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Abstract This study extends the GARCH pricing tree in Ritchken and Trevor (J Financ 54:366–402, 1999) by incorporating an additional jump process to develop a lattice model to value options. The GARCH-jump model can capture the behavior of asset prices more appropriately given its consistency with abundant empirical findings that discontinuities in the sample path of financial asset prices still being found even allowing for autoregressive conditional heteroskedasticity. With our lattice model, it shows that both the GARCH and jump effects in the GARCH-jump model are negative for near-the-money options, while positive for in-the-money and out-of-the-money options. In addition, even when the GARCH model is considered, the jump process impedes the early exercise and thus reduces the percentage of the early exercise premium of American options, particularly for shorter-term horizons. Moreover, the interaction between the GARCH and jump processes can raise the percentage proportions of the early exercise premiums for shorter-term horizons, whereas this effect weakens when the time to maturity increases.

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1 Introduction

The stochastic nature of asset prices is of pivotal importance for option valuations. Conventional assumptions of geometric Brownian motion processes are not satisfied because some existing phenomena, such as leptokurtosis and negative skewness found in asset return distributions, cannot be counted as influencing factors. Since abundant empirical findings indicicate that discontinuities in the sample path of financial asset prices are still found even allowing for autoregressive conditional heteroskedasticity, we thus study the option pricing problem assuming the GARCH-jump process for the underlying asset. Different from the simulation-based option pricing models in existing literature, however, we develop a lattice model to value both European and American options.

The proposed lattice option pricing model is developed by incorporating the technique for modeling the jump process in Amin (1993) into the lattice algorithm in Ritchken and Trevor (1999) GARCH model. By performing the dynamic programming method over our lattice model, we can price European as well as American options under the GARCH-jump process. Our lattice model is structurally simple since both the GARCH and jump processes can be modeled in a univariate lattice framework. Moreover, the GARCH-jump process considered in this paper is rather general to encompass several classical models as special cases or adapt to the GARCH-jump model in Duan et al. (2006). As a result, our lattice model can be an efficient tool to conduct empirical tests or numerical analyses on options pricing under different asset price processes.

There are two widely adopted methods to account for the leptokurtosis and negative skewness of the asset return and thus capture the volatility smile effect. The first is to assume the conditional variance in the underlying process to be stochastic, i.e., stochastic variance (SV) models, and the second is to consider autoregressive conditional variances, i.e., GARCH (Generalized Autoregressive Conditional Heteroskedasticity) models.[1] However, attractive as the GARCH or SV models may be, they are unable to account for occasional and large discrete changes embedded in asset prices. It is common to employ the jump-diffusion process to capture the occasionally large movements in asset prices. Merton (1976) first proposes a jump-diffusion option pricing model, where asset returns are generated by a mixture of two processes, including continuous, incremental fluctuations of prices from a Wiener process, and large, infrequent price

[1] The pioneers of SV option pricing models include Hull and White (1987), Wiggins (1987), Scott (1987), Stein and Stein (1991) and Heston (1993). On the other hand, the GARCH model is first proposed by Engle (1982) and Bollerslev (1986), and there are many variations, such as the exponential GARCH (EGARCH) model in Nelson (1991), the nonlinear asymmetric GARCH (NGARCH) model in Engle and Ng (1993), GJR-GARCH model in Glosten et al. (1993), and the threshold GARCH (TGARCH) model in Zakoian (1994).

jumps pertaining to the nonsystematic risk from a Poisson process. Cox and Ross (1976) and Ahn and Thompson (1988) also derive option pricing models by assuming discontinuous jump processes for the underlying assets. In order to price American options under the jump-diffusion process, Amin (1993) develops a discrete-time lattice model, assuming that the asset price can move upward or downward by one tick in each time step to represent the diffusion process, as postulated by Cox et al. (1979) model, and the asset price also changes on account of a rare event (jump) by permitting the underlying asset price to move by multiple ticks in a single time step. Based on the risk-neutral valuation argument and the assumption that the jump risk is diversifiable, his model weakly converges to the theoretical option values under some regularity conditions.

Although both the SV and GARCH models are able to capture the negative skewness and excess kurtosis of asset returns commonly found in the empirical data, this paper focuses on the GARCH models due to several reasons. First, comparing to the SV models, the GARCH models are more intuitive to understand and thus widely used by market participants. Second, for the GARCH models, both the asset price and variance processes are driven by the same random innovation, but for the SV models, the asset price and variance processes have their own random innovations. This characteristic complicates the estimation of the parameters of the SV models since the variances are not observable in the market. It also causes the complexity to develop a lattice model to price American-style options under the SV models. To take both random innovations of the asset price and variance processes into account, it is unavoidable to consider a bivariate lattice model, which is structurally more complex than a univariate lattice model for the GARCH models. Consequently, it is further difficult to incorporate other stochastic features, for example the jump process, with the bivariate lattice model of the SV models to pricing American-style options.[2]

The choice for combining the GARCH and jump processes to model the behavior of asset prices is supported by abundant empirical evidence in the literature.[3] For example, Vlaar and Palm (1993) and Nieuwland et al. (1994) adopt a GARCH-constant jump intensity model to capture foreign exchange rate dynamics. Likewise, Jorion (1988) combines an ARCH model with a jump component to empirically examine both foreign exchange rates and stock returns. Lin and Yeh (2000) also employ a GARCH-constant jump intensity model and provide empirical tests on the Taiwan stock market to examine whether discontinuous price paths exist. Both Jorion (1988) and Lin and Yeh (2000) find that the combined models could provide better explanations for the

[2] Similar to this paper, Chang and Fu (2001) investigate the option pricing problem under the SV model and jump process. They combine the transformation technique of Hilliard and Schwartz (1996) to deal with the SV model and the jump-diffusion model of Amin (1993) to yield a discrete-time bivariate binomial tree model. Due to the complexity of the bivariate lattice structure, their model is difficult to implement and thus has not much practical implication.

[3] There are also many articles examining empirically the necessity to incorporate a jump process into the SV models (thus the SVJ models) for stock indexes, such as Anderson et al. (2002), Jiang (2002), Pan (2002), Chernov et al. (2003) and Kim et al. (2007). In addition, it is commonly identified in the literature that the SVJ models are superior than the SV models on option pricing, such as Bakshi et al. (1997, 2000), Bates (1996, 2000), and Scott (1997) which propose analytic option pricing formulae for European-style options under the SVJ models by solving the characteristic functions of the cumulative probabilities under the risk-neutral measure.

behavior of asset prices. Recently, Duan et al. (2006, 2007) propose a highly general GARCH-jump model which takes the correlated systematic jump into account. In Duan et al. (2007), they empirically test this GARCH-jump model using the S&P 500 index series as the research sample. The results show that the inclusion of jumps significantly improves the fit of historical series of the S&P 500 stock index. In addition, Duan et al. (2006) propose an option pricing model corresponding to their general GARCH-jump model and solve option prices by simulation approaches.

Equipped with our lattice model, we show that both the GARCH and jump effects in the GARCH-jump model are negative on the values of near-the-money options while positive on the values of in-the-money and out-of-the-money options. This pattern confirms the evidence in many empirical studies that the introduction of either the GARCH or the jump process can help explaining the excess kurtosis and thus the phenomenon of the volatility smile implied from option prices. In addition, even when the GARCH model is considered, the jump process impedes the early exercise and thus reduces the percentage of the early exercise premium of the American option, particularly for shorter-term horizons. Moreover, we also discover the positive impact on the percentage proportions of the early exercise premiums from the interaction between the GARCH and jumps, and this positive impact declines as the maturity increases.

The remainder of this paper is organized as follows. In Sect. 2, we construct a lattice model under the generalized GARCH-jump process and discuss its adaption to several existing models. Section 3 describes the option pricing procedure based on our lattice model. In Sect. 4, we conduct several numerical analyses for our GARCH-jump lattice model on pricing options. Section 5 is the conclusion of this paper.

2 The lattice model with GARCH and jumps

2.1 General framework

This paper denotes S_t as the price of the underlying asset on date t. Suppose that under the risk-neutral measure Q, the logarithmic return of the underlying asset price over the period $(t, t+1]$[4] follows the generalized GARCH-jump process as follows:

$$\ln\left(\frac{S_{t+1}}{S_t}\right) = m_t + \sqrt{h_t}\mathbf{X}_t, \tag{1}$$

where

$$m_t = r_f - \frac{h_t}{2} - \lambda[K_t - 1],$$

$$\mathbf{X}_t = Z_t + \sum_{l=1}^{q_t^Q} J_t^{(l)},$$

[4] Without loss of generality, the time step is fixed to be one day in this paper for the brevity of the notation system.

$$Z_t \sim N^Q(0, 1),$$
$$J_t^{(l)} \sim N^Q(\mu_J(h_t), \sigma_J^2(h_t)),$$
$$K_t = E_J(\exp(\sqrt{h_t} J_t^{(l)})).$$

In the above postulation, m_t and h_t denote the daily drift rate and conditional variance of the asset price process over the period $(t, t+1]$. In addition, r_f is the daily risk-free interest rate, and $\mathbf{X}_t$ is a compound Poisson normal process, which is a mixture of a standard normal process Z_t and a Poisson jump process. The notation q_t^Q represents the total number of Poisson events occurring in $(t, t+1]$ with the daily jump intensity λ and the independent (with respect to different l and time point t) normally distributed jump magnitude $J_t^{(l)}$. The mean and standard deviation of $J_t^{(l)}$, denoted as $\mu_J(h_t)$ and $\sigma_J(h_t)$, are assumed to be generally dependent on the conditional variance h_t. Finally, K_t is defined as the average rate of jump plus 1, and the reason to include $\lambda[K_t - 1]$ in the drift term is to maintain the martingale property of the underlying asset price under the risk-neutral measure Q. In the equation for evaluating K_t, E_J is the expectation operator with respect to the distribution of $J_t^{(l)}$.

The variance process of the asset price returns is assumed to follow a generalized GARCH process with an updating function:

$$h_{t+1} - h_t = f(\nu_{t+1}, h_t), \tag{2}$$

where

$$\nu_{t+1} = \frac{(\ln S_{t+1} - \ln S_t - m_t)/\sqrt{h_t} - E^Q(\mathbf{X}_t)}{\sqrt{Var^Q(\mathbf{X}_t)}},$$

is the standardized innovation of the logarithmic asset price process and

$$E^Q(\mathbf{X}_t) = \lambda\mu_J(h_t),$$
$$Var^Q(\mathbf{X}_t) = 1 + \lambda[\mu_J^2(h_t) + \sigma_J^2(h_t)],$$

are the mean and variance of the compound Poisson normal process, $\mathbf{X}_t$, under the risk-neutral measure Q.

Next, this paper establishes a lattice model for simulating the dynamics of the above GARCH-jump process. Given $y_t = \ln(S_t)$, the logarithmic asset price after one day, y_{t+1}, can be approximated with a grid of nodes in the lattice space as follows:

$$y_{t+1} \in \{y_t, y_t \pm \gamma_n, y_t \pm 2\gamma_n, y_t \pm 3\gamma_n, \ldots\}, \tag{3}$$

where γ_n represents the tick movement of the logarithmic asset price on the lattice and will be defined later. Note that the asset price changes can be driven by a local component and a jump component, where the local component indicates the variation of the asset price follows the assumption of a GARCH diffusion process, and the jump component means that the asset price can change to an arbitrary level, either within or beyond the local change levels.

For the local component to simulate the GARCH model, given the logarithmic asset price y_t and the conditional variance h_t, we follow the assumption in Ritchken and Trevor (1999) that the conditional normal distribution of the logarithmic asset price for the subsequent time step is approximated by a discretely random variable that takes $2n + 1$ values on the lattice. More specifically, Ritchken and Trevor (1999) partition each time step (that is one day in their model) into n subintervals, and for each subintervals, the trinomial tree model is employed to simulate the evolution of the logarithmic asset price. The cumulative effect of the trinomial tree model in n subintervals generates $2n + 1$ branches for each time step. For example, if $n = 2$, for the possible logarithmic price levels at the next time step, there will be two levels higher than and two levels lower than the current logarithmic price level in addition to one unchanged level. In addition, we also need the size control parameter, η, which is defined as the smallest integer that allows the mean and variance of the next period's logarithmic price to match the moments of the posited distribution and ensures the probabilities of all $2n + 1$ branches of each node are in the interval [0,1]. The size control parameter η can be chosen such that

$$(\eta - 1) < \frac{\sqrt{h_t}}{\gamma} \le \eta, \tag{4}$$

where we set $\gamma = \sqrt{1.5h_0}$, in which h_0 is the initial daily variance of the logarithmic asset price process. Finally, the tick size for the change of the logarithmic asset price in Eq. (3) is defined as:

$$\gamma_n = \frac{\gamma}{\sqrt{n}}. \tag{5}$$

To complete the approximation for the GARCH model with the local price changes on the lattice model, the remaining problem is the assignment of probabilities for the branches of each node.

In order to take into account both the GARCH and jump characteristics, we adopt a technique similar to that proposed in Amin (1993) to simulate the jump process in the multinomial tree of Ritchken and Trevor (1999) GARCH pricing model. In our model, we assume that both the update of the conditional variance and the jump events take place only at the end of each day, and we further allow for the concurrence of local price changes and jumps.[5] As a consequence, the probabilities of the branches representing the GARCH model in Ritchken and Trevor (1999) model are modified as

$$\Pr(y_{t+1} - y_t = \theta\eta\gamma_n) = (1 - \lambda)P(\theta) + \lambda\phi(\theta\eta), \ \theta = 0, \pm 1, \ldots, \pm n, \tag{6}$$

[5] This assumption is consistent with practical conditions since it is almost impossible to distinguish a small change in the asset price coming from the diffusion or jump components. In contrast, Amin (1993) permits the two price changes to be mutually exclusive for the expositional convenience. However, in the continuous time limit, i.e., when the length of the time step approaches zero, it is irrelevant whether they are mutually exclusive.

where

$$P(\theta) = \sum_{j_u, j_m, j_d} \binom{n}{j_u\, j_m\, j_d} p_u^{j_u} p_m^{j_m} p_d^{j_d}, \text{ for } \theta = j_u - j_d \text{ and } j_u + j_m + j_d = n,$$

$$P_u = \frac{h_t/(1-\lambda)^{1/n}}{2\eta^2\gamma^2} + \frac{m_t\sqrt{1/n}/(1-\lambda)^{1/n}}{2\eta\gamma},$$

$$P_m = 1 - \frac{h_t/(1-\lambda)^{1/n}}{\eta^2\gamma^2},$$

$$P_d = \frac{h_t/(1-\lambda)^{1/n}}{2\eta^2\gamma^2} - \frac{m_t\sqrt{1/n}/(1-\lambda)^{1/n}}{2\eta\gamma},$$

and

$$\phi(j) = \Phi\left(\left(j + \frac{1}{2}\right)\gamma_n\right) - \Phi\left(\left(j - \frac{1}{2}\right)\gamma_n\right)$$

is the marginal probability that the asset price changes to the jth level relative to the level of y_t, where $\Phi(\cdot)$ is the cumulative normal distribution function with the mean of $\mu_J(h_t)\sqrt{h_t}$ and the variance of $\sigma_J^2(h_t)h_t$. The details to derive P_u, P_m, and P_d are presented in Appendix A.

The probabilities for price changes other than the local change levels, i.e., the probabilities caused only by the jump component, are

$$\Pr(y_{t+1} - y_t = j\gamma_n;\, j \neq \theta\eta) = \lambda\phi(j), \quad j = 0, \pm 1, \pm 2, \ldots, \pm w, \tag{7}$$

where w is defined as $3\sqrt{\sigma_J^2(h_t)h_t}/\gamma_n$ (rounding to the nearest integer greater than or equal to that number), since the probability that the asset price jumps to the level outside the range of $[-3\sqrt{\sigma_J^2(h_t)h_t}, 3\sqrt{\sigma_J^2(h_t)h_t}]$ is very small and can be negligible. In addition, the entire probability mass outside the region $[-3\sqrt{\sigma_J^2(h_t)h_t}, 3\sqrt{\sigma_J^2(h_t)h_t}]$ is assigned to the truncation points. More specifically, for the cases of $j \neq \pm w$, the definition of $\phi(j)$ is as it is; however, when $j = \pm w$, the jump distribution is trimmed such that the entire probability mass lower and higher than the range $[-w, w]$ is assigned to the node $j = -w$ and $j = w$, respectively, i.e., $\phi(-w) = \Phi((-w + 1/2)\gamma_n)$ and $\phi(w) = 1 - \Phi((w - 1/2)\gamma_n)$.

Figure 1 illustrates the probabilities we assign to any state j relative to the level of y_t in our lattice model. For expositional purposes, $n = 1$ is considered in this figure, which means the GARCH process is approximated by a trinomial model for each time step. The size control variable η is assumed to be 2, meaning the possible levels of local price changes to simulate the GARCH model for the next period prove to be $j = -2, 0, 2$. Other price changes are for the jump component only. Note that since the triple of the volatility of jump magnitude, $3\sqrt{\sigma_J^2(h_t)h_t}$, is significantly larger than the spacing parameter γ_n of the lattice model in both theory and practice, the range of $y_{t+1} = y_t + j\gamma_n$, for $j = 0, \pm 1, \pm 2, \ldots, \pm w$ is wider than $y_{t+1} = y_t + \theta\eta\gamma_n$, for

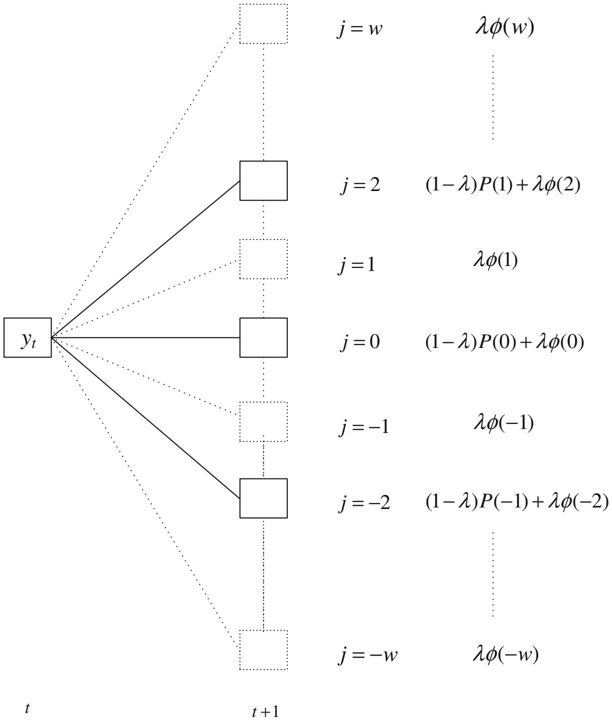

Fig. 1 The probabilities of the GARCH-jump lattice model. This figure illustrates the probabilities of the local price changes and jump components of our lattice model, which are assigned to branches from the node with the logarithmic asset price y_t to each relative state j at time $t+1$. The case of $n = 1$ and $\eta = 2$ is considered in this figure. The *dotted rectangles* represent the price levels corresponding to the jump component only, and the assigned probabilities are reported on their right. The *solid rectangles* indicate the price levels to include both types of changes. The combined probabilities are also shown on their right

$\theta = 0, \pm 1, \pm 2, \ldots, \pm n$. Also note that in this paper, we allow the jump component to cause the price changes to all levels, including the local price change levels.

2.2 Special cases of the general GARCH-jump process

The framework of the proposed GARCH-jump model is quite general and able to encompass many special cases. This section introduces a constant-parameter GARCH-jump model first, and then discusses several nested models of it. Suppose $\mu_J(h_t)\sqrt{h_t} = \mu_J$ and $\sigma_J^2(h_t)\,h_t = \sigma_J^2$, where μ_J and σ_J^2 are constants. Then the jump magnitude follows a normal distribution with the constant mean and standard deviation to be μ_J and σ_J, and the GARCH-jump process in Eq. (1) becomes

$$\ln\left(\frac{S_{t+1}}{S_t}\right) = m_t + \sqrt{h_t}\mathbf{X}_t = m_t + \sqrt{h_t}Z_t + \sum_{l=1}^{q_t^Q}\sqrt{h_t}J_t^{(l)}, \tag{8}$$

where

$$m_t = r_f - \frac{h_t}{2} - \lambda[K_t - 1],$$
$$\sqrt{h_t} J_t^{(l)} \sim N^Q(\mu_J, \sigma_J^2),$$
$$K_t = E_J(\exp(\sqrt{h_t} J_t^{(l)})) = \exp(\mu_J + \sigma_J^2/2).$$

Moreover, we consider the NGARCH process following Ritchken and Trevor (1999), and the variance updating function can be specified as follows:

$$h_{t+1} - h_t = f(\nu_{t+1}, h_t) = \beta_0 + (\beta_1 - 1)h_t + \beta_2 h_t(\nu_{t+1} - c^Q)^2, \tag{9}$$

where $\beta_0, \beta_1, \beta_2$, and c^Q are constants. The non-negative value of c^Q indicates a negative correlation between the innovations of the logarithmic asset price and its conditional volatility under the risk-neutral measure Q. In addition, $\beta_0 > 0, \beta_1 \geq 0$, and $\beta_2 \geq 0$ are required to ensure the positive conditional volatility. As to the innovation of the logarithmic asset price ν_{t+1}, it can be rewritten as follows.

$$\nu_{t+1} = \frac{(\ln S_{t+1} - \ln S_t - m_t)/\sqrt{h_t} - E^Q(\mathbf{X}_t)}{\sqrt{Var^Q(\mathbf{X}_t)}} = \frac{\ln S_{t+1} - \ln S_t - m_t - \lambda\mu_J}{\sqrt{[h_t + \lambda(\mu_J^2 + \sigma_J^2)]}}.$$

In Sect. 4 of this paper, our numerical results are mainly according to this constant-parameter GARCH-jump model.

Furthermore, based on the above constant-parameter GARCH-jump process, if we do not consider the feature of GARCH by setting parameters to be

$$\beta_0 = 0, \beta_1 = 1, \text{ and } \beta_2 = 0,$$

then our constant-parameter GARCH-jump process will reduce to the jump-diffusion process in Amin (1993) and converge to Merton (1976) option pricing model in continuous time. Similarly, if we nullify the jump parameters, leaving the GARCH parameters only, that is:

$$\lambda = \sigma_J = \mu_J = 0,$$

then the constant-parameter GARCH-jump process in Eqs. (8) and (9) can be simplified to the GARCH model of Ritchken and Trevor (1999) and converge to Duan (1995) continuous time framework. Finally, if neither features of GARCH and jump processes are considered, that is,

$$\beta_0 = 0, \beta_1 = 1, \text{ and } \beta_2 = 0, \text{ and } \lambda = \sigma_J = \mu_J = 0,$$

then the constant-parameter GARCH-jump process will degenerate to the pure-diffusion models in Cox et al. (1979) and the Kamrad and Ritchken (1991), whilst converging to the continuous time model in Black and Scholes (1973).

In addition to simulating the constant-parameter GARCH-jump, jump-diffusion, GARCH, and pure-diffusion processes, our GARCH-jump process can adapt to the general GARCH-jump model in Duan et al. (2006), where the asset price and the pricing kernel follow a GARCH-jump process and a jump-diffusion process, respectively, and the jumps in the asset price and pricing kernel are governed by the same Poisson process and the jump magnitudes of them are correlated. The details to rewrite their model to fit into our framework are presented in Appendix B.

3 Option pricing with the GARCH-jump lattice model

For pricing options, apart from the lattice model for the underlying asset price process, one also requires the variance table for each node to account for the GARCH evolutions for different asset price paths. Once the lattice model for the underlying asset price and the variance table for each node have been constructed during the forward-building process, we can apply the standard backward recursive procedure to calculate option values with the lattice model.

The crucial task of the backward recursive procedure for GARCH option pricing is to keep track of the change of the conditional variance for each node to capture the path dependence of GARCH evolutions. However, in order to avoid the exponential growth in the number of possible variances, Ritchken and Trevor (1999) approximate the state space of conditional variances at each node by M linearly interpolated values selected to span the range between the maximum and minimum conditional variances instead of tracking all conditional variances of that node. Moreover, option prices are then computed in accordance with these M levels of conditional variances at each node. Following their approach, we define $h_t^{\max}(i)$ and $h_t^{\min}(i)$ to be the maximum and minimum conditional variances at node (t, i), where t denotes the examined date and i indicates the level of the logarithmic asset price relative to the root node. Next, we introduce $h_t(i, k)$ to denote the kth level of the conditional variance at node (t, i) as follows.

$$h_t(i,k) = \frac{M-k}{M-1} h_t^{\max}(i) + \frac{k-1}{M-1} h_t^{\min}(i), \text{ for } k = 1, 2, \ldots, M. \tag{10}$$

At each node, we need to compute option prices over a grid of M points, i.e., to compute option prices corresponding to all $h_t(i, k)$ at each node (t, i). Hence we define $C_t(i, k)$ as the kth option price at node (t, i) (for $k = 1, 2, \ldots, M$) when the underlying asset price of that node is $S_t(i) = \exp[y_t(i)]$ and the examined variance is $h_t(i, k)$. Suppose that T denotes the maturity date of the examined option, and for each node at maturity, the boundary condition for a standard call option with a strike price Xcan be expressed as[6]

$$C_T(i,1) = C_T(i,2) = \cdots\cdots = C_T(i,M) = \max(0, S_T(i) - X). \tag{11}$$

[6] Here the call option is employed as an illustrative example. To value put options, the boundary condition is $C_T(i,1) = C_T(i,2) = \cdots\cdots = C_T(i,M) = \max(0, X - S_T(i))$.

During the backward recursion for each $h_t(i,k)$, we need to compute the updating variance for each possible successor node at $(t+1)$. More specifically, for the transition from the kth variance entry of node (t,i) to node $(t+1, i+j)$, for $j = 0, \pm 1, \ldots, \pm w$, the resulting variance is

$$h^{next}(j) = h_t(i,k) + \beta_0 + (\beta_1 - 1)\, h_t(i,k) + \beta_2 h_t(i,k)[v_{t+1}(j) - c^Q]^2, \quad (12)$$

where the corresponding innovation of the logarithmic asset price is

$$v_{t+1}(j) = \frac{(j\gamma_n - m_t)/\sqrt{h_t} - E^Q(\mathbf{X}_t)}{\sqrt{Var^Q(\mathbf{X}_t)}}.$$

Since only M different conditional variance levels are stored at node $(t+1, i+j)$, there may not be a variance entry exactly identical to $h^{next}(j)$ during the backward recursive procedure. In this situation, the linear interpolation of the two option prices with conditional variances nearest to $h^{next}(j)$ is employed to obtain the approximated option price for the conditional variance $h^{next}(j)$. Let L be an positive integer smaller than M, and

$$h_{t+1}(i+j, L) < h^{next}(j) \le h_{t+1}(i+j, L+1). \quad (13)$$

The interpolated option price is

$$C^{\text{interp}}(j) = \varpi(j) C_{t+1}(i+j, L) + (1 - \varpi(j)) C_{t+1}(i+j, L+1), \quad (14)$$

where

$$\varpi(j) = \frac{h_{t+1}(i+j, L+1) - h^{next}(j)}{h_{t+1}(i+j, L+1) - h_{t+1}(i+j, L)}.$$

In this way, we can obtain the option prices over all branches of $h_t(i,k)$, and thus derive the option's continuation value $C_t^{con}(i,k)$ by the following equation.

$$C_t^{con}(i,k) = \exp(-r_f) \sum_j \Pr\left(\ln S_{t+1} - \ln S_t(i) = j\gamma_n\right) C^{\text{interp}}(j), \qquad \text{for } j = 0, \pm 1, \ldots, \pm w. \quad (15)$$

If an American option is considered, the option value corresponding to the variance entry $h_t(i,k)$ will be the larger value between $C_t^{con}(i,k)$ and the exercise value for node (t,i). The current option price, obtained by the above backward recursive procedure, can be given by $C_0(0,1)$.[7]

Cakici and Topyan (2000) propose a relatively efficient method by modifying the forward-building process of Ritchken and Trevor (1999) approach. They employ only

[7] This is because all M entries of $h_0(0,k)$ are equal to the initial variance h_0, and thus all M entries of $C_0(0,k)$ should be the same and equal to the option price today.

the maximum and minimum conditional variances at time t to calculate the maximum and minimum conditional variances at $(t+1)$ in the forward-building process, rather than employing all interpolated conditional variances at t. In our lattice model, we apply Cakici and Topyan (2000) method to capture the spectrum of the conditional variance during the forward-building process. However, since they still employ the interpolated M conditional variances during the backward recursive procedure, this method cannot avoid that the conditional variance $h^{next}(j)$ based on $h_t(i,k)$ may exceed the range of the maximum and minimum conditional variances of node $(t+1, i+j)$ during the backward recursive procedure. To solve this problem, the option prices corresponding to the extreme conditional variance $h_{t+1}^{\max}(i+j)$ or $h_{t+1}^{\min}(i+j)$ are used instead.

In addition, the state space for the asset price in our model is determined by truncating extremely high and low levels of the asset prices to ensure that the state space for the underlying asset price is finite. In practice, we decide the state space in our lattice model in two steps. First, following the forward tree-building process in Cakici and Topyan (2000) and the GARCH evolution rule in Eq. (2), a GARCH option pricing tree is constructed and thus the maximum and minimum asset price levels at maturity associated with the local movements under the GARCH process are obtained. Suppose the number of possible states ranging from the maximum to the minimum asset prices at maturity is R after the first step. Second, we add w (defined in Eq. (7)) additional price levels above the "local" maximum asset price level and w additional price levels below the "local" minimum asset price level in the lattice model. Therefore, a vector of $D = R + 2w$ possible asset price levels is derived, and we use these D possible asset price levels to span the state space at each time step in our lattice model. In addition, the maximum and minimum conditional variances for each node are updated as well while spanning the state space with the GARCH-jump model. As for the nodes near the upper and lower boundaries of the state space, both the movements caused by the GARCH and jump processes are truncated, and the entire probability mass outside the upper and lower bounds is assigned to the upper and lower truncation points, respectively.

To illustrate how the GARCH-jump option pricing lattice works, we follow Ritchken and Trevor (1999) GARCH option pricing example by assuming the current underlying price $S_0 = 1000$, the daily risk-free interest rate $r_f = 0$, the GARCH parameters $\beta_0 = 6.575 \times 10^{-6}, \beta_1 = 0.9, \beta_2 = 0.04, c^Q = 0$, and the time step to be one day. As to the parameters for the jump component, we assume that $\mu_J = -0.0000125, \sigma_J = 0.005$, and the daily jump intensity parameter $\lambda = 5/365$. The initial daily variance h_0 is set to be 0.000109589, which is equivalent to the annual variance of 0.04. In the case of $n = 1$ and $M = 3$, we choose a grid of approximating logarithmic prices with the tick size $\gamma_1 = \gamma = \sqrt{h_0} = 0.0105$[8] around the initial value of the logarithmic price $y_0 = \ln S_0 = 6.9078$. For comparative purposes, we reproduce the illustrative example of the GARCH option pricing in Fig. 2 of Ritchken

[8] According to Ritchken and Trevor (1999), the setting of γ only affects the rate of convergence. Setting γ to be $\sqrt{h_0}$ here is simply for expositional purposes. In practice, we choose $\gamma = \sqrt{1.5h_0}$ to conduct our numerical analyses such that the probabilities of the three local jumps are close to 1/3 and able to improve the convergence rate of pricing results.

and Trevor (1999). Herein a three-period at-the-money European call option is priced, by courtesy of our GARCH-jump lattice model, with all jump parameters set to zero. It is worth noting that as we nullify the jump components, our GARCH-jump lattice model becomes identical to that in Ritchken and Trevor (1999). Figure 3 illustrates the valuation of the same three-period at-the-money European call option under the GARCH-jump process with our lattice model. For both Figs. 2 and 3, the maximum and minimum daily conditional variances over all possible paths reaching each node are shown in the left column of the box for each node. The right column shows the option values corresponding to these maximum and minimum daily conditional variances, as well as the option value corresponding to the midpoint daily conditional variance. In Fig. 2, it can be found that R is 9 (i.e., there are 9 levels ranging from the maximum to the minimum asset prices at maturity in Fig. 2), and since $\sigma_J = 0.005$ and $\gamma_1 = 0.0105$, we can derive w to be 2 from Eq. (7). As a consequence, our lattice model employs $D = R + 2w = 13$ possible logarithmic asset levels for each time point in Fig. 3. Comparing these two figures in details, we can distinguish that due to the extra variance introduced by the jump process, the variance range of each node in Fig. 3 is generally wider than that of the counterpart node in Fig. 2. In addition, for this set of parameters, the option is more valuable when taking the discontinuous jump process into consideration.

4 Numerical analysis

4.1 Validity test for the GARCH-jump lattice model

The first part of the numerical analysis is to verify the validity of our lattice model. To achieve this goal, several experiments are conducted to price call options under the constant-parameter GARCH-jump process specified in Sect. 2.2. To examine the convergence behavior of our lattice model, we resort to a large sample of Monte Carlo simulations since there is no analytic option pricing formula given the GARCH-jump process. Our benchmark involves the examination of 1,000,000 asset price paths so as to obtain accurate theoretical prices of European options. Consider the numerical example as follows. For the jump component, the parameter values $\lambda = 5/365$, $\sigma_J^2 = 0.05$, and $\mu_J = -\sigma_J^2/2 = -0.025$ are from Table 1 of Amin (1993). As for the GARCH parameters, $\beta_0 = 0.000006575$, $\beta_1 = 0.9$, $\beta_2 = 0.04$, $c^Q = 0$, and $h_0 = 0.000109589$ are from Table II of Ritchken and Trevor (1999). In addition, $r_f = 0$, $S_0 = \$100$, and $X = \$100$. Table 1 shows the convergence behavior of at-the-money European call option prices with respect to different number of variances M at each node given $n = 1$.The numerical results indicate that these at-the-money call prices converge into the 95 % confidence interval rapidly even with small values of M regardless of the days to maturity. For all maturities, our lattice model can generate converged option prices when M is larger than or equal to 20. Moreover, the variations of the option prices are extremely minor when M is above 20, which indicates the validity and reliability of our model to price options under the GARCH-jump process.

Moreover, as mentioned in Sect. 2.2, the GARCH-jump process considered in this paper provides a general framework and thus is able to encompass the jump-diffusion

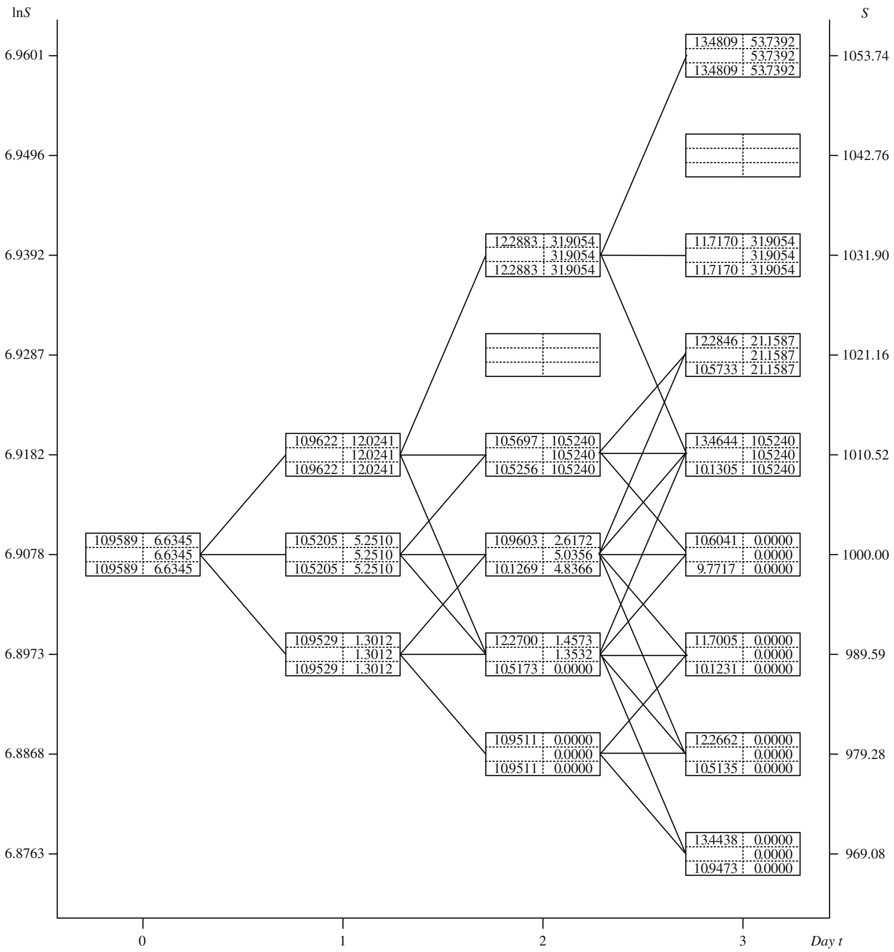

Fig. 2 Lattice model of the GARCH process for a three-period at-the-money call option. This figure reproduces Figure 2 in Ritchken and Trevor (1999), which shows the valuation of a three-period at-the-money European call option, with our GARCH-jump lattice model by setting all jump parameters to be zero. Suppose that the current underlying price $S_0 = 1000$, the daily risk-free interest rate $r_f = 0$, the GARCH parameters $\beta_0 = 6.575 \times 10^{-6}$, $\beta_1 = 0.9$, $\beta_2 = 0.04$, $c^Q = 0$, and the initial daily variance is $h_0 = 0.000109589$. In the case of $n = 1$ and $M = 3$, we choose a grid of approximating logarithmic prices with the tick size $\gamma_1 = \gamma = \sqrt{h_0} = 0.0105$ around the initial value of the logarithmic price $y_0 = \ln S_0 = 6.9078$. For each node, it is represented by a *box* containing five numbers. The top (*bottom*) number in the left column is the maximum (minimum) daily conditional variance (multiplied by 10^5) of each node. In this example, since $M = 3$, three option values are carried at each node, which are shown in the *right column*. The top (*bottom*) number is the option value corresponding to the maximum (minimum) daily conditional variance, and the middle number is the option value corresponds to the middle daily conditional variance

model in Amin (1993) and Merton (1976) and the GARCH process in Ritchken and Trevor (1999) as special cases. Hence, another way to examine the correctness of our lattice model is to price the same option examples in Amin (1993) and Ritchken and Trevor (1999). More specifically, we duplicate the results of Table I in Amin

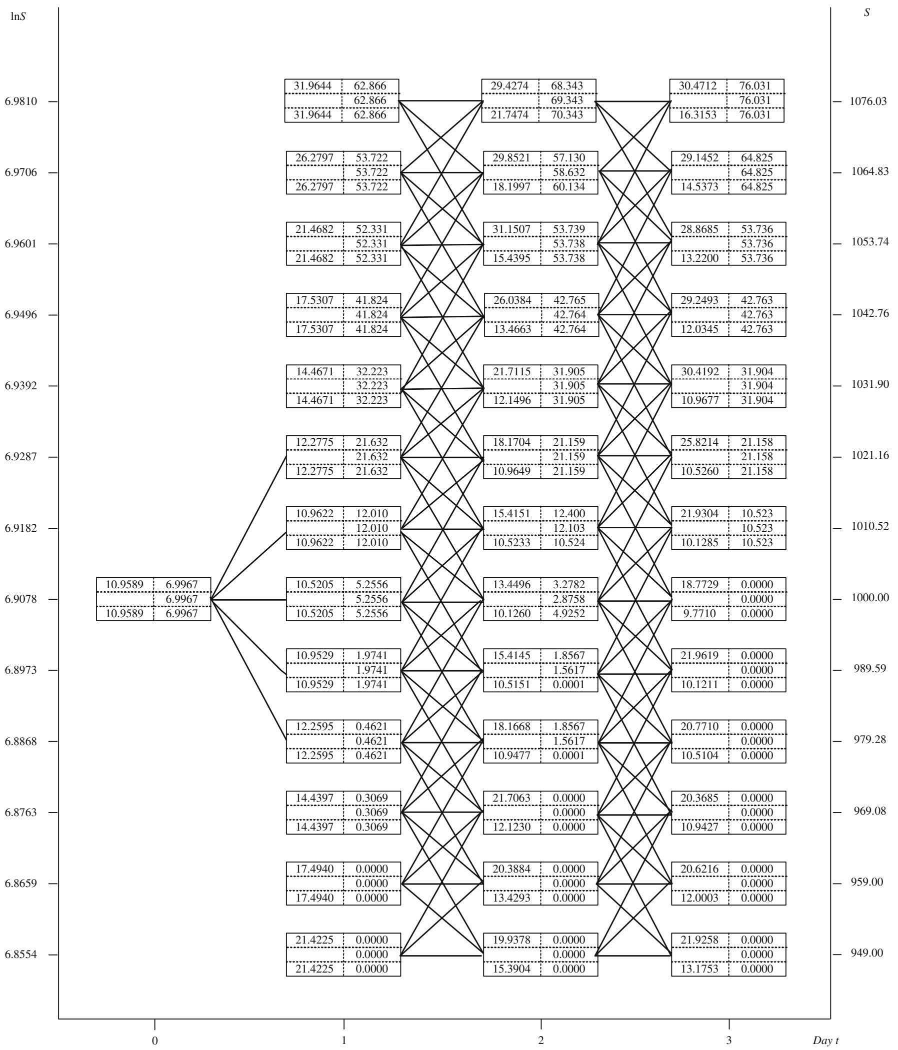

Fig. 3 Lattice model of the GARCH-jump process for a three-period at-the-money call option. This figure shows our GARCH-jump lattice model for pricing the same three-period at-the-money European call option in Fig. 2. The parameters for the GARCH process are identical to those in Fig. 2. In addition, we assume that $\mu_J = -0.0000125$, $\sigma_J = 0.005$, and the daily jump intensity parameter $\lambda = 5/365$. From the GARCH lattice in Fig. 2, it can be found that R is 9 (there are 9 levels ranging from the maximum to the minimum asset prices at maturity in Fig. 2), and since $\sigma_J = 0.005$ and $\gamma_1 = 0.0105$, we can derive w to be 2 from Eq. (7). As a consequence, our lattice model employs $D = R + 2w = 13$ possible logarithmic asset levels for each point in time. For each node, the *left column* represents the daily conditional variances (multiplied by 10^5) and the corresponding option values are reported in the *right column*. Comparing to the GARCH lattice in Fig. 2, the variance range is wider for each node and the option is more valuable in the GARCH-jump process in this figure

(1993) and Table II in Ritchken and Trevor (1999) with our lattice model. Our lattice model can generate almost identical results with these two classical tree models. The maximum differences are less than 2 cents compared with the results in Amin (1993)

Table 1 Convergence of the GARCH-jump lattice model

Number of variances (M)	Maturity of option (days)							
	5	10	20	50	75	100	150	200
5	1.4369	2.2539	3.5535	6.4746	8.3679	9.9735	12.6373	14.8378
10	1.4372	2.2547	3.5570	6.4900	8.3928	10.0063	12.6812	14.8888
20	1.4374	2.2552	3.5591	6.5004	8.4087	10.0263	12.7064	14.9185
30	1.4376	2.2555	3.5597	6.5035	8.4135	10.0323	12.7152	14.9298
40	1.4378	2.2558	3.5600	6.5046	8.4154	10.0349	12.7193	14.9355
50	1.4378	2.2560	3.5602	6.5051	8.4162	10.0364	12.7214	14.9385
∞_L	1.4315	2.2538	3.5550	6.4675	8.4029	10.0014	12.6981	14.9156
∞_U	1.4444	2.2720	3.5807	6.5250	8.4535	10.0600	12.7715	15.0030

This table shows the convergence behavior of at-the-money call options generated by our constant-parameter GARCH-jump lattice model with respect to M, the number of variances at each node. In this numerical example, we simply combine the parameters of the base examples of the jump-diffusion process in Amin (1993) and the GARCH process in Ritchken and Trevor (1999). More specifically, for the jump related parameters, $\lambda = 5/365, \sigma_J^2 = 0.05$, and $\mu_J = -\sigma_J^2/2 = -0.025$, and for the GARCH parameters, $\beta_0 = 0.000006575, \beta_1 = 0.9, \beta_2 = 0.04, c^Q = 0, h_0 = 0.000109589$. In addition, $r_f = 0, S_0 = 100, X = 100$, and the maturities of call options are from 5 to 200 days. Note that we only report the results based on $n = 1$ in this table. The two bottom rows, ∞_L and ∞_U, show the 95 % confidence intervals for the true price based on 1,000,000 simulations. The results show that our lattice model can generate converged option values inside the 95 % confidence interval when $M \geq 20$ even for $n = 1$

and Ritchken and Trevor (1999). The application of the efficient method in Cakici and Topyan (2000) to capture the variance spectrum during the forward-building process may be the reason for the pricing differences under the GARCH process. To streamline this paper, the results of these experiments are not presented but available from the authors upon request.

In conclusion, the above experiments demonstrate the accuracy and reliability of our lattice model to price European-style options under the GARCH, jump-diffusion, and GARCH-jump processes. Note that since our model with $n = 1$ can generate accurate enough results for the experiments in this section,[9] we fix $n = 1$ in the following analyses.

4.2 Jump and GARCH effects on option prices

Based on our GARCH-jump model, one interesting issue is to study the interaction between the GARCH and jump components and their individual effects on option pricing. Equipped with the constant-parameter GARCH-jump process specified in

[9] In fact, the influence of the number of subintervals in one day, n, is minor in our model as well as the GARCH model of Ritchken and Trevor (1999). In their Table I, the maximum difference among the results given $n = 1$ and $n = 25$ is only 0.025 dollars or equivalently 0.426 % of the value of the 200-day at-the-money call option.

Sect. 2.2, we conduct an analysis to compare the GARCH-jump model with the jump-diffusion or GARCH model and thus extract the degree of the GARCH or jump effect individually.

To achieve valid comparison, we need to adjust the volatilities in different models to comparable levels. To extract the GARCH effect on option prices, we first calculate the GARCH option price by nullifying the jump component. We then obtain the corresponding Black-Scholes implied volatility from the GARCH option price and compute the jump-diffusion option price with this implied volatility and the specified jump parameters. The differences between the resulting jump-diffusion option prices and our GARCH-jump option prices reflect the GARCH effect on option prices. Similarly, to extract the jump effect on option prices, we first calculate jump-diffusion option price given the assumption that the daily variance of the diffusion process is fixed at h_0. We then derive the corresponding Black-Scholes implied volatility from the jump-diffusion option price and compute the GARCH option price using the corresponding implied volatility as the initial volatility level and GARCH parameters as previously specified. The differences between the resulting GARCH option prices and our GARCH-jump option prices reflect the jump effect on option prices.

Table 2 exhibits the results of this experiment based on the same parameter values in Table 1. Column (1) shows the results of our GARCH-jump model. The results of the corresponding jump-diffusion and GARCH models are listed in Columns (2) and (3), respectively. All results are derived with our lattice model with $M = 50$ and $n = 1$. Columns (4) and (5) shows the degrees of the GARCH and jump effects expressed as absolute differences in option prices, and Columns (6) and (7) shows the degrees of the GARCH and jump effects expressed as the percentages of the option prices under the GARCH-jump model. From Columns (4), (5), (6), and (7), it is apparent that the GARCH and jump effects exhibit a similar pattern over different strike prices, i.e., the impacts of the GARCH and jump effects on option prices are negative for near-the-money options and positive for out-of-the-money and in-the-money options. This pattern confirms the evidence in many empirical studies that the introduction of either the GARCH or jump process can help explaining the excess kurtosis and thus the phenomenon of the volatility smile implied from option prices. Merton (1976) also indicates the existence of this pattern for the jump effect across different moneyness under the lognormal jump-diffusion process.[10] Therefore, we can conclude that the behavior of the jump effect across different moneyness still retains even when the GARCH model is considered concurrently. Moreover, the percentage differences in Columns (6) and (7) show that the GARCH and jump effects generally become weaker as the maturity increases. These results are in accord with our expectation because the distributions of the stock price generated by either the jump or continuous processes tend to converge to one another for longer period of time.[11]

[10] We appreciate the anonymous referee for reminding us to examine this similarity.

[11] We thank the anonymous referee for pointing out this phenomenon and its underlying reason.

Table 2 Jump and GARCH effects on option prices

Stirke price	GARCH-jump model (1)	Corresponding jump-diffusion model (2)	Corresponding GARCH model (3)	GARCH effect (4) = (1) − (2)	Jump effect (5) = (1) − (3)	GARCH effect (%) (6) = [(1) − (2)]/(1)	Jump effect (%) (7) = [(1) − (3)]/(1)
$T = 20$ days							
85	15.80076	15.79296	15.79012	0.00781	0.01065	0.049	0.067
90	11.20013	11.20359	11.20097	−0.00346	−0.00084	−0.031	−0.007
95	6.85474	6.92851	6.93063	−0.07377	−0.07590	−1.076	−1.107
100	3.56020	3.71765	3.70618	−0.15746	−0.14599	−4.423	−4.101
105	2.00416	2.09301	2.09899	−0.08885	−0.09483	−4.433	−4.732
110	1.41949	1.43141	1.41639	−0.01192	0.00310	−0.840	0.218
115	1.08474	1.07720	1.07999	0.00754	0.00475	0.695	0.438
$T = 50$ days							
80	21.38245	21.36929	21.36226	0.01316	0.02019	0.062	0.094
90	12.97479	13.05207	13.05271	−0.07728	−0.07792	−0.596	−0.601
100	6.50507	6.77152	6.75845	−0.26644	−0.25338	−4.096	−3.895
110	3.50010	3.60902	3.59559	−0.10891	−0.09548	−3.112	−2.728
120	2.17357	2.17220	2.17739	0.00137	−0.00382	0.063	−0.176
130	1.37984	1.35332	1.34815	0.02652	0.03169	1.922	2.297
$T = 100$ days							
70	31.35826	31.33150	31.32400	0.02676	0.03426	0.085	0.109
80	22.94071	22.96524	22.96345	−0.02453	−0.02274	−0.107	−0.099
90	15.62169	15.77899	15.77574	−0.15730	−0.15405	−1.007	−0.986

Table 2 continued

Stirke price	GARCH-jump model (1)	Corresponding jump-diffusion model (2)	Corresponding GARCH model (3)	GARCH effect (4) = (1) − (2)	Jump effect (5) = (1) − (3)	GARCH effect (%) (6) = [(1) − (2)]/(1)	Jump effect (%) (7) = [(1) − (3)]/(1)
100	10.03637	10.31114	10.29883	−0.27477	−0.26246	−2.738	−2.615
110	6.53480	6.72833	6.71786	−0.19352	−0.18306	−2.961	−2.801
120	4.42914	4.49852	4.50620	−0.06938	−0.07707	−1.566	−1.740
130	3.06538	3.07106	3.07933	−0.00568	−0.01394	−0.185	−0.455
140	2.14861	2.11861	2.12774	0.02999	0.02086	1.396	0.971
$T = 150$ days							
60	41.01319	40.96925	40.96901	0.04393	0.04418	0.107	0.108
80	24.49026	24.55746	24.55448	−0.06720	−0.06421	−0.274	−0.262
100	12.72137	12.97346	12.96236	−0.25210	−0.24100	−1.982	−1.894
120	6.56829	6.69162	6.69795	−0.12333	−0.12966	−1.878	−1.974
140	3.57154	3.57509	3.58132	−0.00354	−0.00977	−0.099	−0.274
160	2.03240	1.97822	1.98137	0.05418	0.05103	2.666	2.511

This table reports the effects of the GARCH and jumps on option pricing. Column (1) shows the results of GARCH-jump model with the same parameter values as those in Table 1. The results of the corresponding jump-diffusion and GARCH model are listed in Columns (2) and (3), respectively. The methods to derive the options values of corresponding jump-diffusion and GARCH models are introduced in Sect.4.2. In addition, the number of variances at each node, M, is fixed to be 50 and the number of subperiods in one day, n, is fixed to be 1. From Columns (4) and (5), it can be found that the GARCH and jump effects exhibit a similar pattern across different moneyness—both the GARCH and jump effects on option prices are negative for near-the-money options and positive for out-of-the-money and in-the-money options. These results are consistent with many existing empirical studies that the introduction of either the GARCH or the jump process can help explaining the excess kurtosis and thus the volatility smile implied from option prices. Moreover, the percentage differences in Columns (6) and (7) demonstrate that the GARCH and jump effects reduce gradually as the maturity increases. These results are in accord with our expectation because the distributions of the stock price generated by either the jump or continuous processes tend to converge to one another over a longer-period horizon

Table 3 Sensitivity analysis of the early exercise premium for American puts

Panel A. Percentage proportions of the early exercise premiums for at-the-money puts (%)

GARCH-jump model (1)

$T = 10$ days, $X = 100$							$T = 50$ days, $X = 100$						
Mean of jump magnitude μ_J	Daily jump intensity λ						Mean of jump magnitude μ_J	Daily jump intensity λ					
	GARCH (no jump)	1/365	2/365	3/365	4/365	5/365		GARCH (no jump)	1/365	2/365	3/365	4/365	5/365
−1		0.724	0.161	0.164	0.167	0.171	−1		2.397	1.258	1.087	1.004	0.932
−0.75		0.628	0.140	0.142	0.143	0.145	−0.75		2.220	1.102	0.960	0.883	0.819
−0.5		0.525	0.121	0.121	0.121	0.121	−0.5		2.029	0.958	0.839	0.769	0.710
−0.25		0.418	0.104	0.103	0.101	0.100	−0.25		1.821	0.828	0.725	0.661	0.609
0	2.015	0.309	0.089	0.086	0.083	0.082	0	5.193	1.602	0.711	0.621	0.563	0.516
0.25		0.200	0.075	0.071	0.068	0.066	0.25		1.374	0.606	0.526	0.473	0.432
0.5		0.123	0.063	0.058	0.055	0.053	0.5		1.150	0.512	0.441	0.393	0.358
0.75		0.078	0.052	0.047	0.044	0.042	0.75		0.938	0.429	0.365	0.323	0.292
1		0.054	0.042	0.038	0.034	0.033	1		0.749	0.357	0.300	0.263	0.236

Jump-diffusion model (2)

$T = 10$ days, $X = 100$							$T = 50$ days, $X = 100$						
Mean of jump magnitude μ_J	Daily jump intensity λ						Mean of jump magnitude μ_J	Daily jump intensity λ					
	GBM (no Jump)	1/365	2/365	3/365	4/365	5/365		GBM (no Jump)	1/365	2/365	3/365	4/365	5/365
−1		0.699	0.133	0.139	0.144	0.147	−1		2.354	1.367	1.137	1.039	0.960
−0.75		0.615	0.115	0.120	0.123	0.125	−0.75		2.208	1.206	1.007	0.918	0.846
−0.5		0.524	0.100	0.103	0.104	0.105	−0.5		2.046	1.052	0.884	0.802	0.737
−0.25		0.427	0.087	0.088	0.088	0.087	−0.25		1.872	0.910	0.768	0.694	0.635

Table 3 continued

$T = 10$ days, $X = 100$							$T = 50$ days, $X = 100$						
Mean of jump magnitude μ_J	Daily jump intensity λ						Mean of jump magnitude μ_J	Daily jump intensity λ					
	GBM (no jump)	1/365	2/365	3/365	4/365	5/365		GBM (no jump)	1/365	2/365	3/365	4/365	5/365
0	2.042	0.322	0.074	0.074	0.073	0.072	0	5.247	1.685	0.781	0.660	0.593	0.541
0.25		0.216	0.063	0.062	0.060	0.058	0.25		1.488	0.664	0.561	0.501	0.455
0.5		0.131	0.053	0.051	0.049	0.047	0.5		1.289	0.561	0.472	0.419	0.378
0.75		0.077	0.044	0.041	0.039	0.037	0.75		1.092	0.470	0.393	0.346	0.311
1		0.050	0.037	0.033	0.031	0.029	1		0.904	0.390	0.324	0.282	0.253

Difference between GARCH-jump and jump-diffusion models ((3) = (1) − (2))

$T = 10$ days, $X = 100$							$T = 50$ days, $X = 100$						
Mean of jump magnitude μ_J	Daily jump intensity λ						Mean of jump magnitude μ_J	Daily jump intensity λ					
	GARCH minus GBM (no jump)	1/365	2/365	3/365	4/365	5/365		GARCH minus GBM (no jump)	1/365	2/365	3/365	4/365	5/365
−1		0.025	0.028	0.026	0.024	0.024	−1		0.042	−0.108	−0.050	−0.035	−0.027
−0.75		0.013	0.024	0.021	0.020	0.020	−0.75		0.013	−0.104	−0.048	−0.035	−0.027
−0.5		0.001	0.021	0.018	0.016	0.016	−0.5		−0.017	−0.094	−0.045	−0.034	−0.027
−0.25		−0.009	0.017	0.015	0.013	0.013	−0.25		−0.051	−0.082	−0.042	−0.032	−0.026
0	−0.027	−0.014	0.015	0.012	0.010	0.010	0	−0.054	−0.083	−0.070	−0.039	−0.030	−0.024
0.25		−0.016	0.012	0.009	0.008	0.008	0.25		−0.114	−0.059	−0.035	−0.028	−0.023
0.5		−0.008	0.009	0.007	0.006	0.006	0.5		−0.139	−0.049	−0.032	−0.025	−0.021
0.75		0.001	0.007	0.006	0.005	0.005	0.75		−0.153	−0.040	−0.028	−0.023	−0.018
1		0.004	0.006	0.004	0.003	0.004	1		−0.155	−0.033	−0.024	−0.020	−0.016

Table 3 continued

Panel B. Percentage proportions of the early exercise premiums for out-of-the-money puts (%)

GARCH-jump model (1)

$T = 10$ days, $X = 96$							$T = 50$ days, $X = 90$						
Mean of jump magnitude μ_J	Daily jump intensity λ						Mean of jump magnitude μ_J	Daily jump intensity λ					
	GARCH (no jump)	1/365	2/365	3/365	4/365	5/365		GARCH (no jump)	1/365	2/365	3/365	4/365	5/365
−1		0.507	0.373	0.317	0.283	0.266	−1		1.952	1.501	1.195	1.010	0.890
−0.75		0.467	0.338	0.283	0.250	0.233	−0.75		1.865	1.372	1.076	0.903	0.791
−0.5		0.425	0.304	0.250	0.219	0.202	−0.5		1.769	1.243	0.961	0.801	0.698
−0.25		0.380	0.270	0.219	0.189	0.174	−0.25		1.665	1.119	0.853	0.705	0.611
0	0.772	0.334	0.238	0.190	0.162	0.147	0	2.634	1.552	0.999	0.751	0.615	0.529
0.25		0.286	0.208	0.163	0.137	0.124	0.25		1.430	0.885	0.656	0.531	0.454
0.5		0.242	0.179	0.138	0.115	0.103	0.5		1.303	0.778	0.567	0.455	0.386
0.75		0.209	0.153	0.116	0.095	0.084	0.75		1.175	0.678	0.486	0.385	0.323
1		0.181	0.129	0.096	0.078	0.068	1		1.048	0.585	0.412	0.322	0.268

Jump-diffusion model (2)

$T = 10$ days, $X = 96$							$T = 50$ days, $X = 90$						
Mean of jump magnitude μ_J	Daily jump intensity λ						Mean of jump magnitude μ_J	Daily jump intensity λ					
	GBM (no jump)	1/365	2/365	3/365	4/365	5/365		GBM (no jump)	1/365	2/365	3/365	4/365	5/365
−1		0.420	0.300	0.262	0.238	0.224	−1		2.299	1.703	1.318	1.083	0.931
−0.75		0.391	0.271	0.233	0.210	0.195	−0.75		2.192	1.559	1.186	0.966	0.826
−0.5		0.358	0.243	0.205	0.183	0.168	−0.5		2.072	1.412	1.056	0.853	0.726
−0.25		0.323	0.216	0.179	0.158	0.144	−0.25		1.940	1.265	0.932	0.747	0.632

Table 3 continued

T = 10 days, X = 96							T = 50 days, X = 90						
Mean of jump magnitude μ_J	Daily jump intensity λ						Mean of jump magnitude μ_J	Daily jump intensity λ					
	GBM (no jump)	1/365	2/365	3/365	4/365	5/365		GBM (no jump)	1/365	2/365	3/365	4/365	5/365
0	0.784	0.284	0.189	0.155	0.135	0.122	0	2.660	1.796	1.121	0.813	0.647	0.544
0.25		0.242	0.165	0.133	0.114	0.102	0.25		1.640	0.982	0.702	0.553	0.463
0.5		0.203	0.142	0.112	0.095	0.084	0.5		1.476	0.850	0.599	0.468	0.389
0.75		0.174	0.121	0.094	0.078	0.068	0.75		1.309	0.728	0.505	0.390	0.322
1		0.151	0.101	0.077	0.063	0.055	1		1.144	0.615	0.420	0.321	0.263

Difference between GARCH-jump and jump-diffusion models ((3) = (1) – (2))

T = 10 days, X = 96							T = 50 days, X = 90						
Mean of jump magnitude μ_J	Daily jump intensity λ						Mean of jump magnitude μ_J	Daily jump intensity λ					
	GARCH minus GBM (no jump)	1/365	2/365	3/365	4/365	5/365		GARCH minus GBM (no jump)	1/365	2/365	3/365	4/365	5/365
−1		0.086	0.074	0.055	0.045	0.042	−1		−0.347	−0.202	−0.124	−0.072	−0.041
−0.75		0.076	0.067	0.050	0.041	0.038	−0.75		−0.327	−0.187	−0.110	−0.063	−0.035
−0.5		0.066	0.061	0.045	0.036	0.034	−0.5		−0.303	−0.169	−0.095	−0.053	−0.028
−0.25		0.057	0.055	0.040	0.032	0.030	−0.25		−0.276	−0.146	−0.078	−0.042	−0.022
0	−0.012	0.049	0.049	0.035	0.028	0.026	0	−0.025	−0.245	−0.122	-0.062	−0.032	−0.015
0.25		0.044	0.043	0.030	0.024	0.022	0.25		−0.210	−0.097	−0.046	−0.022	−0.009
0.5		0.039	0.037	0.026	0.020	0.019	0.5		−0.172	−0.073	−0.032	−0.013	−0.003
0.75		0.035	0.032	0.022	0.017	0.016	0.75		−0.134	−0.050	−0.019	−0.005	0.001
1		0.030	0.027	0.019	0.014	0.013	1		−0.096	−0.030	−0.008	0.001	0.005

Table 3 continued

Panel C. Percentage proportions of the early exercise premiums for in-the-money puts (%)

GARCH-Jump Model (1)

$T = 10$ days, $X = 104$							$T = 50$ days, $X = 110$						
Mean of jump magnitude μ_J	Daily jump intensity λ						Mean of jump magnitude μ_J	Daily jump intensity λ					
	GARCH (no jump)	1/365	2/365	3/365	4/365	5/365		GARCH (no jump)	1/365	2/365	3/365	4/365	5/365
−1		1.143	0.112	0.079	0.087	0.095	−1		5.042	1.826	0.967	0.854	0.839
−0.75		1.006	0.077	0.068	0.074	0.080	−0.75		4.680	1.518	0.831	0.751	0.738
−0.5		0.859	0.058	0.058	0.063	0.067	−0.5		4.299	1.240	0.713	0.656	0.644
−0.25		0.712	0.047	0.050	0.053	0.056	−0.25		3.908	1.003	0.610	0.569	0.558
0	3.057	0.562	0.040	0.042	0.044	0.046	0	9.723	3.507	0.807	0.521	0.490	0.480
0.25		0.413	0.034	0.035	0.036	0.037	0.25		3.098	0.649	0.443	0.419	0.409
0.5		0.272	0.028	0.029	0.029	0.030	0.5		2.683	0.522	0.375	0.356	0.347
0.75		0.160	0.024	0.024	0.023	0.024	0.75		2.275	0.421	0.315	0.300	0.291
1		0.086	0.020	0.019	0.019	0.019	1		1.884	0.340	0.264	0.251	0.243

Jump-Diffusion Model (2)

$T = 10$ days, $X = 104$							$T = 50$ days, $X = 110$						
Mean of jump magnitude μ_J	Daily jump intensity λ						Mean of jump magnitude μ_J	Daily jump intensity λ					
	GBM (no jump)	1/365	2/365	3/365	4/365	5/365		GBM (no jump)	1/365	2/365	3/365	4/365	5/365
−1		1.101	0.112	0.071	0.079	0.086	−1		4.364	1.902	1.098	0.929	0.888
−0.75		0.982	0.073	0.061	0.068	0.073	−0.75		4.119	1.654	0.954	0.822	0.787
−0.5		0.855	0.054	0.053	0.058	0.062	−0.5		3.853	1.413	0.825	0.722	0.692
−0.25		0.721	0.043	0.046	0.049	0.051	−0.25		3.566	1.189	0.711	0.630	0.604

Table 3 continued

T = 10 days, X = 104							T = 50 days, X = 110						
Mean of jump magnitude μ_J	Daily jump intensity λ						Mean of jump magnitude μ_J	Daily jump intensity λ					
	GBM (no jump)	1/365	2/365	3/365	4/365	5/365		GBM (no jump)	1/365	2/365	3/365	4/365	5/365
0	3.089	0.584	0.036	0.039	0.041	0.043	0	9.709	3.269	0.988	0.610	0.546	0.523
0.25		0.440	0.031	0.033	0.034	0.035	0.25		2.964	0.814	0.520	0.470	0.449
0.5		0.305	0.026	0.027	0.028	0.028	0.5		2.653	0.667	0.442	0.401	0.383
0.75		0.188	0.022	0.022	0.023	0.023	0.75		2.335	0.544	0.373	0.340	0.324
1		0.102	0.018	0.018	0.018	0.018	1		2.017	0.443	0.313	0.286	0.272

Difference Between GARCH-Jump and Jump-Diffusion Models ((3) = (1) − (2))

T = 10 days, X = 104							T = 50 days, X = 110						
Mean of jump magnitude μ_J	Daily jump intensity λ						Mean of jump magnitude μ_J	Daily jump intensity λ					
	GARCH minus GBM (no jump)	1/365	2/365	3/365	4/365	5/365		GARCH minus GBM (no jump)	1/365	2/365	3/365	4/365	5/365
−1		0.042	0.000	0.008	0.008	0.009	−1		0.678	−0.076	−0.131	−0.075	−0.049
−0.75		0.023	0.004	0.007	0.006	0.007	−0.75		0.561	−0.136	−0.123	−0.071	−0.049
−0.5		0.004	0.005	0.005	0.005	0.005	−0.5		0.446	−0.174	−0.112	−0.066	−0.048
−0.25		−0.008	0.005	0.004	0.004	0.004	−0.25		0.341	−0.186	−0.101	−0.062	−0.046

Table 3 continued

T = 10 days, X = 104							T = 50 days, X = 110						
Mean of jump magnitude μ_J	Daily jump intensity λ						Mean of jump magnitude μ_J	Daily jump intensity λ					
	GARCH minus GBM (no jump)	1/365	2/365	3/365	4/365	5/365		GARCH minus GBM (no jump)	1/365	2/365	3/365	4/365	5/365
0	−0.032	−0.022	0.004	0.003	0.003	0.003	0	0.014	0.238	−0.181	−0.089	−0.056	−0.043
0.25		−0.027	0.003	0.002	0.002	0.002	0.25		0.133	−0.165	−0.078	−0.051	−0.040
0.5		−0.033	0.002	0.002	0.001	0.001	0.5		0.030	−0.144	−0.067	−0.046	−0.036
0.75		−0.027	0.002	0.001	0.001	0.001	0.75		−0.061	−0.123	−0.058	−0.040	−0.032
1		−0.015	0.001	0.001	0.000	0.001	1		−0.133	−0.103	−0.049	−0.035	−0.028

* GBM is the abbreviation of the geometric Brownian motion

This table shows the sensitivity analysis for the percentage proportions of the early exercise premiums of at-the-money, out-of-the-money, and in-the-money American put options with respect to the jump intensity and the expected jump magnitude. In this analysis, all parameter values are the same as those in Table 1 except that the daily risk-free interest rate is set as $r_f = 0.1/365$, and all results are obtained by our lattice model with M fixed to be 50 and n fixed to be 1. It can be found that in both the GARCH-jump and jump-diffusion models, the percentage proportions of the early exercise premium decrease with the increases in the jump intensity and the mean of jump magnitude, which means the introduction of the jump process impedes the early exercise of American options and thus reduces the early exercise premium even after the GARCH model is considered. This negative jump effect on the percentage proportions of the early exercise premiums weakens as the maturity increases. In addition, through analyzing the differences between the results based on the GARCH-jump and jump-diffusion models, we observe the negative GARCH effect without jumps and also discover the positive impact from the interaction between the GARCH and jumps on the percentage proportions of the early exercise premiums. Through comparing the results of 10-day and 50-day American puts, it can be found that this positive impact from the interaction between the GARCH and jumps declines as the maturity increases

4.3 Early exercise premium for American options

In previous subsections, our lattice model is employed to price only European options under the GARCH-jump process. However, note that one important reason to develop this lattice model is its ability to price American options. In this subsection, we not only compute the early exercise premiums of American options, but also analyze the relationship between the early exercise premium and the jump related parameters. In the literature, Amin (1993) indicates that significant jumps reduce the possibility for early exercise of American options and thus reduce the early exercise premium under the jump-diffusion process. This paper intends to reinvestigate this issue when both the jump process and the GARCH model are considered.

To study this issue, we conduct a sensitivity analysis for the early exercise premium with respect to different levels of the jump intensity and expected jump magnitude. Table 3 reports the early exercise premium expressed as the percentage of the American put value given the values of the daily jump intensity as 0, 1/365, …, 5/365 and the values of expected jump magnitude as $-1, -0.75, \ldots, 1$. Other parameter values are identical to those in Table 1 except that the daily risk-free interest rate $r_f = 0.1/365$. To compute these results, M is set to be 50 and n is set to be 1 in our computer program. The results for at-the-money, out-of-the-money, and in-the-money put options are presented in Panels A, B, and C, respectively.[12] In each panel, there are three parts, which contain the results for the GARCH-jump model, the jump-diffusion model, and the differences between them.

First, we can find from Table 3 that under both the GARCH-jump and jump-diffusion models, the percentage proportions of the early exercise premiums decrease with the increase of the jump intensity. Our results not only verify the results in Amin (1993) but also show that this impact of the jump effect on the early exercise premium still exists even when the autoregressive conditional heteroskedasticity is considered. Moreover, when short-term options (the 10-day puts in Table 3) are considered, the impact of the jump effect to impede the early exercise decision is more pronounced. Taking the 10-day puts under the GARCH-jump model in Panel A as example, when the daily jump intensity changes from 0 (i.e., no jump case) to 5/365 given $\mu_J = -1$, the percentage proportion of the early exercise premium changes from 2.015 to 0.0171 %, which is less than one-hundredth of 2.015 % under the no jump case. For the counterpart results of the 50-day puts in Panel A, when the daily jump intensity changes from 0 to 5/365 given $\mu_J = -1$, the percentage proportion of the early exercise premium changes from 5.193 to 0.932 %, which is about one-fifth of 5.193 % under the no jump case. The decay of the jump impacts over the time to maturity is consistent with the conclusions in Bakshi et al. (1997, 2000) and Kim et al. (2007) that the effect of the jump process is more pronounced for shorter-term horizons to capture the non-zero skewness and excess kurtosis.

[12] For 10-day puts, the strike prices of 96, 100, and 104 are examined. For 50-day puts, the values for the strike prices are 90, 100, and 110. The reason to consider different strike prices for 10-day and 50-day puts is because if we consider the strike prices of 90 and 110 for 10-day put options, the put options are extremely in-the-money or out-of-the-money, and thus the time value of the option is very small and too sensitive to be analyzed.

Second, for the expected jump magnitude, if it is negative, the probability for American puts to be in the money and thus early exercised increases during the option life because of the occasional, on average downward jumps in the underlying asset price. Therefore, the percentage proportions of the early exercise premiums are higher for the cases with negative expected jump magnitudes than those for the cases with positive expected jump magnitudes.

Third, when we consider the no jump cases and compare the differences of the percentage proportions of the early exercise premiums between the GARCH-jump and jump-diffusion models in the third part of each panel, it is equivalent to compare the percentage proportions of the early exercise premium under the GARCH and geometric Brownian motion (GBM) models. These differences are in general negative, e.g., −0.027 % for 10-day puts and −0.054 % for 50-day puts in Panel A, which indicate the negative GARCH effect on the early exercise premium given our examined parameter values.[13] However, when we consider the cases with jumps, i.e., λ is nonzero, and then reexamine the differences between the GARCH-jump and jump-diffusion models, the differences of the percentage proportions of the early exercise premiums are in general positive for 10-day puts but negative for 50-day puts. According to these observations, we can infer that the interaction between the GARCH model and the jump process can increase the percentage proportions of the early exercise premiums particularly for shorter term horizons, so the negative GARCH impacts in the no jump cases are reversed to be positive for the cases with jumps for 10-day puts. However, for longer term horizons, i.e., 50-day puts in Panel A, the results imply that the effect of the interaction of the GARCH and jumps on the percentage portions of the early exercise premiums weakens, so the negative GARCH impacts still dominate and thus the differences of the percentage proportions of the early exercise premiums between the GARCH-jump and jump-diffusion models are in general negative. These phenomena may be explained by the first finding in Table 3 that the jump effect declines as the maturity increases, so the weaker jump effect for longer term horizons reduces the net impact from the interaction between the GARCH model and the jump process on the percentage proportions of the early exercise premiums.

In conclusion, Table 3 shows that the early exercise premium expressed as the percentage proportion of the American put value decreases with the increases in the jump intensity and the mean of the jump magnitude under both the GARCH-jump and jump-diffusion models. Consequently, we conclude that the introduction of the jump process impedes the early exercise of American options and thus reduces the early exercise premium even after the autoregressive conditional heteroskedasticity is considered. Moreover, this negative jump effect becomes weaker for longer term horizons. In addition, we also discover the positive impact on the percentage proportions of the early exercise premiums from the interaction between the GARCH and jumps, and this positive impact declines as the maturity increases. Since all of the three panels

[13] The only exception is the in-the-money 50-day puts under the no jump assumption in Panel C. Since the American puts under both the GARCH and GBM models are early exercised at $t = 0$, the American put values are raised to be 10 in both models. Due to these discontinuous rising changes, the degree of the GARCH effect on the percentage proportions of the early exercise premiums is distorted to be positive. Note the early exercise occurs only for the no jump cases, as for other 50-day American puts with nonzero λ in Panel C, they are not early exercised at $t = 0$.

in Table 3 share these common characteristics, we conclude that all the above findings are robust across different moneyness.

4.4 Convergence to Duan et al. (2006) model

Another merit of our GARCH-jump process and the corresponding lattice model is the ability to adapt to Duan et al. (2006) highly general GARCH-jump process with appropriate settings. To confirm this, the example in their Table 4.1 is considered, and their GARCH-jump model is rewritten to fit into our model in Appnedix B. Duan et al. (2006) estimate the values of the parameters in their GARCH-jump model based on the series of S&P 500 stock index. There are three pricing kernel related parameters, b, μ, and δ in their model. The estimation of $b = -0.0723$, and since not all of pricing kernel related parameters can be fully identified from the time series data, both the pricing kernel related parameter δ and the jump-intensity adjusting factor κ are restricted to be 1. The restriction for κ results in the irrelevance for the value of the pricing kernel related parameter μ because μ only appears to determine the jump-intensity adjusting factor κ in their model under the risk-neutral measure. For the parameters associated with the underlying asset price, $\bar{\mu} = 0.0332$ and $\bar{\gamma} = 2.096$. The correlation coefficient for the jumps in the asset price and in the pricing kernel, ρ, is assumed to be 1, and the daily jump intensity parameter is $\lambda = 2.2/365$. For the GARCH parameters, $\beta_0 = 0.000000165$, $\beta_1 = 0.844$, $\beta_2 = 0.0756$, $c^P = 0.7714$,

Table 4 Convergence behavior given Duan et al. (2006) GARCH-jump model

Number of variances	Maturity of option (days)					
(M)	10	20	30	40	50	75
5	17.8574	29.2508	39.1542	48.1150	56.3608	74.5720
10	17.8577	29.2537	39.1653	48.1423	56.4147	74.7492
20	17.8578	29.2549	39.1697	48.1536	56.4377	74.8297
30	17.8578	29.2552	39.1709	48.1567	56.4439	74.8529
40	17.8578	29.2553	39.1714	48.1581	56.4468	74.8631
50	17.8578	29.2554	39.1717	48.1588	56.4483	74.8687
∞_L	17.7902	29.1118	39.0328	47.9934	56.0658	74.7780
∞_U	18.0553	29.4968	39.5013	48.5413	56.6812	75.6223

This table shows the convergence behavior of the call option values based on our lattice model with $n = 1$ to Duan et al. (2006)'s simulation-based results with respect to different number of variance M at each node. As per Table 4.1 in Duan et al. (2006), we set the pricing kernel related parameters $b = -0.0723$ and for the asset price related parameters, we consider $\bar{\mu} = 0.0332$ and $\bar{\gamma} = 2.096$. Since not all parameters can be identified from the time series data, especially the parameters associated with the pricing kernel, Duan et al. (2006) restrict that $\kappa = 1$ and $\delta = 1$. The correlation coefficient for the jumps in the asset price and in the pricing kernel, ρ, is assumed to be 1, and the daily jump intensity parameter is $\lambda = 2.2/365$. For the GARCH parameters, $\beta_0 = 0.000000165$, $\beta_1 = 0.844$, $\beta_2 = 0.0756$, and $c^P = 0.7714$, the initial daily variance is $h_0 = 0.09/365$. In addition, the daily risk-free interest rate $r_f = 0.05/365$, the initial asset price $S_0 = 500$, the strike price $X = 500$, and the time increment Δt is set to be one day. The lower and upper bounds (denoted as ∞_L and ∞_U) represent 95 % confidence interval based on 2,000,000 simulations. The results demonstrate that our model can adapt to Duan et al. (2006) model and converge to theoretical option values with a moderate value of M

and the initial daily variance is $h_0 = 0.09/365$. In addition, the daily risk-free interest rate $r_f = 0.05/365$, the initial asset price $S_0 = 500$, the strike price $X = 500$, and the time increment Δt is set to be one day. Table 4 shows the convergence behavior of the option values based on our lattice model with $n = 1$ to Duan et al. (2006) simulation-based results. The lower and upper bounds for the simulations represent the 95 % confidence interval based on 2,000,000 simulations. For all days to maturity except 75 days, call option prices obtained by our lattice model converge into the 95 % confidence interval even when M equals 5. The option values tend to increase as M increases, until they converge to a stable value. For the maturity equal to 75 days, option prices according to our lattice model converge to the simulated prices when M is larger than or equal to 20. Generally speaking, for longer maturities, a larger value of M is needed to generate convergent option values.

5 Conclusion

In this paper, we develop a generalized GARCH-jump lattice model by extending the GARCH option pricing tree in Ritchken and Trevor (1999) to incorporate a jump process. Owing to the difficulty in obtaining an analytical option pricing formula as the price of underlying security follows a diffusion process with the GARCH and jump effects incorporated, our integrated lattice model contributes to the literature by providing an efficient option pricing method other than the traditional method of Monte Carlo simulation. Numerical results indicate that the option values generated by our model are consistent with the results based on the Monte Carlo simulation for pricing European options under the GARCH-jump process.

Our analyses find that both the GARCH and jump effects are negative on the values of near-the-money options but positive on the values of in-the-money and out-of-the-money options. These results confirm the evidence in existing literature that the introduction of either the GARCH or jump process can help explaining the excess kurtosis and thus the phenomenon of the volatility smile implied from option prices. Moreover, this lattice model enables us to effectively price, in particular, American-style options. For American options, our numerical results show that even after the autoregressive conditional heteroskedasticity is considered, the introduction of the jump process impedes the early exercise and thus reduces the percentage proportions of the early exercise premium on the price of American options, particularly for shorter-term horizons. Moreover, the interaction between the GARCH and jump processes can increase the percentage proportions of the early exercise premiums on American option prices, whereas this effect becomes weaker as the time to maturity increases.

The GARCH-jump process considered in this paper and our corresponding lattice model are quite general, which can adapt to Duan et al. (2006) sophisticated GARCH-jump model for pricing both European- and American-style options. In addition, several classical option pricing tree models, such as Amin (1993) for the jump-diffusion process, Ritchken and Trevor (1999) for the GARCH process, as well as Cox et al. (1979) and Kamrad and Ritchken (1991) for the pure-diffusion model, are nested to our generalized lattice model. Therefore, our lattice model can be a useful tool to conduct empirical studies or numerical experiments to compare option prices under different assumptions of the underlying asset price process. This feature again demonstrates the necessity and importance to develop the GARCH-jump option pricing lattice model in this paper.

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Appendix A

This appendix presents the details to derive P_u, P_m, and P_d. Suppose that each time step (that is one day in this paper) is partitioned into n subintervals of equal length, $1/n$. The drift rate and conditional variance are constant over each of these subintervals and updated only at the end of each day. Therefore, for the diffusion part of our GARCH-jump model, the logarithmic asset price over each subinterval follows a normal distribution:

$$y_{t+1/n} \sim N^Q(y_t + \mu_t/n, \sigma_t^2/n)$$

where μ_t and σ_t^2 represent the daily conditional mean and variance.

Suppose this normal process of $y_{t+1/n}$ can be approximated by a trinomial tree model given three discrete values at time $(t + 1/n)$: $y_t + \eta\gamma_n$, y_t, and $y_t - \eta\gamma_n$. To match the mean and the variance of the normal distribution of $y_{t+1/n}$, the following system of linear equations with unknowns P_u, P_m, and P_d can be obtained.

$$\begin{cases} P_u(y_t + \eta\gamma_n) + P_m y_t + P_d(y_t - \eta\gamma_n) = y_t + \mu_t/n & (16) \\ P_u\,(y_t + \eta\gamma_n)^2 + P_m(y_t)^2 + P_d(y_t - \eta\gamma_n)^2 - (y_t + \mu_t/n)^2 = \sigma_t^2/n & (17) \\ P_u + P_m + P_d = 1 & (18) \end{cases}$$

Substituting Eq. (18) into Eq. (16) leads to

$$P_u\eta\gamma_n - P_d\eta\gamma_n = \mu_t/n \Rightarrow P_u = P_d + \frac{1}{\eta\gamma_n}\frac{\mu_t}{n}. \tag{19}$$

Substituting Eq. (18) into Eq. (17) can generate

$$P_u\left(2y_t\eta\gamma_n + \eta^2\gamma_n^2\right) + P_d\left(-2y_t\eta\gamma_n + \eta^2\gamma_n^2\right) - 2y_t\frac{\mu_t}{n} - \left(\frac{\mu_t}{n}\right)^2 = \frac{\sigma_t^2}{n}. \tag{20}$$

Rewrite Eq. (20) by replacing P_u with $P_d + \frac{1}{\eta\gamma_n}\frac{\mu_t}{n}$ according to Eq. (19),

$$\begin{aligned} &2P_d\eta^2\gamma_n^2 + 2y_t\tfrac{\mu_t}{n} + \eta\gamma_n\tfrac{\mu_t}{n} - 2y_t\tfrac{\mu_t}{n} - \left(\tfrac{\mu_t}{n}\right)^2 = \tfrac{\sigma_t^2}{n} \\ &\Rightarrow 2P_d\eta^2\gamma_n^2 + \eta\gamma_n\tfrac{\mu_t}{n} - \left(\tfrac{\mu_t}{n}\right)^2 = \tfrac{\sigma_t^2}{n} \\ &\Rightarrow 2P_d\eta^2\gamma_n^2 + \eta\gamma_n\tfrac{\mu_t}{n} \approx \tfrac{\sigma_t^2}{n}. \end{aligned} \tag{21}$$

The last equation holds due to the relatively small value of $(\mu_t/n)^2$. In fact, if we explicitly consider the variable Δt (rather than a fixed interval of one day) in our calculation process, the counterpart of the term $(\mu_t/n)^2$ is proportional to Δt^2, which approaches zero and can be ignored when Δt is small. Finally, we can solve P_d as follows.

$$P_d = \frac{\sigma_t^2/n}{2\eta^2\gamma_n^2} - \frac{\mu_t/n}{2\eta\gamma_n} = \frac{\sigma_t^2}{2\eta^2\gamma^2} - \frac{\mu_t\sqrt{1/n}}{2\eta\gamma},$$

where the second equation is according to the definition of $\gamma_n = \gamma/\sqrt{n}$ in Eq. (5). Consequently, P_u can be solved via $P_u = P_d + \frac{1}{\eta\gamma_n}\frac{\mu_t}{n}$, and next P_m can be solved via Eq. (18). The solutions of P_u, P_m, and P_d are summarized as follows.

$$\begin{cases} P_u = \frac{\sigma_t^2}{2\eta^2\gamma^2} + \frac{\mu_t\sqrt{1/n}}{2\eta\gamma} \\ P_m = 1 - \frac{\sigma_t^2}{\eta^2\gamma^2} \\ P_d = \frac{\sigma_t^2}{2\eta^2\gamma^2} - \frac{\mu_t\sqrt{1/n}}{2\eta\gamma} \end{cases}.$$

The above system of probabilities are consistent with the results in Ritchken and Trevor (1999).

However, these probabilities should be adjusted when they are incorporated with a jump process in the way specified in Eqs. (6) and (7). Note that the adjustment $(1-\lambda)P(\theta)$ in Eq. (6) causes effectively the trinomial probabilities for simulating the diffusion process in each subinterval reduced by a multiplying factor of $(1-\lambda)^{(1/n)}$ and these smaller probabilities will underestimate the mean and the variance in each subinterval by a multiplying factor of $(1-\lambda)^{(1/n)}$.[14] Similar to the solution proposed in Amin (1993), the drift and variance terms of the diffusion process should be adjusted by dividing $(1-\lambda)^{(1/n)}$ in order to guarantee the correct mean and variance of the diffusion part. Consequently, we set the per-subinterval drift rate μ_t/n and conditional variance σ_t^2/n to be $(m_t/n)/(1-\lambda)^{(1/n)}$ and $(h_t/n)/(1-\lambda)^{(1/n)}$, or equivalently $\mu_t = m_t/(1-\lambda)^{(1/n)}$ and $\sigma_t^2 = h_t/(1-\lambda)^{(1/n)}$, in the above system to derive P_u, P_m, and P_d in Eq. (6).

Appendix B

In Duan et al. (2006), their GARCH-jump model postulates that the asset price process and the pricing kernel process are jointly distributed: the asset price follows a GARCH-jump process and the pricing kernel follows a diffusion-jump process. Moreover, the jumps in the asset price and pricing kernel processes are governed by the same Poisson process and the jump magnitudes of them are correlated. The framework in this paper

[14] The mean reduces exactly by the multiplying factor of $(1-\lambda)^{1/n}$, but for the variance, it reduces approximately by the multiplying factor of $(1-\lambda)^{1/n}$ if the considered time step is small. According to our experiments, the subinterval with the length of $1/n$ days is small enough such that this approximation for the variance is reliable.

allows us to construct a lattice model to price European-style as well as American-style options for the sophisticated GARCH-jump models in Duan et al. (2006). Here we take the NGARCH-jump model in Duan et al. (2006) for example to show the generality of our pricing framework. Different from Duan et al. (2006), we fix the time step Δt to be one day. Thus their NGARCH-jump model can be rewritten as follows to fit into our framework.

$$\ln\left(\frac{S_{t+1}}{S_t}\right) = m_t + \sqrt{h_t}\mathbf{X}_t,$$

where

$$\begin{aligned}
m_t &= r_f - \frac{h_t}{2} - \lambda\kappa[K_t - 1] = r_f - \frac{h_t}{2} - \lambda\exp(b\mu + b^2\delta^2/2)[K_t - 1],\\
\mathbf{X}_t &= Z_t + \sum_{l=1}^{q_t^Q} J_t^{(l)}\\
Z_t &\sim N^Q(0, 1),\\
J_t^{(l)} &\sim N^Q\left(\frac{\bar{\mu} + b\rho\delta\bar{\gamma}}{\sqrt{\Delta t}}, \frac{\bar{\gamma}^2}{\Delta t}\right),\\
K_t &= \exp(\sqrt{h_t}(\bar{\mu} + b\rho\delta\bar{\gamma})/\sqrt{\Delta t} + h_t\bar{\gamma}^2/(2\Delta t)),
\end{aligned}$$

and q_t^Q counts the number of Poisson events occurring over the period $(t, t+1]$ with the daily jump intensity $\lambda\kappa$. In addition, the variance updating function is[15]

$$\begin{aligned}
h_{t+1} - h_t &= f(\nu_{t+1}, h_t) = \beta_0 + (\beta_1 - 1)\,h_t\\
&\quad + \beta_2\left(\frac{1 + \lambda\kappa\tilde{\gamma}^2/\Delta t}{1 + \lambda(\bar{\mu}^2 + \bar{\gamma}^2)/\Delta t}\right) h_t\left(\nu_{t+1} - c^Q\right)^2,
\end{aligned}$$

where

$$\begin{aligned}
c^Q &= c^P\sqrt{\frac{1 + \lambda(\bar{\mu}^2 + \bar{\gamma}^2)/\Delta t}{1 + \lambda\kappa\tilde{\gamma}^2/\Delta t}} + \frac{\lambda[\bar{\mu} - \kappa(\bar{\mu} + b\rho\delta\bar{\gamma})]/\sqrt{\Delta t} - b\rho\sqrt{\Delta t}}{\sqrt{1 + \lambda\kappa\tilde{\gamma}^2/\Delta t}},\\
\tilde{\gamma}^2 &= (\bar{\mu} + b\rho\delta\bar{\gamma})^2 + \bar{\gamma}^2,
\end{aligned}$$

and

$$\nu_{t+1} = \frac{(\ln S_{t+1} - \ln S_t - m_t)/\sqrt{h_t} - E^Q(\mathbf{X}_t)}{\sqrt{Var^Q(\mathbf{X}_t)}},$$

[15] The counterpart of this variance updating function under the physical probability measure is $h_{t+1} - h_t = \beta_0 + (\beta_1 - 1)\,h_t + \beta_2 h_t\left(u_{t+1} - c^P\right)^2$, where u_{t+1} represents the innovation of the logarithmic asset price under the physical probability measure.

with

$$E^Q(\mathbf{X}_t) = \lambda\kappa(\bar{\mu} + b\rho\delta\bar{\gamma})/\sqrt{\Delta t},$$
$$Var^Q(\mathbf{X}_t) = 1 + \lambda\kappa\tilde{\gamma}^2/\Delta t.$$

Note that the above equations follows the specification in Table 4.1 of Duan et al. (2006) except that the notation γ in their model is replace by δ, and the risk-free interest rate r_f, conditional variance h_t, and jump intensity $\lambda\kappa$ are expressed on daily basis in this paper rather than on annual basis in Duan et al. (2006). The additional parameters to adapt our model to Duan et al. (2006) model are the jump-related parameters in the pricing kernel, b, μ and δ^2, the jump magnitude parameters for the underlying asset price, $\bar{\mu}$ and $\bar{\gamma}^2$, the contemporaneous correlation coefficient between the jump magnitudes of the underlying asset price and the pricing kernel, ρ, and the daily jump intensity for the pricing kernel and asset price, $\lambda\kappa$. In addition, it also needs the GARCH parameters, β_0, β_1, β_2, and c^P for the asset price process under the physical probability measure.

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