads and reflect the consensus view for future recovery rates. Therefore, in Exhibit 3, the average of $\hat{\delta}_i$ could be close to or different from $\bar{\delta}$. For example, on January 22, 2014, the averages of $\hat{\delta}_i$ derived under the revised JT model for AA, A, and BBB ratings (58.33%, 50.95%, and 43.32%, respectively) are similar to those of $\bar{\delta}$ (57.04%, 49.54%, and 45.90%, respectively). The average default rates generated by the revised JT model (2.85%, 4.17%, and 6.03%) are also close to those generated under the traditional JT model (2.73%, 4.10%, and 6.89%) for AA, A, and BBB ratings, respectively. On the other hand, the averages of $\hat{\delta}_i$ for BB and B ratings (32.70% and 30.68%, respectively) are clearly lower than those of $\bar{\delta}$ (42.28% and 38.34%, respectively). Furthermore, for these two credit ratings, while the revised JT model still produces reasonable default rates (ranging from 1% to 34%), the traditional JT model obviously overestimates the default rates. Specifically, the maximum per annum default rates for

EXHIBIT 3

Recovery and Default Rates of the Revised JT Model

	AA Rating			A Rating			BBB Rating			BB Rating			B Rating		
	Revised JT Model		Traditional JT Model ($\bar{\delta} = 57.04\%$)	Revised JT Model		Traditional JT Model ($\bar{\delta} = 49.54\%$)	Revised JT Model		Traditional JT Model ($\bar{\delta} = 45.90\%$)	Revised JT Model		Traditional JT Model ($\bar{\delta} = 42.48\%$)	Revised JT Model		Traditional JT Model ($\bar{\delta} = 38.34\%$)
	$\hat{\delta}$ (%)	DR^Q (%)	DR^Q (%)	$\hat{\delta}$ (%)	DR^Q (%)	DR^Q (%)	$\hat{\delta}$ (%)	DR^Q (%)	DR^Q (%)	$\hat{\delta}$ (%)	DR^Q (%)	DR^Q (%)	$\hat{\delta}$ (%)	DR^Q (%)	DR^Q (%)
January 22, 2014															
Year 1	77.49	1.07	0.56	71.56	1.37	0.77	62.94	1.98	1.36	52.49	3.18	2.63	50.87	3.44	2.74
2	71.20	1.39	0.92	66.21	1.72	1.13	58.38	2.42	1.85	49.26	3.71	3.25	45.60	4.44	3.88
3	75.33	1.17	0.63	67.28	1.64	1.01	57.90	2.48	1.87	46.69	4.21	3.83	42.14	5.29	4.86
4	75.78	1.15	0.53	66.27	1.71	1.02	55.66	2.75	2.12	43.39	4.96	4.74	39.10	6.19	5.93
5	69.98	1.46	0.83	61.74	2.09	1.37	51.82	3.29	2.69	39.89	5.94	6.00	36.32	7.18	7.15
6	60.40	2.21	1.74	54.54	2.89	2.28	46.87	4.17	3.76	37.40	6.77	7.15	33.85	8.21	8.50
7	56.71	2.62	2.21	51.17	3.39	2.81	43.87	4.84	4.56	34.76	7.81	8.66	31.24	9.50	10.29
8	55.44	2.77	2.42	48.95	3.77	3.28	42.14	5.29	5.15	33.47	8.39	9.65	31.91	9.15	9.96
9	52.57	3.17	2.94	46.18	4.32	3.94	39.60	6.03	6.14	31.13	9.56	11.66	29.86	10.28	11.70
10	49.77	3.62	3.56	43.48	4.94	4.72	37.08	6.89	7.36	28.72	10.98	14.40	27.65	11.69	14.07
11	50.88	3.43	3.33	42.52	5.19	5.11	35.95	7.32	8.12	25.26	13.48	19.44	24.68	13.97	18.14
12	48.71	3.81	3.89	40.23	5.84	6.03	33.70	8.28	9.69	22.65	15.83	25.86	22.18	16.31	23.32
13	45.91	4.37	4.78	37.13	6.87	7.58	30.53	9.89	12.50	18.61	20.52	40.80	18.43	20.77	34.50
14	43.60	4.91	5.72	34.72	7.83	9.18	28.03	11.43	15.70	15.31	25.66	73.44	15.14	25.97	55.57
15	41.24	5.54	6.95	32.25	8.97	11.32	25.40	13.37	20.48	11.44	33.87	100.00	11.21	34.42	100.00
mean	58.33	2.85	2.73	50.95	4.17	4.10	43.32	6.03	6.89	32.70	11.66	22.10	30.68	12.45	20.71
median	55.44	2.77	2.42	48.95	3.77	3.28	42.14	5.29	5.15	33.47	8.39	9.65	31.24	9.50	10.29
max	77.49	5.54	6.95	71.56	8.97	11.32	62.94	13.37	20.48	52.49	33.87	100.00	50.87	34.42	100.00
min	41.24	1.07	0.53	32.25	1.37	0.77	25.40	1.98	1.36	11.44	3.18	2.63	11.21	3.44	2.74
January 22, 2013															
mean	55.33	3.08	3.37	48.75	4.57	4.98	42.95	5.80	6.61	33.74	9.93	17.13	30.55	11.81	20.44
median	55.49	2.77	2.65	50.38	3.52	3.14	44.03	4.80	4.74	36.33	7.17	8.11	31.26	9.49	10.72
max	76.67	5.81	8.45	70.92	10.34	14.91	62.24	12.10	18.16	51.08	24.82	81.00	51.27	28.81	100.00
min	40.31	1.11	0.60	29.77	1.40	0.81	27.07	2.04	1.42	15.79	3.40	2.89	13.67	3.37	2.66
January 23, 2012															
mean	52.87	3.58	4.44	46.83	4.94	5.72	39.40	6.88	9.26	31.73	10.07	21.86	24.71	19.99	33.96
median	55.28	2.79	2.74	48.87	3.78	3.54	40.72	5.69	6.19	33.26	8.48	11.27	29.25	10.65	13.38
max	74.54	7.49	13.17	67.24	11.12	17.41	58.53	14.30	28.42	48.97	19.56	100.00	37.89	81.59	100.00
min	35.52	1.21	0.72	28.50	1.64	1.07	24.30	2.41	1.85	19.34	3.77	3.34	0.81	6.60	6.65
January 24, 2011															
mean	51.90	3.73	4.77	46.01	5.17	6.03	40.97	6.41	7.83	33.58	9.37	16.80	29.86	12.36	21.62
median	52.08	3.25	3.33	47.34	4.08	3.83	41.97	5.33	5.40	34.82	7.78	9.30	34.06	8.12	9.11
max	70.47	7.73	14.56	66.14	11.90	19.23	56.31	13.98	24.03	49.47	20.46	74.08	39.98	34.41	100.00
min	34.95	1.43	0.98	27.35	1.72	1.15	24.67	2.66	2.15	18.66	3.68	3.23	11.22	5.91	5.84
January 22, 2010															
mean	50.56	4.03	5.65	45.86	5.19	6.07	40.63	6.46	8.04	62.46	4.17	6.90	31.54	9.90	17.22
median	49.44	3.68	4.06	45.32	4.50	4.39	39.79	5.97	6.31	52.57	3.17	6.23	30.81	9.74	11.84
max	70.98	8.31	17.79	63.49	11.46	18.53	57.08	13.09	23.10	100.00	15.54	21.58	41.04	16.94	61.44
min	33.63	1.40	0.94	27.99	1.93	1.40	25.75	2.57	2.04	22.95	0.45	0.00	21.57	5.60	5.35

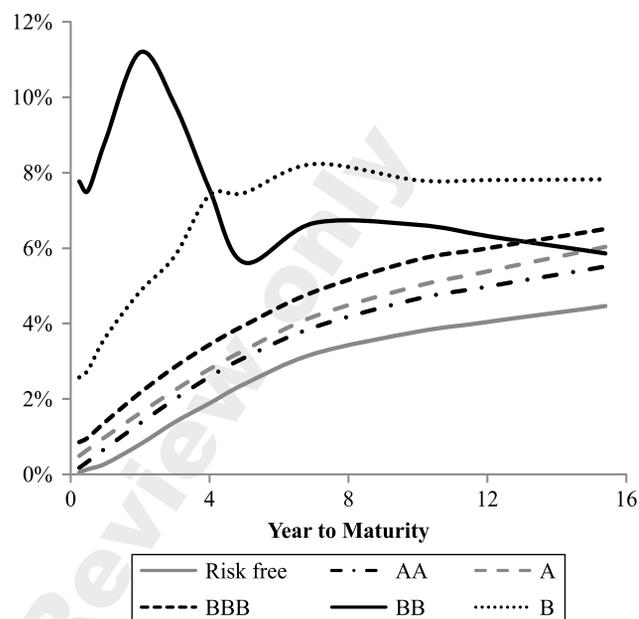
Notes: This exhibit reports the risk-neutral default rates (DR^Q) and the corresponding conditional expected recovery rate ($\hat{\delta}$) of the revised JT model for different ratings on January 22 of 2010, 2011, 2012, 2013, and 2014. The risk-neutral default rates of the traditional JT model with a constant recovery rate $\bar{\delta}$ are also reported for comparison. The results for January 22, 2014, are fully presented for illustration. For all the other examined dates, we report only the means, medians, maxima, and minima of the recovery and default rates.

BB– or B-rated firms are more than 74% (often reaching 100%) for all examined dates, except January 22, 2010. These results imply that a BB– or B-rated firm is almost certainly going to default within 15 years. These abnormally high default rates render the traditional JT model impractical to estimate long-term, say 15-year, survival rates for BB and B ratings. Finally, we observe that both the revised and traditional JT models generate relatively overestimated recovery and underestimated default rates for the BB rating on January 22, 2010. We attribute those unreasonable phenomena to the possibly problematic BB zero-rate curve on that day. In Exhibit 4, it can be observed that the BB zero-rate curve reaches too high for times to maturity shorter than 4 years (even higher than that of the B rating) and is abnormally low for times to maturity longer than 12 years (even lower than those of the BBB and A ratings). Moreover, it also exhibits a steep hump shape for times to maturity shorter than 4 years. The combination of these unusual characteristics yields anomalous results for the BB rating on that day.

To highlight the overestimation problem of default rates pertaining to the traditional JT model with a constant recovery rate, Exhibit 5 reports the 15-year survival rates of the revised and traditional JT models. For comparison, we also present the 15-year survival rates based on the most basic reduced-form model, which simply fits the credit spread with the product of the default and loss rates. Hereafter, we call this type of reduced-form model a credit spread model.¹¹ Given an estimated constant recovery rate ($\bar{\delta}$), the traditional credit spread model calibrates the risk-neutral default rate (DR^Q) by matching the observed credit spread (CS) through $DR^Q = CS/(1 - \bar{\delta})$. In addition, we modify the credit spread model by replacing the constant recovery rate $\bar{\delta}$ with the conditional expected recovery rate $\hat{\delta}$ in Equation (3). The 15-year survival rate for each model is computed as the product of per-period conditional survival rates, which equals one minus the per-period conditional default rates derived by that model.

In Exhibit 5, we find first that for AA, A, and BBB ratings, the revised and traditional JT models generate similar 15-year survival rates. The revised and traditional credit spread models also generate similar survival rates that are significantly higher than those generated by the JT models. We attribute this phenomenon to the fact that the credit spread models do not account for the effect of stochastic interest rates.¹² Stochastic movements in the interest rates in the JT models introduce another

EXHIBIT 4
Zero Curves on January 22, 2010



Note: This exhibit reports the zero curves of all examined credit ratings and the risk-free interest rate on January 22, 2010.

source of risk, which should result in lower survival rates. Second, unlike the revised and traditional credit spread models, the revised and traditional JT models generate diverse survival rates for BB and B ratings. Compared with the results of the revised and traditional credit spread models and the revised JT models, the 15-year survival rates of the traditional JT model for BB and B ratings are clearly underestimated due to the abnormally high default rates observed in Exhibit 3.

PRICING CBS SUBJECT TO DEFAULT RISK

Defaultable Stock Price–Interest Rate Tree

This section proposes a defaultable bivariate tree that can price CBs by simultaneously simulating the evolution of the stock price and interest rate processes. Since the event of jumping to default is taken into account, the prior-to-default share price of the CB issuer, S_t , is assumed to follow a geometric Brownian motion under the risk-neutral probability measure as follows:¹³

$$\frac{dS_t}{S_t} = (r_t - q + \lambda_t)dt + \sigma_s dW_s, \quad (11)$$

EXHIBIT 5

15-Year Survival Rates of the JT and Credit Spread Models

15-Year Survival Rate (%)		January 22, 2014	January 22, 2013	January 23, 2012	January 24, 2011	January 22, 2010
AA	Revised JT model	64.74	62.43	57.75	56.41	53.76
	Traditional JT model ($\bar{\delta} = 57.04\%$)	65.79	59.56	50.09	47.48	41.03
	Revised credit spread model	75.68	71.77	68.15	71.68	68.74
	Traditional credit spread model ($\bar{\delta} = 57.04\%$)	78.21	73.49	67.83	73.34	68.89
A	Revised JT model	52.57	49.31	46.44	44.76	44.63
	Traditional JT model ($\bar{\delta} = 49.54\%$)	52.90	45.80	40.59	38.46	38.25
	Revised credit spread model	64.63	60.03	58.41	61.48	60.93
	Traditional credit spread model ($\bar{\delta} = 49.54\%$)	66.80	60.34	58.19	62.75	61.95
BBB	Revised JT model	38.96	40.52	34.04	36.68	36.44
	Traditional JT model ($\bar{\delta} = 45.90\%$)	33.37	35.16	22.16	28.43	27.57
	Revised credit spread model	53.48	53.05	48.48	55.38	54.79
	Traditional credit spread model ($\bar{\delta} = 45.90\%$)	53.67	53.35	47.22	56.81	56.00
BB	Revised JT model	14.40	20.11	20.07	22.45	51.95
	Traditional JT model ($\bar{\delta} = 42.48\%$)	0.00	2.55	0.00	3.50	33.29
	Revised credit spread model	32.97	35.92	36.77	43.81	52.43
	Traditional credit spread model ($\bar{\delta} = 42.48\%$)	28.36	32.50	34.00	43.37	46.54
B	Revised JT model	12.60	14.48	1.48	12.87	20.74
	Traditional JT model ($\bar{\delta} = 38.34\%$)	0.00	0.00	0.00	0.00	4.31
	Revised credit spread model	31.72	31.14	20.45	35.36	42.12
	Traditional credit spread model ($\bar{\delta} = 38.34\%$)	29.76	29.18	15.89	34.70	42.63

Notes: This exhibit shows the 15-year survival rates of the revised and traditional JT models. The survival rates of the traditional and revised credit spread models are also reported for comparison.

where r_t is the stochastic interest rate that follows the BDT model in Equation (4), q is the dividend yield, the default intensity λ_t in the drift term is introduced to ensure the discounted stock price is a martingale process, σ_s is the constant volatility, and dW_s is the standard Wiener process under the risk-neutral probability measure. Furthermore, we follow the CL model by assuming that the correlation between the Wiener processes dW_s in Equation (11) and dW_r in Equation (4) is a constant ρ .

Given the condition that the issuing firm does not default prior to time $t + \Delta t$, the CL model assumes each arbitrary stock price S_t either moves upward to become $S_t u$ ($u = e^{\sigma_s \sqrt{\Delta t}}$) with probability P_u or moves downward to $S_t d$ ($d = u^{-1}$) with probability $1 - P_u$ at the subsequent time point; the upward probability is assigned as

$$P_u = \frac{e^{(r_t - q + \lambda_t)\Delta t} - d}{u - d}. \quad (12)$$

Combining the defaultable interest rate tree illustrated in Panel B of Exhibit 2 with the aforementioned CRR binomial tree results in the defaultable pentanomial tree shown in Exhibit 6. Notations P_{uu} , P_{ud} , P_{du} , and P_{dd} represent the conditional joint probabilities for the combinations of upward and downward movements of S_t and r_t , given that the firm survives at time $t + \Delta t$. The CL model calibrates P_{uu} , P_{ud} , P_{du} , and P_{dd} to match the correlation ρ as follows:

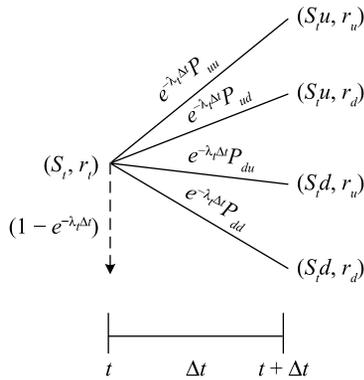
$$\begin{cases} P_{uu} = 0.5P_u - \xi = 0.5 \left[P_u + \rho \sqrt{P_u(1 - P_u)} \right] \\ P_{ud} = 0.5P_u + \xi = 0.5 \left[P_u - \rho \sqrt{P_u(1 - P_u)} \right] \\ P_{du} = 0.5P_d + \xi = 0.5 \left[(1 - P_u) - \rho \sqrt{P_u(1 - P_u)} \right] \\ P_{dd} = 0.5P_d - \xi = 0.5 \left[(1 - P_u) + \rho \sqrt{P_u(1 - P_u)} \right], \end{cases} \quad (13)$$

where $\xi = -0.5\rho\sqrt{P_u(1 - P_u)}$ is introduced to adjust the conditional joint probability to match ρ without

changing the marginal probability for both the stock price and interest rate processes.

To ensure that the values of these conditional joint probabilities are within the range of $[0, 1]$, the CL model proposes the mild condition that

EXHIBIT 6
Pentanomial Tree Structure Used When the Mild Condition Holds



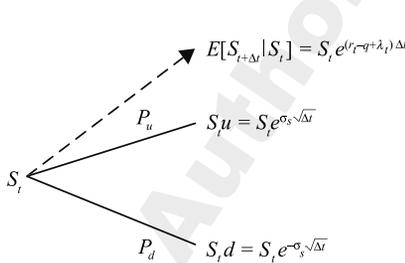
Notes: The pentanomial structure is employed to simulate the defaultable stock price and interest rate processes if the mild condition proposed by the CL model is satisfied. The branching probability from node (S_t, r_t) at time t to each node at time $(t + \Delta t)$ is listed next to the branch. The conditional joint probabilities P_{uu} , P_{ud} , P_{du} , and P_{dd} are determined using Equation (13). The branch plotted by the downward-pointing arrow denotes the default event.

$\rho^2/(1 + \rho^2) \leq P_u \leq 1/(1 + \rho^2)$. However, this condition could be violated in two ways. The first type of violation occurs when the upward stock price probability P_u is not within the range of $[0, 1]$ due to either a high interest rate r_t or a high default intensity λ_t . Thus P_{uu} , P_{ud} , P_{du} , and P_{dd} become imaginary numbers, because the term $\sqrt{P_u(1 - P_u)}$ is an imaginary number. The second type of violation occurs when the probability of P_u is valid but very close to the feasible boundaries 0 and 1. Thus, these conditional joint probabilities are not guaranteed to be within the range $[0, 1]$ after adding or subtracting the adjustment term $\xi = -0.5\rho\sqrt{P_u(1 - P_u)}$.

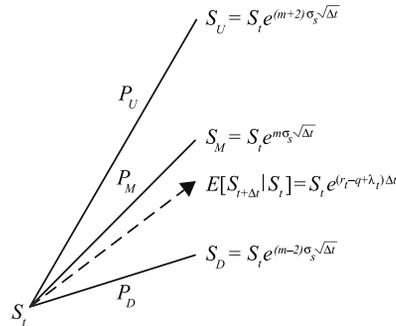
It is noteworthy that the first type of violation occurs frequently due to the stochastic nature of the interest rate r_t and the default intensity λ_t in Equation (12). Specifically, P_u exceeds unity when the conditional expected stock price at the next time point is too high, as illustrated in Exhibit 7, Panel A. Lyuu and Wang [2011] prove that this violation is not merely a discretization problem and cannot be completely resolved by picking a proper Δt . Instead, they suggest that a valid tree can be constructed by adjusting the outgoing branches for the tree nodes that have high r_t or λ_t . Our article borrows their idea to show that the first type of violation, illustrated in Exhibit 7, Panel A, can be avoided by adopting the trinomial tree structure (see Exhibit 7, Panel B) constructed according to

EXHIBIT 7
Employing the Mean-Tracking Method of Dai [2009] to Solve the Infeasible Branching Probability Problem

Panel A: Infeasible Probabilities in the CRR Binomial Tree



Panel B: Trinomial Tree Constructed by the Mean-Tracking Method



Notes: Panel A illustrates the infeasible probability problem under the CRR binomial tree structure, where P_u and P_d denote the branching probabilities for the stock price moving from S_t to S_u and S_d , respectively. Panel B illustrates the trinomial tree structure of the mean-tracking method proposed by Dai [2009], where P_U , P_M , and P_D denote the branching probabilities for the stock price moving from S_t to S_U , S_M , and S_D , respectively. The conditional mean $E[S_{t+\Delta t} | S_t]$ is illustrated by an arrow with a dashed line.

the mean-tracking method of Dai [2009]. This method first identifies one node located at the next time point in the grid of the CRR binomial tree that is nearest to the conditional expected stock price $E[S_{t+\Delta t} | S_t]$. For simplicity, we refer to this point as the “middle node.” Next, the trinomial tree structure is formed with the middle node and its two adjacent nodes. Specifically, suppose the stock price of the middle node S_M is set to $S_t e^{m\sigma_s \sqrt{\Delta t}}$. The stock prices of two adjacent nodes can then be expressed as $S_D = S_t e^{(m-2)\sigma_s \sqrt{\Delta t}}$ and $S_U = S_t e^{(m+2)\sigma_s \sqrt{\Delta t}}$. These three nodes are united to establish the trinomial tree structure in the mean-tracking method, which is illustrated in Exhibit 7, Panel B.

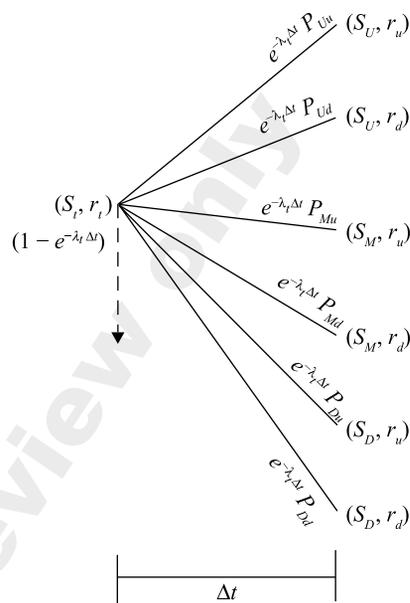
The branching probabilities P_U , P_M , and P_D are determined to match the mean and the variance of $\ln S_{t+\Delta t}$ conditional on $\ln S_t$, which are $E[\ln S_{t+\Delta t} | \ln S_t] = \ln S_t + (r_t - q - 0.5\sigma_s^2 + \lambda_t)\Delta t$ and $\text{Var}(\ln S_{t+\Delta t} | \ln S_t) = \sigma_s^2 \Delta t$, respectively. For convenience, let X , Y , and Z denote the logarithmic stock prices of the upper, middle, and lower nodes minus the conditional expected logarithmic stock price, that is, $X = \ln S_U - E[\ln S_{t+\Delta t} | \ln S_t] = Y + 2\sigma_s \sqrt{\Delta t}$, $Y = \ln S_M - E[\ln S_{t+\Delta t} | \ln S_t]$, and $Z = \ln S_D - E[\ln S_{t+\Delta t} | \ln S_t] = Y - 2\sigma_s \sqrt{\Delta t}$. Valid branching probabilities, P_U , P_M , and P_D , can be obtained by using Dai’s [2009] method as follows.

$$\begin{cases} P_U = (Y - \sigma_s \sqrt{\Delta t})^2 / (8\sigma_s^2 \Delta t) \\ P_M = (3\sigma_s^2 \Delta t - Y^2) / (4\sigma_s^2 \Delta t) \\ P_D = (Y + \sigma_s \sqrt{\Delta t})^2 / (8\sigma_s^2 \Delta t). \end{cases} \quad (14)$$

Combining the mean-tracking method illustrated in Panel B of Exhibit 7 with the defaultable interest rate tree in the JT model illustrated in Panel B of Exhibit 2 yields the heptanomial tree structure illustrated in Exhibit 8.

In Exhibit 9, Panel A, we follow Hull and White [1994] to introduce a probability adjustment term ϵ to adjust the conditional joint probability to match ρ , the correlation between the dW_s of the stock price process and the dW_r of the interest rate process, without changing the marginal probability for both the stock price and the interest rate processes. Since $\rho = \text{Corr}(dW_s, dW_r) = \text{Corr}(d \ln S_t, d \ln r_t)$, ρ can be approximated in the corresponding discrete-time model as

EXHIBIT 8 Heptanomial Tree Structure Used When the Mild Condition Does Not Hold



Notes: The heptanomial tree structure is employed to simulate the defaultable stock price and interest rate processes if the mild condition is violated. The branching probability for each branch is listed next to that branch. The conditional joint probabilities, P_{Uu} , P_{Ud} , P_{Mu} , P_{Md} , P_{Du} , and P_{Dd} , are determined based on Exhibit 9 and Equations (14) and (15). The branch plotted by a downward-pointing arrow denotes the default event.

$$\begin{aligned} \text{Corr}(\ln S_{t+\Delta t}, \ln r_{t+\Delta t}) &= \frac{E[\ln S_{t+\Delta t} \ln r_{t+\Delta t}] - E[\ln S_{t+\Delta t}] E[\ln r_{t+\Delta t}]}{\sqrt{\text{Var}(\ln S_{t+\Delta t})} \sqrt{\text{Var}(\ln r_{t+\Delta t})}}. \end{aligned}$$

Note that $E[\ln S_{t+\Delta t}]$, $\text{Var}(\ln S_{t+\Delta t})$, $E[\ln r_{t+\Delta t}]$, $\text{Var}(\ln r_{t+\Delta t})$, and $E[\ln S_{t+\Delta t} \ln r_{t+\Delta t}]$ can be expressed in terms of conditional joint probabilities and possible outcomes of the stock price and the interest rates illustrated in Exhibit 9, Panel A as follows:

$$\begin{aligned} E[\ln S_{t+\Delta t}] &= P_U \ln S_U + P_M \ln S_M + P_D \ln S_D \\ &= \ln S_t + (r_t - q - 0.5\sigma_s^2 + \lambda_t)\Delta t, \\ \text{Var}(\ln S_{t+\Delta t}) &= P_U (\ln S_U)^2 + P_M (\ln S_M)^2 \\ &\quad + P_D (\ln S_D)^2 - E[\ln S_{t+\Delta t}]^2 = \sigma_s^2 \Delta t, \\ E[\ln r_{t+\Delta t}] &= \pi \ln r_u + (1 - \pi) \ln r_d, \\ \text{Var}(\ln r_{t+\Delta t}) &= \pi (\ln r_u)^2 + (1 - \pi) (\ln r_d)^2 - E[\ln r_{t+\Delta t}]^2 \\ &= \sigma_r^2 \Delta t, \end{aligned}$$

and

$$\begin{aligned}
& E[\ln S_{t+\Delta t} \ln r_{t+\Delta t}] \\
&= (\pi P_U - \varepsilon) \ln S_U \ln r_u + [(1 - \pi)P_U + \varepsilon] \ln S_U \ln r_d \\
&\quad + (\pi P_M) \ln S_M \ln r_u + [(1 - \pi)P_M] \ln S_M \ln r_d \\
&\quad + (\pi P_D + \varepsilon) \ln S_D \ln r_u + [(1 - \pi)P_D - \varepsilon] \ln S_D \ln r_d \\
&= (\pi P_U) \ln S_U \ln r_u + (1 - \pi)P_U \ln S_U \ln r_d \\
&\quad + (\pi P_M) \ln S_M \ln r_u + (1 - \pi)P_M \ln S_M \ln r_d \\
&\quad + (\pi P_D) \ln S_D \ln r_u + (1 - \pi)P_D \ln S_D \ln r_d \\
&\quad + \varepsilon(-\ln S_U \ln r_u + \ln S_U \ln r_d + \ln S_D \ln r_u - \ln S_D \ln r_d) \\
&\equiv \Lambda + \varepsilon \Gamma.
\end{aligned}$$

Thus ε can be solved as

$$\varepsilon = \frac{\rho \sqrt{\text{Var}(\ln S_{t+\Delta t})} \sqrt{\text{Var}(\ln r_{t+\Delta t})} + E[\ln S_{t+\Delta t}] E[\ln r_{t+\Delta t}] - \Lambda}{\Gamma}. \quad (15)$$

Note that using this procedure to determine the conditional joint probabilities could still result in the second type of violation of the mild condition. If the probabilities P_U and P_D obtained in Equation (14) are close enough to the boundary 1 (or 0), the conditional joint probabilities P_{Uu} , P_{Ud} , P_{Du} , or P_{Dd} could become invalid after the addition or subtraction of the adjustment term ε . To solve this problem, we further exploit the flexibility of the mean-tracking method's trinomial tree structure. Under this trinomial tree structure, it is not necessary to adjust the four-corner conditional joint probabilities— P_{Uu} , P_{Ud} , P_{Du} , or P_{Dd} —as illustrated in Exhibit 9, Panel A. Panels B and C of Exhibit 9 present two alternative arrangements of the adjustment term ε for calibrating the correlation ρ if the conditional joint probabilities cannot be feasibly solved with the arrangement in Exhibit 9, Panel A. The arrangement in Exhibit 9, Panel B is adopted for correlation calibration when P_U is sufficiently close to 0 or 1 such that P_{Uu} and P_{Ud} become infeasible after the addition or subtraction of ε . Under this arrangement, ε can be solved using Equation (15) by redefining Γ as $\Gamma \equiv -\ln S_M \ln r_u + \ln S_M \ln r_d + \ln S_D \ln r_u - \ln S_D \ln r_d$. Similarly, the arrangement in Exhibit 9, Panel C is adopted when P_D is sufficiently close to 0 or 1 such that P_{Du} and P_{Dd} become infeasible after applying the adjustment term ε . Under this arrangement, ε can again be solved using Equation (15) by redefining

EXHIBIT 9

Calibrating the Conditional Joint Probabilities of the Stock Price–Interest Rate Tree for ρ between $d \ln S_t$ and $d \ln r_t$ When the Mean-Tracking Method Is Applied

$S_{t+\Delta t} r_{t+\Delta t}$	r_u	r_d	Marginal Prob. for $S_{t+\Delta t}$
Panel A: Normal Condition			
S_U	$P_{Uu} = \pi P_U - \varepsilon$	$P_{Ud} = (1 - \pi)P_U + \varepsilon$	P_U
S_M	$P_{Mu} = \pi P_M$	$P_{Md} = (1 - \pi)P_M$	P_M
S_D	$P_{Du} = \pi P_D + \varepsilon$	$P_{Dd} = (1 - \pi)P_D - \varepsilon$	P_D
Marginal Prob. for $r_{t+\Delta t}$	π	$(1 - \pi)$	1
Panel B: If P_{Uu} or P_{Ud} is Outside the Range of [0, 1]			
S_U	$P_{Uu} = \pi P_U$	$P_{Ud} = (1 - \pi)P_U$	P_U
S_M	$P_{Mu} = \pi P_M - \varepsilon$	$P_{Md} = (1 - \pi)P_M + \varepsilon$	P_M
S_D	$P_{Du} = \pi P_D + \varepsilon$	$P_{Dd} = (1 - \pi)P_D - \varepsilon$	P_D
Marginal Prob. for $r_{t+\Delta t}$	π	$(1 - \pi)$	1
Panel C: If P_{Du} or P_{Dd} is Outside the Range of [0, 1]			
S_U	$P_{Uu} = \pi P_U - \varepsilon$	$P_{Ud} = (1 - \pi)P_U + \varepsilon$	P_U
S_M	$P_{Mu} = \pi P_M + \varepsilon$	$P_{Md} = (1 - \pi)P_M - \varepsilon$	P_M
S_D	$P_{Du} = \pi P_D$	$P_{Dd} = (1 - \pi)P_D$	P_D
Marginal Prob. for $r_{t+\Delta t}$	π	$(1 - \pi)$	1

Notes: This exhibit shows the marginal and joint probabilities of the stock price and interest rate processes conditional on the issuing firm not defaulting in the time period that immediately follows. The interest rate movements, r_u and r_d , and the corresponding marginal probabilities, π and $1 - \pi$, are determined based on the BDT tree. The mean-tracking method determines the stock price movements S_U , S_M , and S_D , and the corresponding marginal probabilities P_U , P_M , and P_D . In normal conditions, the probability adjustment term ε is introduced to calibrate the conditional joint probabilities to match the correlation ρ in Panel A. If either P_{Uu} or P_{Ud} solved by matching the correlation ρ in Panel A is infeasible, the arrangement in Panel B is used instead to calibrate the conditional joint probabilities. Similarly, the arrangement in Panel C is used to calibrate the correlation ρ if either P_{Du} or P_{Dd} solved based on Panel A is infeasible.

Γ as $\Gamma \equiv -\ln S_U \ln r_u + \ln S_U \ln r_d + \ln S_M \ln r_u - \ln S_M \ln r_d$. We find that the performance of these new arrangements is satisfactory and no infeasible probability problem is observed in any of our numerical experiments in the fourth section.

It is worth noting that the mild condition is satisfied for most nodes in our bivariate tree. For these nodes, the pentanomial tree structure illustrated in Exhibit 6 is adopted and Equations (12) and (13) are employed to determine the branching probabilities. If the mild condition is violated, the heptanomial tree structure

illustrated in Exhibit 8 is used instead. Exhibit 9 and Equations (14) and (15) are employed to determine the branching probabilities. This arrangement maintains both the computational efficiency and the validity of the proposed bivariate tree.

Pricing CBs with the Proposed Bivariate Tree

This subsection introduces the algorithm for pricing CBs based on the bivariate tree discussed in the previous subsection. For simplicity, this subsection considers a callable CB without coupon payments. Extending our model to price CBs with coupon payments is straightforward. The face value of the CB is denoted as F , the conversion ratio is denoted as θ , and the call price is represented by CP . Finally, the time span from the current time point to CB's maturity is T , which is partitioned into N equal-length time periods; the length of a time period Δt is T/N .

Our model involves four phases: 1) Constructing a BDT tree to match the market prices of riskless zero-coupon bonds with different maturities (discussed in a subsection of the second section on calibrating the BDT tree to match $P(T)$); 2) calibrating the default intensities and the recovery rates to match the market prices of risky zero-coupon bonds for different maturities with the conditional expected recovery rate equation (see Equation (3)) and the revised JT model (see Equation (10)) (discussed in a subsection of the second section on calibrating the defaultable tree to match $V(T)$); 3) establishing a bivariate tree based on the pentanomial tree structure in Exhibit 6 or the heptanomial tree structure in Exhibit 8 (discussed in the previous subsection); and 4) pricing the CB with the backward induction procedure, which is presented in this subsection.

During the backward induction procedure, each node of our bivariate tree model should carry four pieces of information. The first is the stock price, S_t , and the second is the interest rate, r_t , applied in the time period that immediately follows. The third piece of information is the continuation value (CV), which is the CB value obtained from the backward induction procedure. The last piece of information is the CB value of this node, determined by evaluating $\max(\min(CV, CP)$ and θS_t). Note that the issuer can redeem CBs at the prespecified call price CP if the continuation value CV exceeds CP . In other words, the issuer's call option reduces the CB value to $\min(CV, CP)$. On the other hand, CB holders

can convert their CBs into θ shares of stock even if the issuing firm activates the call back plan. Thus, the CB value cannot be less than its conversion value, which is equal to the product of the conversion ratio (θ) and the prevailing stock price (S_t). Consequently, the CB value is determined by the maximum of $\min(CV, CP)$ and θS_t .

A prerequisite step in the backward induction procedure is determining CB values for all nodes at maturity. For simplicity, these nodes are called "terminal nodes." Since this zero-coupon CB will be redeemed for face value F if it is not converted (by bond holders) or called back (by the issuer), the continuation value for each terminal node is equal to the face value F . Thus, the CB value for each terminal node can be evaluated by $\max(\min(F, CP), \theta S_T)$. Moreover, in practice, the call price (CP) is never lower than the face value at maturity. Consequently, the CB values for all terminal nodes can be expressed as $\max(F, \theta S_T)$ to reflect the possible conversion.

There are two different types of the backward induction procedures in our CB pricing model. The first type is for the final time period, while the second type is for all other time periods. For nodes at time $(T - \Delta t)$, it is not necessary to consider the evolution of the interest rate for terminal nodes because the recorded interest rate $r_{T-\Delta t}$ serves as the discount rate for the final time period. As a result, only the stock price movements in the final time period need to be modeled. If the conditional probability P_u defined in Equation (12) can be feasibly solved, the defaultable CRR binomial tree model is used, and the continuation value can be evaluated by the following backward induction formula:

$$\begin{aligned} CV(S_{T-\Delta t}, r_{T-\Delta t}) &= e^{-r_{T-\Delta t}\Delta t} [(1 - e^{-\lambda_N\Delta t})F\hat{\delta}_N + e^{-\lambda_N\Delta t}(P_uCB_u + P_dCB_d)], \end{aligned} \quad (16)$$

where λ_N and $\hat{\delta}_N$ denote the default intensity and conditional expected recovery rate for the N -th time period (i.e., the final time period). Otherwise, we employ the trinomial-tree-based mean-tracking method (discussed in Exhibit 7, Panel B) with an extra default branch to guarantee that the branching probabilities are feasible. The formula for computing the continuation value is

$$\begin{aligned} CV(S_{T-\Delta t}, r_{T-\Delta t}) &= e^{-r_{T-\Delta t}\Delta t} [(1 - e^{-\lambda_N\Delta t})F\hat{\delta}_N + e^{-\lambda_N\Delta t}(P_U CB_U \\ &\quad + P_M CB_M + P_D CB_D)]. \end{aligned} \quad (17)$$

Once the continuation value of each node at time $T - \Delta t$ is derived, the corresponding CB value for each node can be evaluated as $\max(\min(CV, CP), \theta S_t)$.

For any time period i other than the final one, if the mild condition proposed in the CL model is satisfied, the continuation value for each node at time t is computed as

$$\begin{aligned} CV(S_t, r_t) &= e^{-r_t \Delta t} [(1 - e^{-\lambda_i \Delta t}) F \hat{\delta}_i + e^{-\lambda_i \Delta t} (P_{uu} CB_{uu} \\ &+ P_{ud} CB_{ud} + P_{du} CB_{du} + P_{dd} CB_{dd})]. \end{aligned} \quad (18)$$

Otherwise, if the mild condition is violated, the continuation value is found by using

$$\begin{aligned} CV(S_t, r_t) &= e^{-r_t \Delta t} [(1 - e^{-\lambda_i \Delta t}) F \hat{\delta}_i + e^{-\lambda_i \Delta t} (P_{Uu} CB_{Uu} + P_{Ud} CB_{Ud} \\ &+ P_{Mu} CB_{Mu} + P_{Md} CB_{Md} + P_{Du} CB_{Du} + P_{Dd} CB_{Dd})]. \end{aligned} \quad (19)$$

Finally, the corresponding CB value can be determined by evaluating $\max(\min(CV, CP), \theta S_t)$ after the continuation value (CV) is obtained.

The aforementioned backward induction procedure is then repeated from the final time period to the first one. The pricing result of CB can be obtained at the root node of the proposed tree.

Illustrative Example for Pricing CBs

This subsection presents a hypothetical CB example to illustrate our CB pricing model. The three-year CB example adopted in the CL model is used in this article. For the purpose of the example, the number of time periods, N , is three, and thus the length of a time period Δt is one year. All parameter values are taken from the CL model other than the stock price volatility σ_s , which changes from 20% to 19%. By slightly lowering the stock price volatility, the conditional expectation $E[S_{t+\Delta t} | S_t]$ of some nodes exceed $S_u = S_t e^{\sigma_s \sqrt{\Delta t}}$, violating the mild condition. The values of all other parameters are summarized as follows. The initial stock price S_0 is 30, and the dividend payout rate q is assumed to be zero. The current term structures of riskless and risky zero rates are flat at 10% and 15%, respectively. The interest rate volatility σ_r is 10%, and the correlation between the interest rate and stock price processes, ρ , is -0.1 . This hypothetical zero-coupon CB has a face value F of 100 and a conversion ratio θ of three. In addition, this CB

EXHIBIT 10

Three-Period CB Pricing Example

Panel A: Three-Period BDT Interest Rate Tree

$r(i, j)$	$i = 0$	$i = 1$	$i = 2$
$j = 1$	$r(0, 1) = 10.00\%$	$r(1, 1) = 11.00\%$	$r(2, 1) = 12.12\%$
$j = 2$		$r(1, 2) = 9.01\%$	$r(2, 2) = 9.92\%$
$j = 3$			$r(2, 3) = 8.12\%$

Panel B: Calibration Results Based on the Conditional Expected Recovery Rate

	1st Time Period	2nd Time Period	3rd Time Period
λ_i in the i -th time period	0.0785	0.0843	0.0914
DR_i in the i -th time period	0.0755	0.0808	0.0873
$\hat{\delta}_i$ in the i -th time period	0.3539	0.3414	0.3274

Panel C: Calibration Results Based on the Constant Recovery Rate $\bar{\delta} = 32\%$

	1st Time Period	2nd Time Period	3rd Time Period
λ_i in the i -th time period	0.0744	0.0805	0.0883
DR_i in the i -th time period	0.0717	0.0773	0.0845

Panel D: CB Values of Different Models

Our model with $\hat{\delta}_i$ in the i -th time period	92.6672
Constant recovery rate $\bar{\delta} = 32\%$ with infeasible probability correction	92.4094

Notes: For the illustrative example, Panel A shows the three-period ($N = 3$) BDT interest rate tree. Panels B and C shows the calibration results based on our conditional expected recovery rate model and a constant recovery rate fixed at 32%. In addition to the default intensities λ_i and the default rates DR_i , the corresponding conditional expected recovery rates $\hat{\delta}_i$ are also reported. Panel D compares the CB values generated by our model and a constant recovery rate model with correction of the infeasible probability problem.

is callable at any time point with call price $CP = 105$. We implement our model with Matlab on a PC with an Intel i7-6700K processor (4GHz) and 16GB RAM.

The three-time-period default-free interest rate tree is shown in Panel A of Exhibit 10. The default intensities, default rates, and recovery rates in different time periods under the conditional expected recovery rate assumption of our model are shown in Panel B of Exhibit 10. Comparing with the default rates given $\bar{\delta} = 32\%$, which is assumed in both Hung and Wang

[2002] and the CL model, one can find that the magnitudes of the default and recovery rates derived by our model are in reasonable ranges. In addition, incorporating the conditional expected recovery rate into the JT model can result in an inverse relationship between the recovery and default rates in different time periods. The default rates in the three time periods increase to be 0.0755, 0.0808, and 0.0873, while the corresponding conditional expected recovery rates decrease to be 0.3539, 0.3414, and 0.3274.

The three-time-period bivariate CB pricing tree is illustrated in Exhibit 11. For the final time period, the mild condition holds at nodes *J*, *M*, and *P*. The branching probabilities are determined via Equation (12) and the continuation values can be evaluated using Equation (16). For the remaining nodes in the final time period, the mild condition is violated. The branching probabilities are determined by Equation (14), and the continuation values are derived with Equation (17). For other time periods, since the mild condition holds at the nodes *A*, *C*, and *E*, the branching probabilities can be derived using Equations (12) and (13) and the continuation values are derived with Equation (18). On the other hand, the mild condition is violated at nodes *B* and *D*. Equations (14) and (15) and Exhibit 9 are used to derive the branching probabilities, and Equation (19) is employed to compute the continuation values. It is optimal for the issuer to exercise the redemption call at nodes *B*, *C*, *F*, *G*, *H*, *I*, and *J*; thus, the CB holders are forced to convert the CB into three equity shares at these nodes. The final pricing result is shown in the last field of node *A*. Based on our conditional expected recovery rate model, the CB price is 92.7329 when $N=3$, which is also shown in Exhibit 10, Panel D.

The computational time for our CB pricing tree model grows nearly proportionally to N^3 as illustrated in Exhibit 12; that is, our model is an $O(N^3)$ algorithm. This is because the backward induction is performed for each node, the size of which also grows in $O(N^3)$ due to the three dimensions: the stock price, the interest rate, and the timeline, being modeled in our CB pricing tree. Note that the exact size of the number of nodes cannot be easily estimated before the tree building process. This is because varying tree construction procedures illustrated in Exhibit 6 and Exhibit 8 are employed for guaranteeing feasible branching probabilities due to different levels of interest rates and default intensities implied from the prevailing zero curves.

DANAHER'S CB EXAMPLE

In this section, we employ our model to price a 20-year zero-coupon CB issued by Danaher Corporation on January 22, 2001. Danaher Corporation (DHR), an S&P 500 component stock, is a large global company that designs, manufactures, and markets industrial and consumer products. DHR comprises four major segments: professional instrumentation, medical technologies, industrial technologies, and tool components.

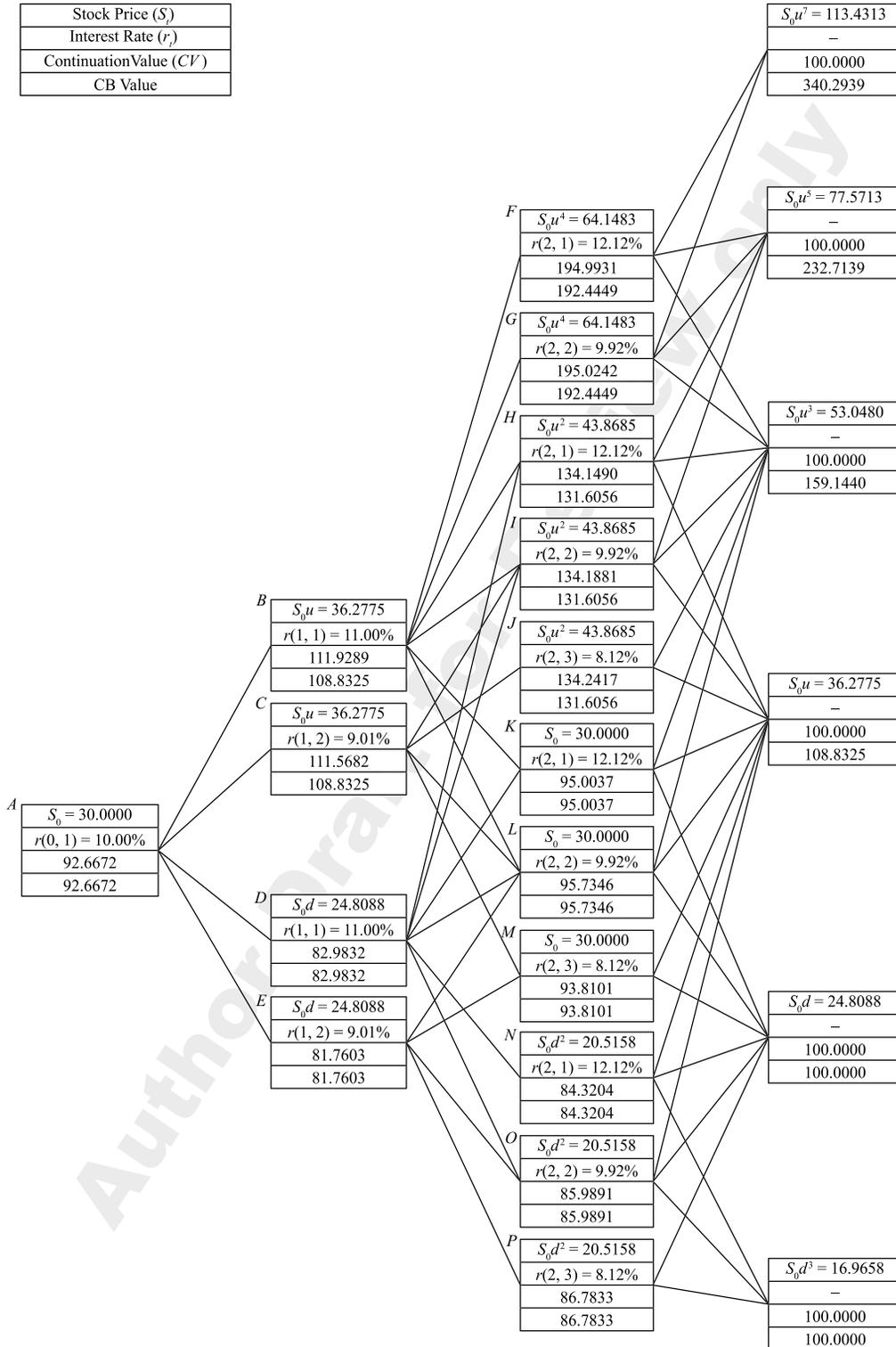
We price this DHR CB on January 22, 2009, for the following reasons. First, choosing January 22 as the pricing date makes the time span between the pricing date to the maturity date work out to multiples of a year. That makes it easier for our discrete time pricing model to handle call and put schedules,¹⁴ which in the DHR CB contract are specified on an annual basis. Second, we wish to select a relatively recent pricing date to guarantee the availability of necessary historical data. Third, we need a pricing date on which the CB price is not too close to its conversion value (the value of stock shares obtained by exercising the conversion option of a CB). This is because when the CB price is very close to its conversion value, it usually reflects the fact that the CB is forced to be converted in response to the redemption request of the CB issuer. In this situation, the default risk matters little and almost all CB pricing models yield the same CB price, which is the conversion value, making it difficult to distinguish performance among different models. According to the Bloomberg database, the differences between the CB prices and the conversion values on January 22 of 2010, 2011, 2012, 2013, and 2014 are very minor and less than 1% of the CB prices on these dates. Hence, we discard these five most recent candidates and focus on January 22, 2009.

The parameter values used to price the DHR CB on January 22, 2009, are listed in Exhibit 13. This table also shows the data source for the required parameters and the calibration method for estimating their values when necessary. The parameter values for this CB contract and the risk-free and A-credit-rating zero-rate curves are collected from the Bloomberg database, and the stock price series and the information about the dividend payment are downloaded from the Yahoo! finance website. To estimate the stock price volatility (σ_s) and dividend yield (q), we collect the data of daily stock prices and dividend payment amounts for

EXHIBIT 11

Three-Period Bivariate CB Pricing Tree for the Hypothetical CB Example

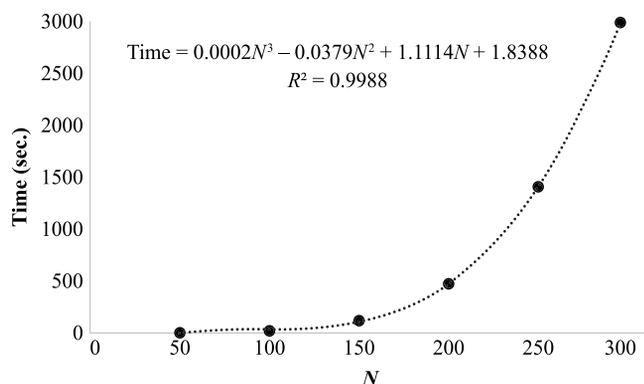
Stock Price (S_t)
Interest Rate (r_t)
Continuation Value (CV)
CB Value



Notes: The three-dimensional structure of our bivariate tree on a two-dimensional plane is illustrated by sorting the nodes at the same time point first by stock price and then by interest rate. The final pricing result of this example is 92.6672, as shown in the last field of node A.

EXHIBIT 12

Computational Time Analysis of Our Model



Notes: This exhibit analyzes the computational time of our bivariate tree for pricing the hypothetical three-year CB. The computational time of our CB pricing model grows proportionally to the cubic of time periods N (i.e., $O(N^3)$).

the three-year period prior to January 22, 2009. The dividend yield is computed as the ratio of the average annual cash dividends over the average stock price for this three-year period. To estimate σ_r in Equation (4) and the correlation between $d\ln S_t$ and $d\ln r_t$, we use the daily data on the six-month Treasury yield over the three-year period prior to January 22, 2009.

Comparisons among Different Models

Exhibit 14, Panel A reports pricing results for the DHR CB on January 22, 2009, based on different recovery rate assumptions. The percentage difference between each pricing result and the actual market price is shown in parentheses under that pricing result. We observe that pricing the DHR CB with a 600 time-period tree under the conditional expected recovery rate assumption generates a fairly accurate though slightly overestimated value (higher than the market price by 0.38%). The accurate CB price estimated using our model is obtained without needing to estimate the recovery rate. In contrast, the constant recovery rate assumption of $\bar{\delta} = 49.54\%$ yields a CB price of 85.1231 dollars, which is significantly higher than the CB market price by 1.34%.

The aforementioned pricing error becomes more pronounced if we price the options embedded in the DHR CB contract. This problem is critical since many CBs are divided and sold as several components

in financial markets. For example, a CB asset swap (CBAS) contract separates the conversion option from the CB contract. The value of the options embedded in the CB contract can be estimated by deducting the value of the corresponding corporate straight bond from the CB price. Note that the adoption of different recovery rate assumptions does not influence the pricing of the corresponding corporate straight bonds. This is because even if the default rate is calibrated poorly due to a poorly estimated recovery rate, the coordinated effect of the distorted default and recovery rates still accurately captures the expected loss of the corporate straight bond. The experiments in Exhibit 14, Panel B demonstrate how different recovery rate assumptions influence the values of the embedded conversion option, which is the most important option in CBAS contracts. To evaluate the conversion option, we first nullify the call and put options in the DHR CB contract by setting the call price CP to infinity and the put price PP to zero. Under these modifications, the difference between the CB pricing result and the market value of the risky 12-year zero-coupon bond (which is $F \times V(12) = 43.1038$ U.S. dollars) reflects only the value of the conversion option embedded in the DHR CB contract. Here we use the pricing result generated under the conditional expected recovery rate with 600 time periods as the benchmark. The percentage difference between each pricing result and the benchmark is reported below the pricing result. Compared with the percentage differences in Exhibit 14, Panel A, we observe that the impact of inaccurate constant recovery rate estimates is more highly pronounced on the embedded conversion option than it is on the CB value. The value of the conversion option is 53.6525 dollars under the conditional expected recovery rate, whereas the value of the conversion option is 72.2829 dollars under a constant recovery rate of 49.54%. That is, if the recovery rate cannot be estimated correctly, it tends to significantly misprice the embedded conversion option.

Other Analyses

In this section, we describe several experiments conducted to analyze the sensitivity of the DHR CB value with respect to the parameters $\bar{\delta}$, σ_s , σ_r , and ρ given $N = 144$. Since we use historical data to estimate these parameters and extrapolate them into the future, the sensitivity analyses for those parameters help us to

EXHIBIT 13

Parameters for Pricing DHR CB on January 22, 2009

Name	Data Source	Value and Explanation						
Panel A: CB Parameters								
Market price	Bloomberg	84.00						
Years to maturity (T)		12						
Face value (F)		100						Normalized to 100
Conversion ratio (θ)		1.45352 shares						For $F = 100$
Rating and seniority		A (senior and unsecured)						
Call schedule and call price (CP)		01/22/10	01/22/11	01/22/12	01/22/13	01/22/14	01/22/15	
		77.128	78.970	80.857	82.789	84.767	86.792	
		01/22/16	01/22/17	01/22/18	01/22/19	01/22/20		
		88.865	90.989	93.162	95.388	97.667		
Put schedule and put price (PP)		01/22/11						
		78.970						
Panel B: Stock Price Parameters								
Prevailing stock price (S_0)	Yahoo! finance	51.74						
Stock price volatility (σ_S)	website	28.21%						Standard deviation of the log difference in daily stock prices ($\Delta \ln S$) in the prior 3 years
Dividend yield (q)		0.07%						Ratio of the average annual cash dividends over the average stock price in the prior 3 years
Panel C: Interest Rate Parameters								
Interest rate volatility (σ_r)	Website of U.S. Treasury Department	6.08%						Standard deviation of the log difference in daily 6-month Treasury yields ($\Delta \ln r_t$) in the prior 3 years
Correlation between $d \ln S_t$ and $d \ln r_t$ (ρ)		17.92%						Correlation of daily $\Delta \ln S_t$ and $\Delta \ln r_t$ in the prior 3 years
Term structure of risk-free zero rates (%)	Bloomberg	3M	6M	1Y	2Y	3Y	4Y	
		0.10	0.29	0.40	0.72	1.10	1.39	
		5Y	7Y	10Y	12Y	15Y		
		1.62	2.06	2.69	3.08	3.60		
Term Structure of Risky Zero Rates of Rating A (%)		3M	6M	1Y	2Y	3Y	4Y	
		4.09	4.11	4.15	4.52	4.99	5.34	
		5Y	7Y	10Y	12Y	15Y		
		5.59	6.04	6.65	7.01	7.47		
Panel D: Recovery Rate Parameters								
Constant recovery rate	Hamilton et al. [2007]	49.54%						Average recovery rate in their Exhibit 20
(a, b)	Altman et al. [2005]	(0.0022, -0.1133)						Our Equation (1)
(α, β, γ)	Hull et al. [2005]	(0.1336, 0.8822, -0.1435)						Our Equation (2)

Notes: This exhibit lists the parameters values for pricing DHR CB on January 22, 2009.

assess pricing discrepancies due to estimation errors or changes in financial markets.

Exhibit 15 depicts the default rates over different time periods by setting the constant recovery rates $\bar{\delta}$ at 32%, 40%, 45%, 49.54%, and 55%. We examine the

values of the recovery rate centering around 49.54%, which is the average recovery rate for A-rated firms used in pricing DHR CB. The recovery rates 32% and 40% are included because these values are commonly assumed to be constant recovery rates in reduced-form

EXHIBIT 14

DHR CB Values under Different Recovery Rate Assumptions on January 22, 2009

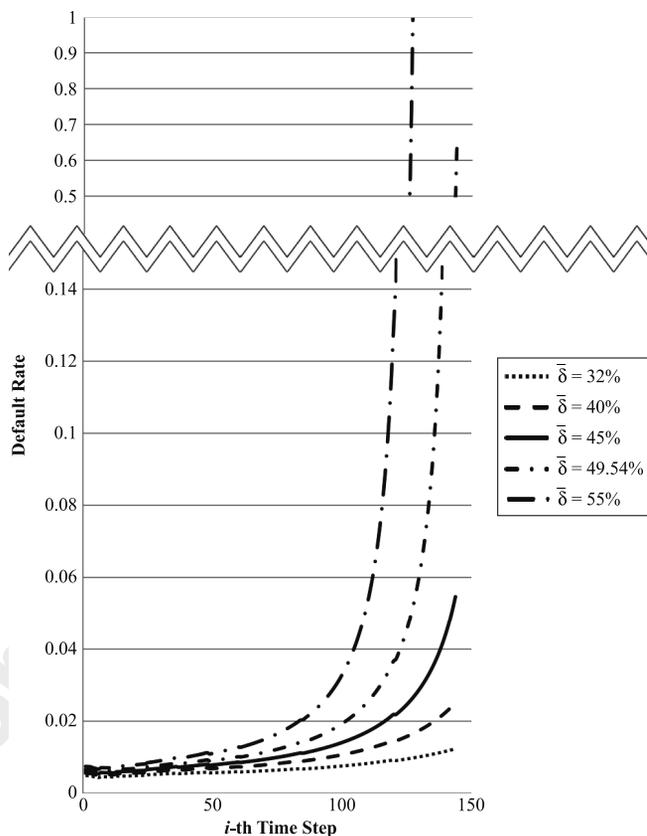
	$N = 12$	$N = 600$
Panel A: CB Value (percentage difference vs. actual market price)		
Our model with $\hat{\delta}_i$ in the i -th time period	85.5090 (1.80%)	84.3198 (0.38%)
Constant recovery rate $\bar{\delta} = 49.54\%$ with infeasible probability correction	86.2871 (2.72%)	85.1231 (1.34%)
Panel B: Conversion Option Value (= $CB - F \times V(12)$ given $CP = \infty$ and $PP = 0$) (percentage difference vs. our model with $\hat{\delta}_i$ under $N = 600$)		
Our model with $\hat{\delta}_i$ in the i -th time period	53.1416 (-0.95%)	53.6525 (0.00%)
Constant recovery rate $\bar{\delta} = 49.54\%$ with infeasible probability correction	72.8444 (35.77%)	72.2829 (34.72%)

Notes: Panel A reports the results for pricing DHR CB on January 22, 2009, based on different recovery rate assumptions. The percentage difference between each pricing result and the actual market price (84 U.S. dollars) is shown in parentheses below that pricing result. Panel B compares the values of the conversion option embedded in DHR CB on January 22, 2009, under different recovery rate assumptions. In addition to the values of the conversion option, the percentage differences versus the pricing results generated by our model with $N = 600$ are also reported below the values of the conversion option.

models. It is apparent that in all time periods, the default rates increase with the recovery rate. Thus, Exhibit 15 demonstrates that if one overestimates recovery rates, overvalued default rates are derived based on the traditional JT model,¹⁵ and a minor deviation in recovery rates results in a disproportionately large difference in default rates. Moreover, the constant-recovery-rate model generates abnormally high default rates for later time periods when the constant recovery rate increases. For example, in the case of a constant recovery rate $\bar{\delta}$ of 55%, the conditional default rate approaches almost 100% before maturity, surprisingly implying that DHR will certainly default as the CB is about to mature. Even when $\bar{\delta}$ is set to 49.54%, the traditional JT model for pricing DHR's CB reports that the conditional default rate exceeds 60% near the maturity, which is obviously too high as compared with results when $\bar{\delta}$ is 32%, 40%, and 45%. According to our analyses of the default and survival rates in Exhibits 3 and 5, we have reason to believe that these conditional default rates for $\bar{\delta}$ at 49.54% and 55% are abnormally high for an A-rated firm like DHR.

EXHIBIT 15

Default Rates in Each Time Period Generated under Different Constant Recovery Rates

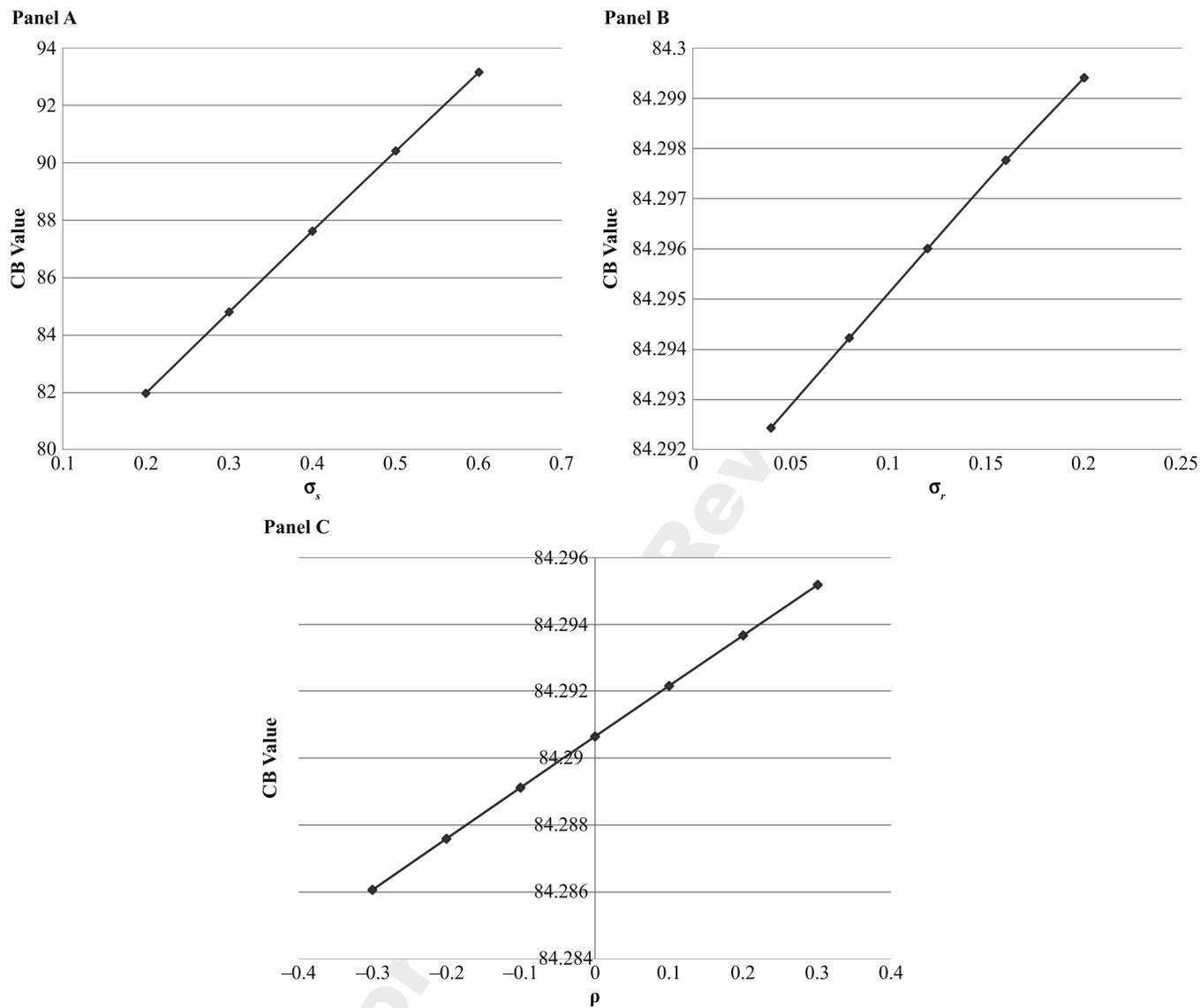


Notes: The curves in this exhibit show the default rates in each period under different constant recovery rates. For the 12-year time horizon, the number of total periods N is 144. This exhibit shows that, under the traditional JT model, higher recovery rates imply higher default rates. Moreover, the default rates in later periods, when $\bar{\delta}$ is 49.54% and 55%, are abnormally high.

Next, based on our bivariate tree model, the results of the sensitivity analyses of the CB values with respect to σ_s , σ_r , and ρ are shown in Exhibit 16. First, we observe that all three parameters have a positive effect on CB value. Since option values increase with increments in the underlying asset's volatilities, it is reasonable to expect that σ_s and σ_r will have a positive impact on the embedded options, and thus on the total CB value. The positive effect of the correlation between the stock price and interest rate processes on CB prices is consistent with the results in the CL model. Second, the impact made by varying σ_s is pronounced, while that made by varying σ_r and ρ is fairly minor. The CB value increases

EXHIBIT 16

Sensitivity Analyses of CB Values with Respect to σ_s , σ_r , and ρ



Notes: Examined in Panels A, B, and C, respectively, are the impacts on the CB value (represented by the y-axis) brought by changes in the stock price volatility σ_s , the interest rate volatility σ_r , and the correlation ρ (represented by the x-axis). There exists a significantly positive relationship between σ_s and the CB value, whereas the positive reactions in CB value to σ_r and ρ are relatively minor.

by more than 10 dollars when σ_s increases from 20% to 60% in Exhibit 16, Panel A. In contrast, the CB value increases by less than 0.01 dollars when σ_r dramatically increases from 4% to 20% in Exhibit 16, Panel B, or when ρ increases from -30% to 30% in Exhibit 16, Panel C. Consequently, we infer that even for inaccurate estimates of σ_r and ρ , our model still produces reliable estimates of CB values.

A POSSIBLE EXTENSION FOR OUR MODIFIED REDUCED-FORM MODEL

The relationship between the physical and the risk-neutral default rates described in Equation (2) does not consider changes in global economic environments or in investors' risk attitudes. Therefore, a regime-switching model that can describe time-varying relationships for

these two rates under, for example, economic expansions or recessions seems to be a potential solution.¹⁶ Although our tree model can implement regime-switching relationships with a forest composed of trees for modeling different relationships as suggested in Liu, Dai, and Wang [2016], a small number of observations of the physical and the risk-neutral default rates is not sufficient to generate reliable calibration results for a regime-switching model. For physical default rates of different credit ratings, we can access only the annual data provided by Moody's since 1970. On the other hand, the Merrill Lynch bond indexes, which are required to estimate risk-neutral default rates, have been available since 1997. Consequently, there are fewer than 20 available observations for the pairs of these two rates. Since calibrating parameters of a complex model with few observations would result in unpredictable estimation errors, our article does not consider a time-varying version for Equation (2).

However, to capture how changes in economic environments influence recovery rates like low recovery rates during financial crises, we follow Altman et al. [2005] by adding another economic variable, such as the growth rates of the S&P 500 Index, as an explanatory variable to modify Equation (1) as

$$\hat{\delta} = a + b \ln(DR^P) + cR^P + v. \quad (20)$$

Here the growth rate of the S&P 500 Index R^P is defined as the difference of the logarithmic stock index level, i.e., $\ln M_t - \ln M_{t-1}$ ¹⁷; the subscript t for R^P is suppressed to keep the following equations simple. The magnitudes of time-varying R^P and its coefficient c are used to proxy how the varying economic environments influence the recovery rate. We pick R^P instead of other macroeconomic variables like GDP as an explanatory variable because the risk-neutral counterpart for R^P is widely discussed in financial engineering literature. In contrast, there are no well-known risk-neutral stochastic process models for GDP, and it is also difficult to calibrate the risk-neutral parameters for these models. According to the same dataset in 1982–2001 used in Altman et al. [2005], the estimated results for Equation (20) are $a=0.0019$, $b=-0.1167$, and $c=-0.0943$.

The relationship between the recovery rate and the risk-neutral default probability is obtained by substituting the right-hand sides of Equation (2) for $\ln(DR^P)$ in Equation (20). In addition, the logarithmic S&P 500

Index return under the physical measure R^P is replaced by $R^Q + \Lambda$, where R^Q is the logarithmic S&P 500 return in the last year under the risk-neutral measure and Λ represents the risk premium for the logarithmic return. We estimate Λ to be 9.43% by calculating the average difference between returns of the S&P 500 Index and the risk-free interest rates (approximated by six-month Treasury yields) in 1982–2001. By mimicking the procedure in the second section, the expected recovery rate $\hat{\delta}$ can be expressed as

$$\hat{\delta} = a + b[\alpha + \beta \ln(1 - e^{-\lambda^Q}) + \gamma(\ln(1 - e^{-\lambda^Q}))^2] + c(R^Q + \Lambda), \quad (21)$$

which depends on the two risk-neutral factors λ^Q and R^Q .

Now we can extend the defaultable interest rate tree illustrated in Exhibit 2, Panel B by adding a new dimension that models the evolution of S&P 500 Index, M_t . Following the common practice in the literature, the stochastic process for the S&P 500 Index M_t under the risk-neutral measure is posited as

$$\frac{dM_t}{M_t} = (r_t - q_M)dt + \sigma_M dW_M, \quad (22)$$

where q_M and σ_M denote the dividend yield and the volatility of the S&P 500 Index, respectively. dW_M is a standard Wiener process under the risk-neutral measure, and the correlation between the Wiener processes dW_M and dW_r in Equation (4) is denoted as ρ_M . The evolutions of the two stochastic processes in Equations (4) and (22) can be simulated by taking advantage of our tree construction procedure for the CB pricing tree in Section 3. Note that since we model a stock index process M_t rather than a price process for an individual stock, it is not necessary to add the default-intensity adjustment (i.e., λ_t in Equation (11)) into the drift term of M_t in Equation (22).

After constructing $M_t - r_t$ tree, for each node, we calculate possible S&P 500 Index growth rates R^Q as the difference between the logarithmic S&P 500 Index of the current node and the logarithmic index levels of all its predecessor nodes.¹⁸ Default rates and conditional expected recovery rates can be calibrated by matching the risky bond price evaluated by the $M_t - r_t$ tree with its market observation, similar to what we did in Equation (10). The key difference is that the relationship among expected recovery rates, default rates,

and S&P 500 Index growth rates listed in Equation (21) is adopted, and index growth rates are endogenously determined by the $M_t - r_t$ tree. Note that for all nodes (each of which may have multiple R^Q values) in the same time period, they share a common risk-neutral default intensity λ^Q , but distinct conditional expected recovery rates are determined by Equation (21) for different R^Q of each node. As a consequence, the recovery rate of our modified reduced-form model can thus vary according to prevailing economic conditions, which is presumably approximated by the growth rate of the S&P 500 Index.

We then compare our extension model discussed in this section and the revised JT model discussed in the second section for estimating the recovery rates, default rates, and 15-year survival rates for AA, A, BBB, BB, and B ratings on January 22 in 2010, 2011, 2012, 2013, and 2014 as illustrated in Exhibits 17 and 3, respectively. The S&P 500 Index dividend yield q_M , the volatility σ_M , and the correlation between the short rate ρ_M for each examined date are estimated through daily data on S&P 500 Index levels and six-month Treasury yields in the past three years. The time period of the $M_t - r_t$ tree is set to be one year. Note that the results for $\hat{\delta}$ in Exhibit 17 are the probability-weighted average (adjusted by the survival rate) of the conditional expected recovery rates over all possible values of R^Q of all nodes in the same time period. When the factor of the S&P 500 return is added into Equation (1) to form Equation (20), the extension of our modified reduced-form model (extension model for short hereafter) still maintains all desired features that can be observed in Exhibit 3. That is, the calibrated recovery rates change over time, there is an obvious inverse relationship between calibrated recovery and default rates, and our extension model calibrates higher (lower) default rates and lower (higher) recovery rates for lower (higher) credit rating reference entities. In addition, by comparing Exhibits 17 and 3, we find that the impact of introducing the S&P 500 return is minor for both calibrated default rates and the corresponding conditional expected recovery rates. On average, the extension model generates slightly higher default rates than those in the original model (our modified reduced-form model that does not involve the S&P 500 return) by 0.07%, 0.07%, 0.06%, 0.00%, and 0.03% for AA, A, BBB, BB, and B ratings, respectively. Furthermore, the extension model on average generates slightly higher conditional expected recovery rates than those of the original model by 0.29%, 0.23%,

0.15%, and 0.29% for AA, A, BBB, and BB ratings, respectively. The only exception is the B rating, for which the extension model on average generates slightly lower conditional expected recovery rates than those of the original model by 0.04%. Consequently, the survival rates of the extension model are close to those of the original model (i.e., revised JT model) in Exhibit 5. The minor impact from the S&P 500 return is consistent with the results in Altman et al. [2005]. They find that the negative relationship between the default and recovery rates is more robust and significant than the statistical relationship between the default rate and other macroeconomic factors such as the GDP or S&P 500 return. Even so, this section still illustrates the potential of our model for introducing another representative economic/market explanatory variable to determine the conditional expected recovery rate.

CONCLUSION

Traditional reduced-form models commonly estimate a constant recovery rate in advance. However, it is difficult to obtain an accurate estimation of the recovery rate; an inaccurate recovery rate significantly influences the calibrated default rate and sometimes distorts it. In addition, the assumption of the independence of the constant recovery rate from the default rate contradicts empirical findings. Moreover, our numerical analyses also suggest that the improper setting of constant recovery rates may generate unreasonable default rates.

To solve these problems, this article suggests that any traditional reduced-form model can integrate our notion of conditional expected recovery rate, which is based on an empirical regression relationship between the recovery and default rates and a transformation between default rates under the physical and risk-neutral measures. Our model can maintain the simplicity and applicability of traditional reduced-form models. In addition, both the default and recovery rates can be endogenously determined during the calibration process. Consistent with many empirical studies, the resulting default and recovery rates are time-varying and negatively correlated. Furthermore, our conditional expected recovery rate can incorporate other macroeconomic factors as explanatory variables in addition to the default rate.

EXHIBIT 17

Recovery, Default, and Survival Rates of the Extension of Our Model

	AA Rating		A Rating		BBB Rating		BB Rating		B Rating	
	$\hat{\delta}(\%)$	$DR^e(\%)$								
January 22, 2014										
Year 1	77.16	1.06	70.74	1.37	62.30	1.94	51.56	3.12	49.90	3.37
2	71.66	1.45	67.07	1.75	59.12	2.47	49.88	3.76	46.16	4.49
3	76.59	1.19	68.24	1.67	58.59	2.52	47.23	4.25	42.60	5.33
4	76.49	1.19	66.99	1.75	56.21	2.79	43.79	5.00	39.42	6.22
5	70.27	1.52	62.15	2.14	52.28	3.32	40.13	5.97	36.49	7.20
6	60.96	2.25	55.01	2.92	47.16	4.21	37.53	6.80	33.90	8.23
7	56.98	2.67	51.41	3.44	44.07	4.89	34.80	7.84	31.20	9.52
8	55.78	2.81	49.19	3.81	42.24	5.33	33.42	8.41	31.85	9.15
9	52.75	3.23	46.32	4.37	39.66	6.08	31.05	9.58	29.76	10.29
10	49.90	3.68	43.54	4.99	37.05	6.93	28.54	10.99	27.46	11.68
11	50.94	3.50	42.54	5.23	35.86	7.36	24.96	13.47	24.38	13.95
12	48.82	3.87	40.20	5.90	33.58	8.33	22.34	15.81	21.87	16.28
13	45.88	4.45	37.01	6.94	30.31	9.95	18.18	20.47	18.01	20.69
14	43.50	4.99	34.52	7.90	27.73	11.48	14.76	25.54	14.61	25.81
15	41.05	5.63	31.95	9.05	24.99	13.42	10.78	33.61	10.58	34.09
mean	58.58	2.90	51.13	4.22	43.41	6.07	32.60	11.64	30.55	12.42
median	55.78	2.81	49.19	3.81	42.24	5.33	33.42	8.41	31.20	9.52
max	77.16	5.63	70.74	9.05	62.30	13.42	51.56	33.61	49.90	34.09
min	41.05	1.06	31.95	1.37	24.99	1.94	10.78	3.12	10.58	3.37
January 22, 2013										
mean	55.68	3.14	49.01	4.63	43.14	5.86	33.76	9.98	30.51	11.85
median	55.82	2.83	50.76	3.57	44.34	4.86	36.47	7.22	31.29	9.54
max	76.73	5.91	70.61	10.44	61.96	12.20	50.67	24.87	50.87	28.82
min	40.16	1.11	29.48	1.42	26.74	2.04	15.23	3.37	13.06	3.34
January 23, 2012										
mean	53.23	3.65	47.13	5.02	39.58	6.95	31.77	10.14	24.68	20.10
median	55.69	2.85	49.21	3.84	40.98	5.75	33.34	8.54	29.49	10.70
max	75.09	7.62	67.67	11.26	58.92	14.46	49.19	19.74	37.91	82.54
min	35.31	1.24	28.19	1.67	23.88	2.44	18.83	3.79	0.93	6.60
January 24, 2011										
mean	52.16	3.81	46.23	5.26	41.13	6.50	33.62	9.45	29.86	12.39
median	52.39	3.31	47.64	4.14	42.19	5.40	34.91	7.85	33.25	8.16
max	70.02	7.90	65.75	12.08	55.76	14.17	48.75	20.66	40.53	34.40
min	34.63	1.43	26.96	1.71	24.23	2.63	18.11	3.62	10.61	5.96
January 22, 2010										
mean	50.80	4.10	46.06	5.26	40.76	6.53	63.89	3.99	31.55	9.92
median	49.74	3.74	45.56	4.56	39.98	6.02	54.63	2.99	30.96	9.76
max	69.89	8.46	62.45	11.61	55.85	13.24	100.00	15.54	39.31	16.97
min	33.39	1.38	27.69	1.88	25.42	2.50	23.29	0.04	21.29	5.44

(continued)

EXHIBIT 17 (continued)

Recovery, Default, and Survival Rates of the Extension of Our Model

15-Year Survival Rate (%)	January 22, 2014	January 22, 2013	January 23, 2012	January 24, 2011	January 22, 2010
AA	64.22	61.86	57.12	55.69	53.15
A	52.17	48.85	45.91	44.16	44.13
BBB	38.71	40.14	33.61	36.18	36.04
BB	14.46	19.96	19.80	22.14	53.41
B	12.70	14.39	1.39	12.80	20.67

Notes: This exhibit reports the risk-neutral default rates (DR^Q), the corresponding conditional expected recovery rate ($\hat{\delta}$), and the 15-year survival rates of the extension of our model, which incorporates the change in the logarithmic S&P 500 Index in the previous year as an explanatory factor for δ . The results for January 22, 2014, are fully presented. For all the other examined dates, we report only the means, medians, maxima, and minima of the recovery and default rates.

To demonstrate the applicability of our idea, we revise the JT model and use the revised model to price CBs. Another contribution of this article is our employing Dai's [2009] mean-tracking method and modifying the correlation calibration method in Hull and White [1994] to solve the problem of infeasible branching probabilities in the CL model. As an example, we examine the pricing problem of the Danaher CB. The numerical results demonstrate that our revised JT model produces a more accurate CB value (only 0.38% higher than the prevailing market price) than the original constant recovery rate model. In addition, we also show that the constant recovery rate model overvalues the conversion option in the Danaher CB by 34.72%. Reliable sensitivity analysis results also confirm the robustness of our revised model.

ENDNOTES

The authors thank Hung-Ting Chen for the preliminary study and financial support from the Ministry of Science and Technology of Taiwan.

¹According to a Moody's Investors Service report (Hamilton et al. [2007]), average recovery rates change over time, ranging from 20% to 60%, from 1982 to 2006.

²Similar arguments can be found in Duffie and Singleton [1999].

³Conversely, Hu and Perraudin [2006] and Acharya, Bharath, and Srinivasan [2007] use default rate as well as other explanatory variables to predict the recovery rate. Our model can be extended to include other explanatory variables for the recovery rate. However, to maintain the simplicity and applicability of our model and focus on the improvements of our core idea on endogenous determinations of recovery and

default rates, the introduction of our model does not consider other explanatory variables.

⁴The R-squared values of the linear regression equation between the recovery and default rates in Hamilton et al. [2007] and Altman et al. [2005] are 0.55 and 0.51, respectively.

⁵We thank the editor for pointing out the negative default risk premium problem and for the suggestion to find a high-R-squared value regression relationship between DR^P and DR^Q that simultaneously keeps the default risk premium positive when DR^Q is within a reasonable range. Indeed, it is challenging to find a high-R-squared-value regression that keeps $DR^Q - DR^P$ positive for all possible values of DR^Q . Our quadratic regression between $\ln DR^P$ and $\ln DR^Q$ is a practical compromise between the two needs.

⁶Berndt et al. [2011] also suggest estimating the risk-neutral default rates from CDS spreads.

⁷We follow the JT model in assuming that the occurrence of defaults and the interest rate movements are independent.

⁸The JT model atypically assumes that the bondholders receive the recovery on the bond maturity date given the default event occurs. This article, however, follows Duffie and Singleton [1999] by assuming that bondholders receive the recovery immediately after the issuer defaults.

⁹Since both January 22, 2011, and January 22, 2012, are holidays, the zero rates on January 24, 2011, and January 23, 2012, are used instead.

¹⁰The resource center on the website of the U.S. Treasury Department (<http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/default.aspx>) also provides daily data of one-month and three-month Treasury yields, which are more appropriate candidates for estimating the interest rate volatility. However, due to the missing data problem, we do not consider these two series.

¹¹This approach is commonly used in practice to estimate default rates of corporate bonds with different credit ratings. For details on this approach, refer to Hull [2014].

¹²For the same reason, the credit spread model is inappropriate for pricing option-embedded defaultable bonds. This is also the major reason why in this article we choose the JT model to illustrate our notion of the conditional expected recovery rate and apply the revised JT model to pricing CBs.

¹³This process can be derived from Merton [1976], a seminal work in considering the option pricing problem under the jump-diffusion process. By considering the jump-to-default process in Equation (17) of Merton [1976], the jump intensity should be added into the drift term of the stock price process to maintain its martingale property.

¹⁴To take the put (sell back) option into consideration, the CB value of each node is determined by evaluating $\max(\min(CV, CP), \theta S_t, PP)$, where PP denotes the put price.

¹⁵To the best of our knowledge, this phenomenon is a common characteristic for constant-recovery-rate reduced-form models. Although a rigorous and general proof is not available, we can take the credit spread reduced-form model for example to illustrate our argument. Note that the credit spread model calibrates the risk-neutral default rate from the credit spread given a recovery rate through $DR^Q = CS / (1 - \bar{\delta})$. By analyzing this formula, one can find that if a recovery rate is overestimated, the calibrated default rate in turn is overvalued for this inaccurate recovery rate.

¹⁶We thank the anonymous reviewer for pointing out the implicit stationarity assumption in Equation (2) and the editor's constructive suggestion regarding this issue.

¹⁷In contrast to Altman et al. [2005], who use the contemporary annual return on the S&P 500 Index as an explanatory variable, our R^P is calculated as the S&P 500 Index return over the period just prior to the examined time point to avoid introducing the foresight problem into our modified reduced-form model.

¹⁸The index growth rate for the root node is simply determined as $\ln M_0 - \ln M_{-\Delta t}$, where M_0 and $M_{-\Delta t}$ are the actual S&P 500 Index levels on the examined date and on the date one period before the examined date, respectively.

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